CHAPTER I
PRELIMINARIES

Notations and terminology required in the sequel are developed in this chapter. We also list here some definitions and results so as to make the thesis self-contained.

This chapter is divided into four sections. In Section 1, the basic results about categories are discussed. The concept of biordered set is introduced in Section 2. In the remaining sections we discuss the concepts of inductive groupoids and Rees groupoids and state some of their properties. For well-known concepts and definitions, not explicitly given here, one may refer to [10] and [12].

1. CATEGORIES

In this section we summarise some definitions and results regarding categories which are required in the sequel. As far as possible, we follow [12] for definitions, notations, concepts etc.

If \( C \) is a category we write \('A \in C'\) to mean that \( A \) is an object of \( C \) and, \'\( f \) in \( C \)\' to mean
that \( f \) is a morphism in \( C \). A morphism \( f : A \rightarrow B \)
in \( C \) is said to be an isomorphism if there exists
\( g : B \rightarrow A \) in \( C \) such that \( fg = 1_A \) and \( gf = 1_B \)
where \( 1_A \) and \( 1_B \) denote the identity morphisms on \( A \)
and \( B \) respectively. In this thesis we shall use the
following convention regarding composition of morphisms.
The composition \( fg \) means \( f \) acts first. That is the
morphisms are composed in the order in which they appear in
a commutative diagram. It may be noted that the convention
adopted in [12] is the left-right dual of this.

A small category in which every morphism is an
isomorphism is called a groupoid. In dealing with groupoids
we shall often identify the objects with the corresponding
identity morphisms. To be explicit, in this case \( 'f \in C' \)
will be used to denote the fact that \( f \) is a morphism in \( C \).

If \( C \) is any category, then the dual category \( C^* \)
is defined as follows. The class of objects of \( C^* \) is the
same as that of \( C \) and morphisms of \( C^* \) are \( f^* \) such
that \( f \rightarrow f^* \) is a one-to-one correspondence between the
morphisms of \( C \) and morphisms of \( C^* \) with the following
properties. If \( f : A \rightarrow B \) in \( C \) then \( f^* : B \rightarrow A \)
in \( C^* \) and \( (fg)^* = g^*f^* \).
A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be full if for each pair $(A, B)$ of objects of $\mathcal{C}$, the restriction of $F$ to $[A, B]_\mathcal{C}$ is a mapping onto $[F(A), F(B)]_\mathcal{D}$, where $[A, B]_\mathcal{C}$ denotes the morphism set of all morphisms from $A$ to $B$ in $\mathcal{C}$. $F : \mathcal{C} \to \mathcal{D}$ is said to be faithful if the above mapping is one-to-one. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is said to be a full subcategory if the inclusion functor is full.

The following examples of categories and functors are of interest.

We denote by $\text{Set}$ the category whose class of objects is the class of all small sets and morphisms are mappings between these sets. If $\mathcal{C}$ is any category, for each $\alpha \in \mathcal{C}$, there exists a functor $H^\alpha : \mathcal{C} \to \text{Set}$ defined by $\beta \mapsto [\alpha, \beta]_\mathcal{C}$ and for $\Theta : \beta \to \gamma$ in $\mathcal{C}$, $H^\alpha(\Theta) : [\alpha, \beta]_\mathcal{C} \to [\alpha, \gamma]_\mathcal{C}$ is the mapping defined by $u \mapsto u\Theta$. $H^\alpha$ is called the covariant hom-functor corresponding to $\alpha$. Similarly there is a contravariant hom-functor $H_\alpha : \mathcal{C} \to \text{Set}$ defined by $H_\alpha(\beta) = [\beta, \alpha]_\mathcal{C}$ and for $\Theta : \beta \to \gamma$, $H_\alpha(\Theta) : [\gamma, \alpha]_\mathcal{C} \to [\beta, \alpha]_\mathcal{C}$ defined by $u \mapsto \Theta u$. 
The following proposition characterises natural transformations to any functor from a hom-functor.

**PROPOSITION 1.1.** (cf. [12], Yoneda Lemma, p.61). If \( K : \mathcal{D} \rightarrow \text{Set} \) is a functor and \( R \) is an object of \( \mathcal{D} \), then there exists a bijection \( Y : \text{Nat}(H^R, K) \rightarrow K(R) \)

where \( \text{Nat}(H^R, K) \) is the set of all natural transformations from \( H^R \) to \( K \). Further \( Y \) sends each natural transformation \( \alpha : H^R \rightarrow K \) to the element \((1_R) \alpha_R \) in \( K(R) \). Further the assignment \( R \mapsto H^R \) and \( f : S \rightarrow R \mapsto H^f : H^R \rightarrow H^S \) defines a faithful functor \( Y : \mathcal{D}^* \rightarrow \text{Set}^\mathcal{D} \). \( Y \) is called the Yoneda functor \( \square \)

This proposition directly gives the following characterization of natural transformations between hom-functors.

**PROPOSITION 1.2.** Let \( \alpha, \beta \in C \). Every natural transformation \( \gamma : H^\alpha \rightarrow H^\beta \) is of the form \( H^f \) for a unique \( f : \beta \rightarrow \alpha \) in \( C \) \( \square \)

By a partial functor \( S \) from a category \( \mathcal{D} \) to a category \( C \), we mean a pair \( (S, \mathcal{D}') \) where \( \mathcal{D}' \) is a full subcategory of \( \mathcal{D} \) and \( S : \mathcal{D}' \rightarrow C \) is a functor.
We write $D' = \text{dom } S$. If $S, T : D \to C$ are partial functors, a natural transformation $\sigma : S \to T$ is a natural transformation between the restrictions of $S$ and $T$ to $\text{dom } S \cap \text{dom } T$.

A natural transformation between two set-valued functors is said to be injective [surjective] if each of the component is injective [surjective].

If $D$ is a category then $\delta : D \to D \times D$ denotes the diagonal functor defined by $\alpha \mapsto (\alpha, \alpha)$ and $f \mapsto (f, f)$ where $\alpha$ stands for objects and $f$ for morphisms. If $S, T : D \to C$ are functors, we denote by $S \times T : D \times D \to C \times C$ the product functor defined componentwise. We shall sometimes use this notation to denote the composite functor $\delta \circ S \times T : D \to C \times C$ also. The exact meaning of the notation will be clear from the context.

If $S$ and $T$ are functors from $D$ to $\text{Set}$ then we denote by $S \times T$ the composite $\delta \circ S \times T \circ \otimes$ where $\otimes : \text{Set} \times \text{Set} \to \text{Set}$ is the cartesian product functor. Thus $S \times T$ is a functor from $D$ to $\text{Set}$. It may be noted that $S \times T$ is the direct product of $S$ and $T$ in the functor category $\text{Set}^{D}$. 

A category $\mathcal{C}$ is said to be a preorder if for each pair $\alpha, \beta$ of objects of $\mathcal{C}$, the morphism set $[\alpha, \beta]_{\mathcal{C}}$ contains at most one element. If $(X, \preceq)$ is a quasiordered set (by a quasiorder we mean a reflexive, transitive relation) then for $x, y \in X$, define morphism sets as follows:

$$[x, y]_X = \begin{cases} \emptyset & \text{if } (y, x) \notin \preceq \\ (x, y) & \text{if } (y, x) \in \preceq \end{cases}$$

Then $X$ becomes a category, which is a preorder. Every quasiordered set can thus be treated as a preorder. This preorder is the dual of the one defined in [12], corresponding to $(X, \preceq)$.

Let $\mathcal{D}$ be an arbitrary category. Identify all the morphisms in each hom-set $[\alpha, \beta]_{\mathcal{D}}$. Then the quotient category is a preorder on the same set of objects as in $\mathcal{D}$. This new category is called the preorder of $\mathcal{D}$.

A preorder in which, for each $\alpha, \beta; [\alpha, \beta] \cup [\beta, \alpha]$ is at most singleton is called a partial order. Every partially ordered set will be treated as a preorder. We do not distinguish between partially ordered sets and preorder with the above property. If $I$ and $\Lambda$ are partially ordered sets, the preorder $I \times \Lambda$ is also a partially
ordered set. In particular we shall denote by \( n \) the preorder on the set \( \{0, 1, \ldots, n-1\} \) with the natural order.

We call a preorder \( \mathcal{D} \) a lower semilattice if it has coproducts for every pair of objects. If product of any two objects exists, \( \mathcal{D} \) is said to be an upper semilattice. Usually we take semilattices to be lower semilattices.

If \( (I, \leq) \) is a partially ordered set we often use the following notations. For \( i \in I \)

\[
\begin{align*}
[\leftarrow, i] &= \{ j \in I : j \leq i \} \\
[i, \rightarrow] &= \{ j \in I : i \leq j \}
\end{align*}
\]

If \( C \) is any category, then by \( C^2 \) we denote the category defined as follows. The objects of \( C^2 \) are morphisms of \( C \) and for \( f : A \to B \) and \( g : C \to D \) in \( C \) a morphism in \( C^2 \) from \( f \) to \( g \) is a pair \((h, k)\) where \( h : A \to C \) and \( k : B \to D \), such that \( fk = hg \). \( C^2 \) is called the category of arrows of \( C \). Note that \( C^2 \) is the functor category of all functors from the preorder \( 2 \) into \( C \).

If \( S : C \to \mathcal{D} \) is a functor and \( R \in \mathcal{D} \), a universal arrow from \( S \) to \( R \) is a pair \((C, u)\)
where \( C \) is an object of \( \mathcal{C} \) and \( u : S(C) \to R \) in \( \mathcal{D} \) such that if \( v \) is an arrow from \( S(C') \) to \( R \) for some \( C' \in \mathcal{C} \), then there exists a unique arrow \( h : C' \to C \) such that \( S(h)u = v \).

Let \( F : \mathcal{D} \to \mathcal{C} \) be a functor and let \( A \in \mathcal{C} \). A \textit{cone} in \( \mathcal{C} \) from \( A \) to \( F \) is a natural transformation \( \sigma_A : A^* \to F \) where \( A^* \) is the constant functor from \( \mathcal{D} \) to \( \mathcal{C} \) which maps each object to \( A \) and each morphism to \( 1_A \).

If \( S, T : \mathcal{C} \to \mathcal{D} \) and \( S', T' : \mathcal{C}' \to \mathcal{D}' \) are functors and \( \alpha : S \to T, \beta : S' \to T' \) are natural transformations, then we define a morphism between natural transformations \( \alpha \) and \( \beta \) as follows. A morphism from \( \alpha \) to \( \beta \) is a pair \((F, G)\) of functors where \( F : \mathcal{C} \to \mathcal{C}' \) and \( G : \mathcal{D} \to \mathcal{D}' \) such that
\[
SG = FS', \quad TG = FT' \quad \text{and} \quad \alpha G = F \beta.
\]
Here \((\alpha G)_A = \alpha_A G\) and \((F \beta)_A = \beta_{F(A)}\) for each \( A \in \mathcal{C} \).

The class of all natural transformations between functors of small categories form a category with morphisms as defined above.
Let $\mathcal{D}$ and $\mathcal{C}$ be categories. An adjunction from $\mathcal{D}$ to $\mathcal{C}$ is a triplet $(F, G, \varphi)$ where $F : \mathcal{D} \to \mathcal{C}, G : \mathcal{C} \to \mathcal{D}$ are functors and $\varphi$ is a function which assigns to each pair of objects $D \in \mathcal{D}$ and $C \in \mathcal{C}$, a bijection

$$\varphi_D, C : \left[ F(D), C \right]_\mathcal{C} \to \left[ D, G(C) \right]_\mathcal{D}$$

which is natural in $D$ and $C$. In this case $F$ is called a left adjoint for $G$, and $G$ is said to be a right adjoint for $F$.

A subcategory $\mathcal{A}$ of $\mathcal{D}$ is said to be a reflective subcategory, if the inclusion functor $\subseteq : \mathcal{A} \to \mathcal{D}$ has a left adjoint. For construction of adjoints it is convenient to use the following result.

**Proposition 1.3.** ([12], Theorem 2, p.81). Each adjunction $(F, G, \varphi) : \mathcal{C} \to \mathcal{D}$ is completely determined by the items in any one of the following.

1. Functors $F, G$ and a natural transformation $\eta : 1_\mathcal{C} \to FG$ such that each $\eta_X : X \to FG(X)$ is universal from $X$ to $G$. $\eta$ is called the unit of the adjunction.

2. The functor $G : \mathcal{D} \to \mathcal{C}$ and for each $X \in \mathcal{C}$ an object $F_0(X) \in \mathcal{D}$ and a universal arrow
$\eta_X : X \to G(F_0(X))$ from $X$ to $G$.

(iii) Functors $F, G$ and a natural transformation $\xi : GF \to 1_D$ such that each $\xi_A : GF(A) \to A$ is universal from $F$ to $A$. $\xi$ is called the co-unit of the adjunction.

(iv) The functor $F : C \to D$ and for each $A \in D$ an object $G_0(A) \in C$ and an arrow $\xi_A : F(G_0(A)) \to A$ which is universal from $F$ to $A$.

(v) Functors $F, G$ and natural transformations $\eta : 1_C \to FG$ and $\xi : GF \to 1_D$ such that the composites

$$G \xrightarrow{G\eta} GF \xrightarrow{\xi G} G \quad \text{and} \quad F \xrightarrow{\eta F} FGF \xrightarrow{F\xi} F$$

are identities. Conversely every adjunction determines the data described in (i), (ii), (iii), (iv) and (v) above.

Since the function $\varphi$ in the adjunction $(F, G, \varphi)$ is completely determined by the unit $\eta$ and the co-unit $\xi$ and conversely, the adjunction may also be represented as $(F, G, \eta, \xi) : C \to D$.

An adjoint equivalence of categories $C$ and $D$ is an adjunction $(F, G, \eta, \xi) : C \to D$ in which both
A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an equivalence of categories $\mathcal{C}$ and $\mathcal{D}$ if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG \cong 1_\mathcal{C}$ and $GF \cong 1_\mathcal{D}$.

The following proposition shows the relation between the two.

**Proposition 1.4.** ([12], Theorem 1, p.91). The following properties of a functor $F : \mathcal{C} \to \mathcal{D}$ are equivalent.

(i) $F$ is an equivalence of categories.

(ii) $F$ is part of an adjoint equivalence $(F, G, \eta, \varepsilon) : \mathcal{C} \to \mathcal{D}$.

We close this section with a description of transformations of adjunctions. Given two adjunctions $(F, G, \varphi) : \mathcal{C} \to \mathcal{D}$ and $(F', G', \varphi') : \mathcal{C}' \to \mathcal{D}'$ we define a transformation of adjunctions from $(F, G, \varphi)$ to $(F', G', \varphi')$ as a pair of functors $(K, L)$ where $K : \mathcal{C} \to \mathcal{C}'$ and $L : \mathcal{D} \to \mathcal{D}'$ are such that

(i) in the following diagram of functors, both squares are commutative.
(D1.1)  

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow K & & \downarrow K' \\
\mathcal{C}' & \xrightarrow{F'} & \mathcal{D}'
\end{array}
\]

(ii) the following diagram of hom-sets and adjunctions is commutative for all objects \( X \in \mathcal{C} \) and \( A \in \mathcal{D} \):

\[
\begin{array}{ccc}
[F(X), A] & \xrightarrow{\varphi} & [X, G(A)] \\
\downarrow L^* & & \downarrow K^* \\
[L(F(X)), L(A)] & \xrightarrow{=} & [K(X), K(G(A))] \\
\end{array}
\]

\[
\begin{array}{ccc}
[F'(K(X)), L(A)] & \xrightarrow{\varphi'} & [X(X), G'(L(A))] \\
\end{array}
\]

Here \( L^* \) and \( K^* \) are the maps given by \( f \mapsto L(f) \) and \( g \mapsto K(g) \) respectively.

2. BIORDERED SETS

In this section we state the basic results about biordered sets which are needed in the sequel.

The concept of a biordered set was introduced by Nambooripad [19]. A set \( E \) together with a partial
binary operation is referred to as a partial algebra. The domain of the partial binary operation will be denoted by $D_E$. Then $D_E$ is a relation on $E$ and $(e, f) \in D_E$ if and only if the product $ef$ exists in the partial algebra $E$. When no confusion is likely, we shall indicate the product in $E$ by juxtaposition. On $E$ define:

\begin{align*}
(1.1) \quad \omega^* &= \{(e, f) : fe = e\}, \quad \omega^f = \{(e, f) : ef = e\}
\end{align*}

and

\begin{align*}
(1.2) \quad R &= \omega^* \cap (\omega^*)^{-1}, \quad L = \omega^f \cap (\omega^f)^{-1} \quad \text{and} \quad \omega = \omega^* \cap \omega^f
\end{align*}

If $\rho$ is any relation on $E$, we write

$$
\rho(e) = \{ f \in E : f \rho e \} \quad \text{for} \quad e \in E.
$$

**DEFINITION 1.5.** ([19], Definition 1.1). Let $E$ be a partial algebra. Then $E$ is a biordered set if the following axioms and their left-right duals hold. (The dual of a statement $T$ will be denoted by $T^*$). In the following, $e, f$ etc. denote arbitrary elements of $E$.

(B1) $\omega^*$ and $\omega^f$ are quasiorders on $E$ and

$$
D_E = (\omega^* \cup \omega^f) \cup (\omega^* \cup \omega^f)^{-1}
$$

(B2) $f \in \omega^*(e) \Rightarrow f R fe \omega e$. 


(B22) \( g \circ f; f, g \in \omega^*(e) \Rightarrow ge \circ fe. \)

(B31) \( g \circ f \circ e \Rightarrow gf = (ge)f \)

(B32) \( g \circ f; f, g \in \omega^*(e) \Rightarrow (fg)e = (fe)(ge). \)

Let \( M(e, f) \) denote the quasiordered set 
\((\omega^*(e) \cap \omega^*(f), \preceq)\) where \( \preceq \) is defined by

\[
(1.3) \quad g \preceq h \iff eg \preceq eh \quad \text{and} \quad gf \preceq hf
\]

Then the set
\[
S(e, f) = \{ h \in M(e, f) : g \preceq h \quad \text{for all} \quad g \in M(e, f) \}
\]
is called the sandwich set of \( e \) and \( f \) in that order.

(B4) \( f, g \in \omega^*(e) \Rightarrow S(f, g)e = S(fe, ge). \)

The biordered set \( E \) is said to be regular if

(R) \( S(e, f) \neq \emptyset \)

for all \( e, f \in E \)

The partial binary operation of a biordered set \( E \)
is called the basic product of \( E \) (cf. [5]).

The following proposition shows that axiom (B4) in the
above definition can be replaced by a more convenient axiom (B4').
PROPOSITION 1.6. ( [19] , Proposition 2.4). Assume that the partial algebra $E$ satisfies all axioms for a biordered set except axioms (B4) and (B4)*. Then $E$ satisfies axiom (B4) if and only if it satisfies the following:

(B4') If $g, h \in \omega^+ (e)$ and $ge \omega^+ h$, then there exists $g_1 \in \omega^+ (e)$ such that $g_1 \omega^+ h$ and $g_1 e = ge$.

Further when $g_1$ exists, it is unique $\Box$

If $S$ is any semigroup, then define a partial algebra on $E(S)$, the set of idempotents of $S$ as follows. For $e, f \in E(S)$,

$$ef = \begin{cases} ef & \text{if } ef = e, ef = f, fe = e \text{ or } fe = f \text{ in } S \\ \text{undefined otherwise} \end{cases} \tag{1.4}$$

The $E(S)$ is a biordered set. Further if $S$ is regular, $E(S)$ will be a regular biordered set. We denote the biordered set $E(S)$ by $E(S)$.

DEFINITION 1.7. ( [19] , Definition, 1.2). Let $E, E'$ be biordered sets and $\theta : E \rightarrow E'$ be a mapping. Then $\theta$ is called a bimorphism if it satisfies the following axiom.

(M) $(e, f) \in D_E \Rightarrow (e \theta, f \theta) \in D_E$, and $(ef) \theta = (e \theta)(f \theta)$

$\theta$ is called a regular bimorphism if, in addition,
A biordered subset \( E' \) of a biordered set \( E \), is a biordered set under the restriction of the operation in \( E \).

If \( \Theta : E \rightarrow E' \) and \( \Theta' : E' \rightarrow E'' \) are bimorphisms, then, clearly \( \Theta \Theta' : E \rightarrow E'' \) is a bimorphism.

Further if \( \Theta \) and \( \Theta' \) are both regular, then so is \( \Theta \Theta' \).

Thus we get a category \( \mathcal{B} \) whose objects are biordered sets and morphisms are bimorphisms; and a subcategory \( \mathcal{RB} \) of \( \mathcal{B} \) whose objects are regular biordered sets and morphisms are regular bimorphisms. A biorder isomorphism is an isomorphism in this category.

Further, if \( \mathcal{RS} \) denotes the category, whose objects are regular semigroups and morphisms are homomorphisms of regular semigroups, then the assignment \( S \mapsto \mathbb{E}(S) \) and \( \varphi \mapsto \mathbb{F}(\varphi) = \varphi \big|_{\mathbb{E}(S)} \) where \( \varphi : S \rightarrow S' \) in \( \mathcal{RS} \) is a functor from \( \mathcal{RS} \rightarrow \mathcal{RB} \).

The following results about biordered sets are often used. In the following \( E \) denotes a biordered set.
PROPOSITION 1.8. ([19] Proposition 2.7). For every \( e \in E \), \( \omega^+(e) \), \( \omega^-(e) \) and \( \omega(e) \) are biordered subsets of \( E \).

PROPOSITION 1.9. ([19], Proposition 2.2). If \( (e, f) \in D_E \), then \( ef \in \mathcal{S}(f, e) \).

PROPOSITION 1.10. Let \( e, e', f, f' \in E \), \( e \perp e' \) and \( f \parallel f' \). Then \( \mathcal{S}(e, f) = \mathcal{S}(e', f') \). Further if \( g, h \in \mathcal{S}(e, f) \), then there exists \( g' \in E \) such that \( g \parallel g' \perp h \) and \( g' \in \mathcal{S}(e, f) \).

Proof. The first statement is the same as Proposition 2.5 of [19], and the second statement follows from Corollary 2.11 of [19].

PROPOSITION 1.11. ([19], Corollary 2.8). Let \( e, f \in E \) be such that either \( e \parallel f \) or \( e \perp f \). For all \( g \in \omega(e) \) define

\[
(1.5) \quad g \mathcal{T}(e, f) = \begin{cases} gf & \text{if } e \parallel f \\ fg & \text{if } e \perp f \end{cases}
\]

Then \( \mathcal{T}(e, f) \) is an isomorphism of the biordered subset \( \omega(e) \) onto \( \omega(f) \) such that if either \( e \parallel f \parallel g \) or \( e \perp f \perp g \), then

\[
\mathcal{T}(e, f) \mathcal{T}(f, g) = \mathcal{T}(e, g)
\]

and
\[ T(e, f) = (T(f, e))^{-1} \]

Further, if \( g, g' \in \omega^+(e) \) and \( g \vdash g' \), then
\[ T(g, g') T(g', g'e) = T(g, ge) T(ge, g'e) \]

Let \( E \) be a biordered set. An \( E \)-square is a
\[ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \]
matrix of elements of \( E \) where \( e \vdash f \), \( g \vdash h \), \( e \vdash g \) and \( f \vdash h \). By a degenerate \( E \)-square we mean any one of the form
\[ \begin{pmatrix} e & e \\ g & g \end{pmatrix}, \begin{pmatrix} e & f \\ g & f \end{pmatrix} \text{ or } \begin{pmatrix} e & e \\ g & e \end{pmatrix} \]

An \( E \)-square of the form \[ \begin{pmatrix} g & h \\ eg & eh \end{pmatrix} \] where \( g \vdash h \), \( g, h \in \omega^l(e) \); is said to be row-singular. Dually if \( g \vdash h \) and \( g, h \in \omega^r(e) \), then the \( E \)-square \[ \begin{pmatrix} g & ge \\ h & he \end{pmatrix} \] is said to be column-singular. A singular \( E \)-square is one which is either row-singular or column-singular.

An \( E \)-square \[ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \] is said to be \( \tau \)-commutative if
\[ T(e, f) T(f, h) = T(e, g) T(g, h) \]
PROPOSITION 1.12. ([19], Proposition 2.9). If \( g, h \in \omega^e(e) \) and \( ge \omega^e he \), then there exists a unique E-square
\[
G = \begin{pmatrix} g & g_1 \\ g_2 & h' \end{pmatrix}
\]
satisfying the following conditions.

(a) \( h' \omega h \), (b) \( ge = g_1e \) and (c) \( g_2e = h'e = (he)(ge) \).

Further \( h' = h \) if and only if \( ge \perp he \).

3. INDUCTIVE GROUPOIDS

Recall that a groupoid is a small category in which every morphism is an isomorphism. If \( G \) is a groupoid, the class of morphisms will be denoted by \( G \) itself, and the class of vertices (or objects) will be denoted by \( V(G) \).

For \( e \in V(G) \), we denote the identity morphism corresponding to \( e \) by \( e \) itself, if there is no ambiguity, otherwise by \( 1_e \). For \( x \in G \), \( e_x [f_x] \) will denote the domain \( [\text{codomain}] \) of \( x \) and for \( e, f \in V(G) \), \( G(e, f) \) will denote the hom-set of all morphisms with domain \( e \) and codomain \( f \). If \( \varphi : G \rightarrow G' \) is a groupoid map, that is a functor, then \( \varphi \) will also denote the morphism map of \( \varphi \) and \( V(\varphi) \) will denote the vertex map of \( \varphi \). \( \varphi \) is said to be \( V \)-injective, \( V \)-surjective or \( V \)-bijective if \( V(\varphi) \) has the corresponding property.
We denote by $D_G$ the domain of the partial binary operation on $G$.

**DEFINITION 1.13.** ([19], Definition 3.1). Let $G$ be a groupoid and $\leq$ be a partial order on $G$. Then $(G, \leq)$ is said to be an ordered groupoid, if the following axioms hold. Here letters $x, y$ etc. denote arbitrary elements of $G$ and $e, f$ etc. those of $V(G)$.

1. **(OG1)** $x' \leq x, y' \leq y$ and $(x, y), (x', y') \in D_G \Rightarrow x'y' \leq xy$.
2. **(OG2)** $x \leq y \Rightarrow x^{-1} \leq y^{-1}$
3. **(OG3)** If $1_e \leq 1_{e_x}$, then there exists a unique element $x \mid e$ (called the restriction of $x$ to $e$) in $G$ such that $x \mid e \leq x$ and $e_x \mid e = e$

We denote by $OG$ the category whose objects are ordered groupoids and morphisms are order preserving functors. Here the axioms (OG1) and (OG2) are self-dual. The dual of (OG3) is the following:

4. **(OG3)*** If $f \leq f_x$, then there exists a unique element $f \mid x$ such that $f \mid x \leq x$ and $f(f \mid x) = f$.

The dual of restriction is called corestriction.
The following result shows that axioms for an ordered groupoid are self-dual.

**PROPOSITION 1.14.** ([19], Proposition 3.1). Let $G$ be a groupoid, and $\leq$ a partial order on $G$ such that $(G, \leq)$ satisfies axioms (OG1) and (OG2). Then it satisfies (OG3) if and only if it satisfies (OG3)*.

Let $E$ be a biordered set. Since $R(\leq E \times E)$ is an equivalence relation on $E$, $R$ is the morphism set of a groupoid whose vertex set is $E$ (cf. [9], p.9). We denote this groupoid also by $R$. $R$ becomes an ordered groupoid if we define partial order on $R$ by:

$$(g, h) \leq (e, f) \iff g \leq e \text{ and } h = gf$$

Similarly we may regard $\mathcal{L}$ also as an ordered groupoid whose vertex set is $E$. Let $1_E$ denote the discrete groupoid with vertex set $E$ and $j_r$ and $j_l$ be the inclusions of the discrete groupoid $1_E$ in $R$ and $\mathcal{L}$ respectively. Then by Proposition 3.3 of [19] we get that there exists an ordered groupoid $\mathcal{Q}(E)$, such that, the following diagram is a pushout in the category $\mathcal{OG}$.
The morphisms of $\mathcal{G}(E)$ and the partial order on $\mathcal{G}(E)$ can be described as follows. If we write $j_R(e, f) = c(e, f)$ for $(e, f) \in R$ and $j_L(e', f') = c(e', f')$ for $(e', f') \in L$, then, we can represent the elements of $\mathcal{G}(E)$ as $c = c(e_0, e_1, \ldots, e_n)$, where $(e_{i-1}, e_i) \in R \cup L$.

For $c = c(e_0, e_1, \ldots, e_n) \in \mathcal{G}(E)$ and $h \in \omega^+(e_0)$, define

\[(1.7) \quad h \ast c = c(h, h_0, h_1, \ldots, h_n),\]

where $h_0 = he_0$, and $h_i = h_{i-1} \tau(e_{i-1}, e_i)$, for $i = 1, \ldots, n$.

Also define $\leq$ on $\mathcal{G}(E)$ as follows. If $c, c' \in \mathcal{G}(E)$ define

\[(1.3) \quad c \leq c' \iff e_c \omega e_{c'}, \quad \text{and} \quad c = e_c \ast c'.\]

It follows easily that the partial order on $\mathcal{G}(E)$ coincides with this partial order (cf. [19]). $\mathcal{G}(E)$ is called the groupoid of $E$-chains of $E$. If $c \in \mathcal{G}(E)$ with $e_c = f_\delta$, 

\[
\begin{array}{ccc}
R & \xrightarrow{j_R} & \mathcal{G}(E) \\
\downarrow j_r & & \uparrow j_L \\
1_E & \xrightarrow{j_1} & L
\end{array}
\]
then $c$ is said to be a cycle in $E$.

If $\Theta : E \to E'$ is a bimorphism, then

$$G(\Theta) : G(E) \to G(E'),$$
defined by,

$$(1.9) \quad G(\Theta)(c(e_0, \ldots, e_n)) = c(e_0 \Theta, \ldots, e_n \Theta)$$

is an order-preserving functor.

Let $G$ be an ordered groupoid and $\mathcal{E} : G(E) \to G$ be a $V$-isomorphism of $G(E)$ into $G$. If $e \not< f$ or $e \not< f$, we write $\mathcal{E}(e, f)$ for $\mathcal{E}(c(e, f))$. If

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$
is an $E$-square in $E$, we say that this $E$-square is $\mathcal{E}$-commutative, if

$$\mathcal{E}(e, f) \mathcal{E}(f, h) = \mathcal{E}(e, g) \mathcal{E}(g, h).$$

DEFINITION 1.15. ([19], Definition 3.2). Let $G$ be an ordered groupoid such that $V(G) = E$ is a regular biordered set and let $\mathcal{E}_G : G(E) \to G$ be a $V$-isomorphism. Then $(G, \mathcal{E}_G)$ is said to be an inductive groupoid if the following axioms and their duals hold.

$(I_G1)$ Let $x \in G$ and $e_1, e_2 \in \omega(e_x)$. If $e_1 \omega^* e_2$, then $f_1 \omega^* f_2$ and
\[ \mathcal{E}_G(e_1, e_1e_2)(x|e_1e_2) = (x|e_1) \mathcal{E}_G(f_1, f_1f_2), \]

where \( f_i = f_{x|e_i} \) for \( i = 1, 2 \).

(IG2) All singular \( E \)-squares are \( \mathcal{E}_G \)-commutative.

\( \mathcal{E}_G \) is called the evaluation of \( \mathcal{G}(E) \) in \( G \).

An inductive subgroupoid of \( (G, \mathcal{E}_G) \) is a pair \((G', \mathcal{E}_{G'})\) where \( G' \) is an ordered subgroupoid of \( G \) such that \( \mathcal{V}(G') \) is a biordered subset of \( \mathcal{V}(G) \) and

\[ \mathcal{E}_{G'} = \mathcal{E}_G \left| \mathcal{G}(\mathcal{V}(G')) \right. \]

Let \((G, \mathcal{E}_G)\) and \((G', \mathcal{E}_{G'})\) be inductive groupoids.

An order preserving functor \( \varphi : G \rightarrow G' \) is said to be an inductive functor if \( \mathcal{V}(\varphi) : \mathcal{V}(G) \rightarrow \mathcal{V}(G') \) is a bimorphism such that the following diagram is commutative.

\[ \begin{array}{ccc}
\mathcal{G}(\mathcal{V}(G)) & \xrightarrow{\mathcal{G}(\mathcal{V}(\varphi))} & \mathcal{G}(\mathcal{V}(G')) \\
\mathcal{E}_G \downarrow & & \mathcal{E}_{G'} \downarrow \\
G & \xrightarrow{\varphi} & G'
\end{array} \]

(D1.4)

We now give some examples of inductive groupoids which we use later.
Let $E$ be a biordered set. If $e, f \in E$, then $\omega(e)$ and $\omega(f)$ are biordered subsets of $E$. A biorder isomorphism of the $\omega$-ideal $\omega(e)$ onto $\omega(f)$ is called an $\omega$-isomorphism of $E$.

Let $T^*(E)$ denote the set of all $\omega$-isomorphisms of $E$. If $\alpha, \beta \in T^*(E)$ and $f_\alpha = e_\beta$, then $\alpha \beta$ is defined to be the composition of the two $\omega$-isomorphisms, and product is not defined otherwise. Then $T^*(E)$ is a groupoid whose vertex set may be identified with $E$. For $\alpha, \beta \in T^*(E)$, define

$$ (1.10) \quad \alpha \leq \beta \iff e_\alpha \omega e_\beta \quad \text{and} \quad \alpha = \beta \mid \omega(e_\alpha), $$

where $e_\alpha$ is the generator of the $\omega$-ideal, $\text{dom} \alpha$.

Then $T^*(E)$ becomes an ordered groupoid.

With the identification of $V(T^*(E))$ with $E$, $e_\alpha$ is the domain of $\alpha \in T^*(E)$. Also the restriction of $\alpha$ to $e \in \omega(e_\alpha)$ is the usual restriction of $\alpha$ to $\omega(e)$. Further for $c = c(e_0, \ldots, e_n) \in \mathcal{L}(E)$, define

$$ (1.11) \quad \tau_E(c) = \tau(e_0, e_1) \cdots \tau(e_{n-1}, e_n). $$

The following is an immediate consequence of definitions (1.7) and (1.11).
PROPOSITION 1.16. If \( c \in \mathcal{G}(E) \) and \( e \omega e_c \), then
\[
e_{\mathcal{T}_E(c)} = f_{e \star c} \]

PROPOSITION 1.17. ([19], Proposition 3.6). Let \( T^*(E) \) be the groupoid of \( \omega \)-isomorphisms of a biordered set \( E \) and let \( V(T^*(E)) = E \). Then \( T^*(E) \) is an inductive groupoid with partial order defined by Equation (1.10) and evaluation defined by Equation (1.11). Further
\[
(\mathcal{T}^*_E, \mathcal{T}_E) \text{ is an inductive subgroupoid of } T^*(E) \text{ where } \mathcal{T}^*_E = \text{im } \mathcal{T}_E \]

PROPOSITION 1.18. ([19], Proposition 3.7). Let \( G \) be an inductive groupoid with \( V(G) = E \). For \( x \in G \), define
\[
\alpha_G(x) : \omega(e_x) \rightarrow \omega(f_x) \]
by
\[
(1.12) \quad e \alpha_G(x) = f_{e \star x} \]
for all \( e \in \omega(e_x) \). Then \( \alpha_G(x) \in T^*(E) \). Further
\[
\alpha_G : G \rightarrow T^*(E) \text{ is a } V\text{-isomorphism} \]

Now we proceed to construct an inductive groupoid from a given regular semigroup.

Let \( S \) be a regular semigroup. Let
\[
G(S) = \{(x, x') : x \in S \text{ and } x' \in V(x)\} \]
where \( V(x) \) denotes the set of all inverses of \( x \).

Define a partial product in \( G(S) \) by

\[
(1.13) \quad (x, x')(y, y') = \begin{cases} 
(xy, y'x) & \text{if } x'x = yy' \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

Then \( G(S) \) is a groupoid whose vertex set may be identified with \( E(S) \), the set of idempotents of \( S \). Define \( \leq \) on \( G(S) \) as follows:

\[
(1.14) \quad (x, x') \leq (y, y') \iff x = (xx')y, \quad x' = y'(xx') \quad \text{and} \quad xx' \omega yy'.
\]

It is easily seen that if \( e \omega xx' \) then \( (x, x')|e = (ex, e'x) \). If \( c = c(e_0, \ldots, e_n) \in G(E) \) where \( E = E(S) \), define \( E_S(c) \) by

\[
(1.15) \quad E_S(c) = (e_0e_1\cdots e_n, e_n e_{n-1} \cdots e_0).
\]

**Proposition 1.19.** ([19], Theorem 3.8). Let \( S \) be a regular semigroup. Then \( G(S) \) is an inductive groupoid in which composition, partial order and evaluation are defined by (1.13), (1.14) and (1.15) respectively.

Further if \( \varphi : S \to S' \) is a homomorphism of regular semigroups, then \( G(\varphi) : G(S) \to G(S') \), defined by

\[
(1.16) \quad G(\varphi)(x, x') = (x\varphi, x'\varphi)
\]
is an inductive functor of \((G(S), E_S)\) into \((G(S'), E_{S'})\).

The following proposition is an immediate consequence of the definition of the inductive groupoid \(G(S)\).

**Proposition 1.20.** If \(G = G(S)\) where \(S\) is a regular semigroup, we denote \(a_G\) by just \(a\). Then

\[ e a(x, x') = x'e x \]

for all \(e \in \omega(xx')\) and if \(e, f \in E(S)\) with \(e R f [e \subseteq f]\), then

\[ a(e, f) = T(f, e) [T(e, f)] \]

Now we describe briefly the construction of semigroups from inductive groupoids.

Let \((G, E)\) be an inductive groupoid. If \(x \in G\) and \(h \in \omega^*(e_x)\), define

\[(1.17) \quad h * x = \xi(h, he_x)(x | he_x)\]

If \(h \in \omega(e_x)\), it can be seen that \(h * x = x | h\). Dually for \(k \in \omega^*(f_x)\), define

\[(1.18) \quad x * k = (f_x k | x) \xi(f_x k, k)\]

where \(f_x k | x\) is the corestriction of \(x\) to \(f_x k\).
Let $p$ be the relation defined on $G$ as follows. If $x, y \in G$

\[(1.19) \quad x \mathrel{p} y \iff e_x \mathcal{R} e_y, f_x \mathcal{L} f_y \quad \text{and} \quad x \in (f_x, f_y) = \mathcal{E}(e_x, e_y).\]

**PROPOSITION 1.21.** ([19], Lemma 4.5). The relation $p$ defined by (1.19) is an equivalence relation on $G$ such that, if $x, y \in G(e, e)$, then $x \mathrel{p} y$ implies $x = y$.

If $x \in G$, we write $\bar{x}$ to denote the $p$-class of $G$ containing $x$. The following is the principal structure theorem for regular semigroups.

**THEOREM 1.22.** ([19], Theorem 4.12). Let $G = (G, \mathcal{E})$ be an inductive groupoid and $p$ be the equivalence relation on $G$ defined by (1.19). On $S(G) = G/p$ define product by

\[(1.20) \quad \bar{x} \bar{y} = (x * h)(h * y)\]

where $x, y \in G$ and $h \in \mathcal{S}(f_x, e_y)$. Then $S(G)$ is a regular semigroup with respect to this product and $G$ is isomorphic to the inductive groupoid $G(S(G))$ of $S(G)$. 
In particular the biordered set \( V(G) \) is isomorphic to \( E(S(G)) \). Conversely if \( S \) is any regular semigroup, then \( S \) is isomorphic to \( S(G(S)) \).

A regular semigroup \( S \) is said to be \textbf{fundamental} if the only congruence on \( S \) contained in the Green's relation \( \mathcal{H} \) is the identity congruence. If \( S \) is a regular semigroup, then

\[
\mu_S = \left\{ (x, y) : a_s(x, x') = a_s(y, y') \text{ for some } x' \in V(x), y' \in V(y) \right\}
\]

is the greatest idempotent separating congruence on \( S \).

A subsemigroup \( T \) of \( S \) is said to be a \textbf{full subsemigroup} if \( E(T) = E(S) \).

The following is a structure theorem for fundamental regular semigroups.

\textbf{Theorem 1.23.} ([19], Theorem 5.2). Let \( E \) be a biordered set. Then \( T(E) = S(T^*(E)) \) is a fundamental regular semigroup whose biordered set is isomorphic to \( E \). Conversely, if \( S \) is any regular semigroup, then \( S \) is fundamental if and only if \( S \) is isomorphic to a full subsemigroup of \( T(E(S)) \). Further when \( S \) is fundamental

\[
\overline{a}_S : S \to T(E(S)) \text{ defined by}
\]

\[
(1.22) \quad \overline{a}_S(x) = a(x, x') \text{ for some } x' \in V(x)
\]
is a well defined mapping, which is an isomorphism of $S$ onto a full subsemigroup of $T(E(S))$.

**REMARK.** By Proposition 1.17, $(\mathcal{T}^*(E), \mathcal{T}_E)$ is an inductive groupoid. We denote $S(\mathcal{T}^*(E))$ by $E$. Then $E$ is idempotent generated and is called the fundamental idempotent generated semigroup on the biordered set $E$. We often denote elements of $E$ by $\overline{e}$ where $e \in \mathcal{J}(E)$; in which $\overline{e}$ will correspond to the element $\mathcal{T}_E(e)$.

We close this section with the description of the natural partial order on a regular semigroup, introduced by Nambooripad [23], which is related to the structure of regular semigroups. Also in the case of inverse semigroups, this partial order coincides with the usual partial order.

**DEFINITION 1.24.** Let $S$ be a regular semigroup. For $x, y \in S$ define

$$x \leq y \iff xS \subseteq yS \quad \text{and} \quad x = fy \quad \text{for some } f \in E(R_x).$$

$\leq$ is called the natural partial order on $S$.

The following characterization of the natural partial order is often useful.
PROPOSITION 1.25. ([25], Proposition 1.2). Let $S$ be a regular semigroup. Then for $x, y \in S$, the following are equivalent.

(a) $x \leq y$

(b) for every $f \in E(R_y)$ there exists $e \in E(R_x)$ such that $e \omega f$ and $x = ey$

(c) for every $f' \in E(L_y)$, there exists $e' \in E(L_x)$ such that $e' \omega f'$ and $x = ye'$. 

4. REES GROUPOIDS

We know that a Rees groupoid is the partial algebra of non-zero elements of a completely o-simple semigroup. The Greens relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ of a Rees groupoid $D$ are the restrictions of the corresponding relations for the completely o-simple semigroup $D^0$, to $D$. If $D$ and $D'$ are two Rees groupoids, a morphism from $D$ to $D'$ is a mapping that preserves the products. It may be noted that any homomorphism $\phi : D^0 \to D'^0$ of the corresponding completely o-simple semigroups will induce a morphism of $D$ into $D'$. But the converse is not true.

Let $D$ be a regular $\mathcal{D}$-class of a semigroup. By the trace of $D$ we mean the partial algebra $D(\ast)$.
defined as follows. For \( x, y \in D \)

\[
x \star y = \begin{cases} 
xy & \text{if } xy \in R_x \cap L_y \\
\text{undefined otherwise.}
\end{cases}
\]

Sometimes for convenience, we denote this trace also by \( D \).

It is known that \( D(*) \) is isomorphic to the partial algebra of non-zero elements of a completely \( 0 \)-simple semigroup. (cf. [6], Theorem 3.5) Thus \( D(*) \) is a Rees groupoid. If \( S \) is a regular semigroup, then

\[
S(*) = \bigcup \{ D(*) : D \in S/\theta \}
\]

is called the pseudo-groupoid associated with \( S \).

The following theorem shows that products in \( S \) can be expressed in terms of products in \( S(*) \) and elements in the appropriate sandwich sets.

**THEOREM 1.26** ([19], Theorem 1.2). Let \( S \) be a regular semigroup and \( x, y \in S \). Let \( e, f \in E(S) \) be such that \( e \mathcal{L} x \) and \( f \mathcal{R} y \) where \( \mathcal{L} \) and \( \mathcal{R} \) are Green's relations. Further, if \( h \in \mathcal{J}(e, f) \), then

\[
xy = (xh) \star (hy)
\]

where the product on the right is a product in \( S(*) \).