CHAPTER VI

EXAMPLES

In this chapter we provide some examples to illustrate the construction of combinatorial regular semigroups described in Chapter III. The first example is the construction of a combinatorial inverse semigroup starting from a small category. The fundamental four-spiral semigroup \( Sp_4 \) provides a bisimple pseudo-inverse case and \( DSp_4 \) presents a general situation.

The verification of the axioms for an admissible category is given in both the above cases. The example on ordered semigroups shows how structure theorems for combinatorial semigroups with additional structure may be obtained by imposing additional structure on the domain/codomain category.

1. COMBINATORIAL INVERSE SEMIGROUP

Let \( \mathcal{D} \) be a strictly skeletal small category with the following properties.

(i) There exists \( \alpha \in \mathcal{D} \) with \( \mathcal{D}[\alpha, \beta] \neq 0 \) for every \( \beta \in \mathcal{D} \) and such that every pair \( (f, g) \) of morphisms in \( \mathcal{D} \) with \( \text{dom} f = \text{dom} g = \alpha \), admits a pushout in \( \mathcal{D} \).
(ii) Every morphism with domain $\alpha$ is an epimorphism.

Note that (i) is satisfied if we assume that $\alpha$ is an initial object in $\mathcal{D}$ and that every pair of objects of $\mathcal{D}$ has a coproduct.

Let $P = H^{\alpha}: \mathcal{D} \to \text{Set}$. Then clearly $P$ is disjoint and non-empty. We prove that $P$ satisfies the conditions in Theorem 4.34. Since every natural transformation from $H^\beta$ to $P$ is of the form $H^f$ for a morphism $f: \alpha \to \beta$ in $\mathcal{D}$, (ii) above implies condition (1) of Theorem 4.34. Now let

$$x = H^f: H^\beta \to P, \ y = H^g: H^\gamma \to P \in (\mathcal{D} \downarrow P).$$

Then from the pushout diagram below

(D6.1)

we get that $u = H^{gt} = H^{fk}: H^s \to P$ is the product of $x$ and $y$ in $(\mathcal{D} \downarrow P)$, with projections $H^t: u \to y$ and $H^k: u \to x$. Therefore by Theorem 4.34

$$S = \bigcup_{\beta \in \mathcal{D}} [H^\beta, H^\alpha] \times [H^\beta, H^\alpha]$$
with product defined by (4.9) is a combinatorial inverse semigroup. Identifying \( x = H^f \in [H^\beta, H^\alpha] \) with the morphism \( f \), we get that

\[
S = \bigcup_{\beta \in \mathcal{D}} [\alpha, \beta] \times [\alpha, \beta]
\]

with product defined by

\[
(f, g)(r, s) = (ft, sk)
\]

where \( f, g : \alpha \to \beta, r, s : \alpha \to \gamma \) in \( \mathcal{D} \) and \( (t, k) \) is given by the pushout diagram below.

\[
\begin{array}{ccc}
\alpha & \xrightarrow{r} & \gamma \\
\downarrow{g} & & \downarrow{k} \\
\beta & \xleftarrow{t} & \delta
\end{array}
\]

(D6.2)

2. THE FUNDAMENTAL FOUR-SPIRAL SEMIGROUP, \( S_4 \)

The semigroup \( S_4 \) (cf. [3]) is defined as the fundamental idempotent generated regular semigroup on a four-spiral biordered set \( E_4 \). The structure of \( E_4 \) may be described by the diagram below. In the diagram

\[
e_i R f_i, f_i L g_i, g_i R h_i, h_i L e_{i+1} \quad \text{for } i = 0, 1, 2, \ldots
\]

and \( e_{i+1} \omega e_i, f_{i+1} \omega f_i, g_{i+1} \omega g_i \) and \( h_{i+1} \omega h_i \).
The definition of basic product and quasiorders are clear from the above diagram. It follows that $E_4$ is a pseudosemilattice. Since the only cycles in $E_4$ are of the form $c(e, e)$ for $e \in E_4$, we get that $Sp_4$ is combinatorial. Let $\varphi = \Phi(S)$ be the combinatorial functor corresponding to $S = Sp_4$ (cf. Theorem 2.12). Then

$$f_o \varphi(e_o, e_1) = e_1 \Delta e_o, \quad f_o = e_1 \Delta (e_o, f_o) = e_1 f_o = f_1$$

so that $\varphi(e_o, e_1) = \varphi(f_o, f_1)$. Similarly

$$\varphi(e_o, e_1) = \varphi(g_o, g_1) = \varphi(h_o, h_1)$$

Since $\varphi(e_o, e_1) = \varphi(h_o, h_1)$ and $e_1 \not\perp h_o$, it follows
Thus we have
\[ e_0 \varphi(e_0, e_2) = e_2 \varphi(e_0, e_1) \varphi(e_1, e_2) = e_0 \varphi(e_0, e_1)^2 \]

Thus we have
\[
\varphi(e_i, e_{i+1}) = \varphi(f_i, f_{i+1}) = \varphi(g_i, g_{i+1}) = \varphi(h_i, h_{i+1})
\]
and
\[
\varphi(e_i, e_{i+j}) = (\varphi(e_i, e_{i+1}))^j
\]

Also since \( \text{Sp}_4 \) is bisimple, \( \text{im} \varphi \) has only one object.

From the above observation, we arrive at the conclusion that \( \text{im} \varphi \) is a cyclic semigroup, generated by \( \varphi(e_0, e_1) \) with an identity.

Now we proceed to construct \( \text{Sp}_4 \) in terms of an admissible category (which will reduce to a P-category, since \( \text{Sp}_4 \) is pseudo-inverse).

Let \( \mathfrak{D} \) be an infinite cyclic semigroup, with an identity. Let \( \Theta \) be the generator of \( \mathfrak{D} \). We regard \( \mathfrak{D} \) as a small category with one object \( \alpha \) and \([\alpha, \alpha]_{\mathfrak{D}}\) is the infinite cyclic semigroup generated by \( \Theta \), with an identity. Clearly \( \mathfrak{D} \) is strictly skeletal. Define \( P_1, P_2, \Delta : \mathfrak{D} \rightarrow \text{Set} \) as follows.
\[ P_1(\alpha) = P_2(\alpha) = \mathbb{N} \], the set of natural numbers.
\[ P_1(\Theta) = P_2(\Theta): n \mapsto n+2 \]
and
\[ \triangle(\alpha) = \{ (n, n), (n, n+1): n \in \mathbb{N} \} \]

It is immediate that and \( \triangle \) extends to functors such that \( \triangle \) is a subdirect product of \( P_1 \) and \( P_2 \) and that \( P_1 \times P_2 \) is faithful.

The induced quasiorders on \( P_1(\alpha) \) and \( P_2(\alpha) \) (cf. Equation (3.2)) may be described as follows. For \( m, n \in I = P_1(\alpha) = \wedge = P_2(\alpha) \)

\[ m \leq n \iff m-n \] is non-negative and even.

Further the categories \( \wedge \) and \( I \) have the following form.

\[ \wedge_n = \{ m: m \geq n \} \]

and

\[ I_n = \{ m: m \geq n \} \]

For \( m \geq n \), define

\[ F(n; m) = P_2(\Theta^k) \]

where \( k = \frac{m-n}{2} \) if \( m-n \) is even and \( k = \frac{m-n-1}{2} \) if
m-n is odd, and

\[ G(n; m) = P_1(\theta^k) \]

where \( k = \frac{m-n}{2} \) if \( m-n \) is even, \( k = \frac{m-n+1}{2} \) if \( m-n \), odd.

Here \( \theta^0 \) is taken to be identity. In what follows for convenience we write \( P_1(\theta^i) = P_2(\theta^i) \) by \( \theta^i \) itself.

To prove that \( F \) defines cones, it is sufficient to show that for \( m \geq n \),

\[ F(n; m \theta^i) = F(n; m \theta^i). \]

Now let \( F(n; m) = \theta^k \). Then

\[ F(n; m \theta^i) = F(n; m + 2i) = \theta^{k+i} = F(n; m) \theta^i. \]

Hence \( F \), and similarly \( G \), is a cone.

Now we prove that \( \langle \mathcal{D}, P_1, P_2, \Delta, F, G \rangle \) satisfies the axioms in Definition 3.2 and is therefore an admissible category. To see \( I(i) \), let \( (m \theta^i, n) \in \Delta \). Then \( (m+2i, n) \in \Delta \), so that \( n = m+2i \) or \( m+2i + 1 \) and hence \( F(m; n) = \theta^i \). Conversely if \( F(m; n) = \theta^i \), then \( n = m+2i \) or \( m+2i+1 \) so that \( (m+2i, n) \in \Delta \). That is \( (m \theta^i, n) \in \Delta \). Thus \( I(i) \) holds.

To verify axiom \( I(ii) \), let \( (m, n) \in \Delta \). Then \( n = m \) or \( n = m+1 \), so that \( F(m; n) = 1 \). Hence \( I(ii) \) holds.
To see axiom II(i), let \( m \in I_n \), \( n \in \Lambda_l \) and \( (1, s) \in \Delta \).

Then \( m \geq n \), \( n \geq l \) and \( s = 1 \) or \( 1 + 1 \). Suppose \( n \leq s \) in the quasiorder of \( P_2(\alpha) \). Therefore \( n-s \) is even and non-negative. Let \( m-n \) be even. Then

\[
\begin{align*}
G(n; m) &= \Theta^k \quad \text{where} \quad k = \frac{m-n}{2}, \\
G(s; m) &= \Theta^j \quad \text{where} \quad j = \frac{m-s}{2} \quad \text{and} \\
G(n; 1G(s; m)) &= G(n; 1\Theta^j) = G(n; 1 + 2j) = \Theta^k
\end{align*}
\]

since \( 1 + 2j - n = 2k + l - s = 2k \) or \( 2k-1 \). The case when \( m-n \) is odd follows similarly. Thus II(i) holds.

To check axiom II(ii), instead of \( n-s \) is even in the hypothesis of II(i), suppose \( m-l \) is even and non-negative. Assume that \( m-n \) is even. Then

\[
\begin{align*}
n-l &= m-l-(m-n) \quad \text{is even and so} \quad F(1; n) = \Theta^i \\
&\text{where} \quad i = \frac{n-1}{2}. \quad \text{Thus}
\end{align*}
\]

\[
G(sF(1; n); m) = G(s+2i; m) = \Theta^k
\]

since \( m-(s+2i) = m-n+l-s = 2k \) or \( 2k-1 \). The case for \( m-n \) is odd, may be proved similarly. Thus II(ii) holds.

Axiom III follows from the following description of \( S(m, n) \) for any \( m \in \Lambda \) and \( n \in I \).
The duals of the axioms may be proved similarly and thus
\[ D = \langle \emptyset, P_1, P_2, \Delta, F, G \rangle \in A'Cat. \] Since \( \Delta \) is connected
\[ D \in A'Cat. \] Now the semigroup \( S = \mathcal{C}(D) \) is combinatorial and idempotent generated whose biordered set is isomorphic to
\( \mathcal{C}(D) \). Further the assignment
\[
\begin{align*}
e_i & \mapsto (2i+1, 2i+1) \\
f_i & \mapsto (2i+1, 2i+2) \\
g_i & \mapsto (2i+2, 2i+2) \\
h_i & \mapsto (2i+2, 2i+3)
\end{align*}
\]
defines an isomorphism of \( E_4 \) into \( \mathcal{C}(D) \). Hence
\[ S = \mathcal{C}(D) \] is isomorphic to \( Sp_4 \).

3. THE DOUBLE FOUR-SPIRAL SEMIGROUP \( DSP_4 \)

\( DSP_4 \) is the fundamental idempotent generated regular semigroup on a generalised four-spiral biordered set (cf. [4]). The biordered set of \( DSP_4 \) may be described by the following diagram.
where $A_n = \{(n, 0), (n, 1), \ldots, (n, 2^k-1)\}$ for $n = 2k+1$ or $n = 2k+2$. Now the biordered set is

$$E = \bigcup_n A_n \times \{n, n+1\}$$

In $E$ any two elements with the same first co-ordinates are $R$-related and any two elements with the same second co-ordinates are $L$-related. Further $\omega$-relation is given by

$$((m', j'), n') \omega ((m, j), n) \iff m'-m = n'-n = 2k,$$

for a non-negative integer $k$ and

$$j' \epsilon \{2^k j, 2^k j+1, \ldots, 2^k (j+1)-1\}.$$
For convenience we use a different notation to denote the elements of the above biordered set. For each \( A_n \), the second co-ordinate of the elements of \( A_n \) may be represented by their binary expressions as a word in the free semigroup over two symbols \( \{0, 1\} \), with the length of the word to be \( \frac{n-1}{2} \) or \( \frac{n-2}{2} \) according as \( n \) is odd or even.

Let \( <0, 1> \) denote the free semigroup on \( \{0, 1\} \). For any word \( X \in <0, 1> \), \( \sigma(X) \) will denote the length of the word. Further we denote by \( X_j \), the subword of \( X \) which is the segment containing the last \( j \) terms of \( X \) and \( X(j-) \) will denote the subword of \( X \) with the last \( j \) terms removed, so that \( X(j-)X_j = X \) for any \( j \leq \sigma(X) \).

With these notations we can reformulate the definition of as follows.

\[(6.1) \quad ((m', X), n') \omega ((m, Y), n) \iff m' - m = n' - n = 2k \]

for a non-negative \( k \) and \( X(k-) = Y \).

It is easily seen that \( \text{DSP}_4 \) is a bisimple combinatorial regular semigroup. Now we proceed to give an explicit construction of \( \text{DSP}_4 \) in terms of an admissible category (cf. Chapter III). Since \( \text{DSP}_4 \) is bisimple, \( \square \) may be taken as a category with one object \( \lambda \). Let
\[ \{ x, y \} = \langle \theta, \varphi \rangle \] be the free semigroup on two symbols \( \theta \) and \( \varphi \); with an identity. We identify the identity with the empty word. Clearly \( \mathcal{D} \) is strictly skeletal. Define \( P_1 : \mathcal{D} \to \text{Set} \) as follows.

\[ P_1(\alpha) = \bigcup_n A_n = I \]

where \( A_n = \{(n, X) : X \text{ is a word of length } k \text{ in } \langle \theta, \varphi \rangle \} \)

for \( n = 2k+1 \) or \( n = 2k+2 \). The empty word is taken as the word of length zero. Further for any morphism \( Y \) in \( \mathcal{D} \) define

\[ P_1(Y) : I \to I \] by \( (n, X) \mapsto (n+2\sigma(Y), XY) \)

Then \( P_1 \) is a functor. We define \( P_2 : \mathcal{D} \to \text{Set} \) by

\[ P_2(\alpha) = \mathbb{N} = \Lambda \] and \( P_2(Y) : n \mapsto n+2\sigma(Y) \).

Clearly \( P_2 \) is a functor and \( P_1 \ast P_2 \) is faithful.

Now define \( \Delta : \mathcal{D} \to \text{Set} \) by

\[ \Delta(\alpha) = \bigcup_n A_n \times \{ n, n+1 \} = \bar{\Delta} \]

Then \( \Delta \) extends to a subdirect product of \( P_1 \) and \( P_2 \).

The quasiorders on \( I \) and \( \Lambda \) induced by \( P_1 \) and \( P_2 \) respectively are as follows. For \( (m, X), (m', Y) \in I \)
and \( n, n' \in \triangle \)

\[ (6.2) \quad (m, X) \leq (m', Y) \iff m-m' = 2k \quad \text{and} \quad Y = X(k-) \]

for an integer \( k \geq 0 \).

\[ (6.2)' \quad n \leq n' \iff n-n' \text{ is even and non-negative.} \]

Now for \( (m, X) \in I \)

\[ \bigtriangleup_{(m, X)} = \{ m+i : i = 0, 1, \ldots \} \]

and for \( n \in \bigtriangleup, n \geq 2 \)

\[ I_n = \bigcup \{ A_{n+i} : i = -1, 0, 1, \ldots \} \quad \text{and} \quad I_1 = I \]

For \( m+i \in \bigtriangleup_{(m, X)} \) define

\[ F((m, X); m+i) = \begin{cases} P_2(\theta_i^1) & \text{if } i \text{ is even} \\ P_2(\theta_i^{i-1}) & \text{if } i \text{ is odd} \end{cases} \]

where \( \theta^0 = 1 \). Again for \( (n+i, X) \in I_n \), define

\[ G(n; (n+i, X)) = P_1(X_i) \]

where \( i = 2j \) or \( 2j-1 \). We now prove that \( F \) and \( G \)
defined above are cones. Let \( (m, X) I, i = 0, 1, 2, \ldots \)
and let \( P_1(\theta) = \theta_1 \quad \text{and} \quad P_2(\theta) = \theta_2 \). Then

\[ F((m, X); (m+i) \theta_2) = F((m, X); m+i+2) = \begin{cases} P_2(\theta_i^\frac{i+1}{2}) & \text{if } i \text{ is even} \\ P_2(\theta_i^{i-1}+1) & \text{if } i \text{ is odd} \end{cases} \]
Since \( \langle \alpha \rangle \mathcal{D} \) is generated by \( \Theta, \varphi \), and since \( P_2(\Theta) = P_2(\varphi) \), it follows that \( F((m, X); -) \) is a cone.

Now let \( n \in \Lambda \) and \( (n+i, X) \in A_{n+i} \) for \( i \in \{-1, 0, 1, \ldots\} \). Let \( i = 2j \) or \( 2j-1 \). Then

\[
G(n; (n+i, X) \Theta_1) = G(n; (n+i+2, X \Theta) ) \\
= P_1(X_{j+1}' \Theta_1) \quad \text{where} \quad X' = X \Theta \\
= P_1(X_j) \Theta_1 \\
= G(n; (n+i, X)) \Theta_1
\]

Similarly

\[
G(n; (n+i, X) \varphi_1) = G(n; (n+i, X)) \varphi_1
\]

Hence \( G(n; -) \) is a cone.

We have seen that \( \mathcal{D} = \langle \mathcal{D}, P_1, P_2, \triangle, F, G \rangle \) satisfies the requirements given in the data for an admissible category. We prove next that \( \mathcal{D} \) satisfies axioms I, II, III and the duals of Definition 3.2. Since \( \Lambda \) is connected, this will imply that \( \mathcal{D} \in \mathcal{A}'\text{Cat} \).
To prove that axiom \( I \) holds, suppose \( \psi \) in \( \mathcal{D} \) and \((m, X) \psi_1, n) \in \Delta \). Further assume that \( \sigma(\psi) = j \).

Then \((m, X) \psi_1 = (m+2j, X \psi)\). Since

\[ ((m+2j, X \psi), n) \in \Delta, \quad n = m+2j \quad \text{or} \quad n = m+2j+1. \]

Therefore \( F((m, X); n) = P_2(\theta^j) \). Now \( \psi_2 = P_2(\psi) = P_2(\theta^j) \) since \( \sigma(\psi) = j \). Hence \((\psi_1, F((m, X); n)) \) is in \( \Delta(\mathcal{D}) \). Conversely suppose that \((\psi_1, F((m, X); n)) \) is in \( \Delta(\mathcal{D}) \). Let \( F((m, X); n) = P_2(\theta^j) \). Then

\[ n = m+2j \quad \text{or} \quad n = m+2j+1 \quad \text{and} \quad \sigma(\psi) = j. \]

Now \((m, X) \psi_1 = (m+2j, X \psi) \) and so \((m, X) \psi_1, n) \in \Delta \).

Thus axiom \( I(i) \) holds.

Dually let \((m, X), n \psi_2) \in \Delta \). If \( \sigma(\psi) = j \) then \( n \psi_2 = n+2j \) and so \((m, X), n+2j) \in \Delta \). Therefore \( m = n+2j \quad \text{or} \quad m = n+2j-1 \). Now by the definition of \( G \),

\[ G(n; (m, X)) = P_1(X_j) \]

Since \( \sigma(\psi) = j \), we have \( P_2(\psi) = P_2(X_j) \) and so

\( G(n; (m, X), \psi_2) \) is in \( \Delta(\mathcal{D}) \). Conversely if

\( G(n; (m, X), \psi_2) \) in \( \Delta(\mathcal{D}) \), then

\[ G(n; (m, X)) = P_1(X_j) \]
since for \( j = \sigma (\psi) \), \( m = n+2j \) or \( m = n+2j-1 \). Now \( \psi_2 = n+2j \) and so \( ((m, X), n \psi_2) \in \Lambda \). Thus dual of I(i) holds. Axiom I(ii) follows directly from the definitions of \( F \) and \( G \). Thus axiom I holds.

To prove that axiom II holds, let \( k \in \Lambda \), \( \mu \in \Lambda \), and \( (i', \lambda) \in \Lambda \). Suppose \( \mu = n \) and \( i' = (m, Y) \).

Then \( k = (n+i, X) \) and \( n = m+j \) for some \( i \in \{-1, 0, 1, \ldots\} \) and \( j \in \{0, 1, \ldots\} \). Since \((i', \lambda) \in \Lambda \), \( \lambda = m \) or \( m+1 \). Let \( \lambda = m \). Suppose that \( \mu \leq \lambda \). Then \( n-m = 2j \) for a non-negative \( j \). Therefore,

\[
G(\mu; i'G(\lambda; k)) = G(n; (m, Y)G(m, (n+i, X)))
\]

\[
= G(n; (m, Y)G(m; (m+2j+i, X)))
\]

\[
= G(n; (m, Y)P_1(X_r)) \text{ where } 2j+i = 2r \text{ or } 2r-1
\]

\[
= G(n; (m+2r, YX_r)) \text{ since } \sigma(X_r) = r
\]

\[
= G(n; (n+2r-2j, YX_r)
\]

\[
= P_1((YX_r)_{i_0}) \text{ where } i_0 = r-j.
\]

Now \( i = 2i_0 \) or \( 2i_0 - 1 \) and so \( i_0 \leq r \). Therefore

\[
G(\mu; k) = G(n; (n+i, X))
\]

\[
= P_1(X_{i_0})
\]
\[= P_1(YX_r)_{i_0}, \text{ since } i_0 \leq r\]
\[= G(\mu; i'G(\lambda; k)).\]

Now let \( k, \mu, i' \) be as given above and let \( k \leq i' \) in \( I \).

That is, \( (n+i, X) \leq (m, Y) \). Therefore \( n+i-m = i+j > 0 \)
and even and \( X(\frac{i+j}{2}) = Y \). Let \( i = 2i_0 \) or \( 2i_0-1 \). Then
\[G(\mu; k) = G(n; (n+i, X)) = P_1(X_{i_0})\]
and
\[G(\lambda F(i'; \mu); k) = G(mF((m, Y); n); (n+i, X))\]
\[= G(m+2i_1; (n+i, X)) \text{ where } j = 2i_1 \text{ or } 2i_1+1\]
\[= G(m+2i_1; (m+i+j, X))\]
\[= P_1(X_{i_0}) \text{ since } j+i-2i_1 = 2i_0.\]

Thus \( G(\mu; k) = G(\lambda F(i'; \mu); k) \). The case for \( \lambda = m+1 \)
and the dual may be similarly proved. Thus axiom II holds.

To verify axiom III, let \( n \in \Lambda \) and \( (m, X) \in I \). Let
\( m = n+i \) for \( i > 0 \). Let \( k = (m, X) \) and \( v = m \) or
\( m+1 \) according as \( i \) is even or odd. Then
\( ((m, X), v) \in S(n, (m, X)) \). For if \( ((m', X'), v') \in \overline{\Lambda} \) with
\( (m', X') \leq (m, X) \) and \( v' \leq n \), then \( m'-m = 2j \) and
\( X'(j-) = X \) for \( j > 0 \). Also \( v' = m' \) or \( m'+1 \)
so that \( v'-n = 2j+1 \) or \( 2j+1-1 \) according as \( v' = m' \)
or \( m'+1 \). Let \( v' = m' \). Then \( m'-n = 2j+1 \) and

\[
G(n; (m', X')) = P_1(X_r) \quad \text{where} \quad 2j+1 = 2r \quad \text{or} \quad 2r-1
\]

\[
G(n; (m, X)) = P_1(X_s) \quad \text{where} \quad i = 2s \quad \text{or} \quad 2s-1
\]

Now \( r = j+s \), so that

\[
X_sX'_j = X'(s+j) = X'_r, \quad \text{since} \quad X'(j-) = X
\]

Therefore

\[
G(n; (m', X')) = G(n; (m, X)P_1(X'_j))
\]

Also

\[
F((m, X); v') = P_2(\Theta^j) \quad \text{and} \quad F((m, X); v) = 1
\]

so that

\[
F((m, X); v') = F((m, X); v)P_2(\Theta^j)
\]

Hence \(((m, X), v) \in S(n, (m, X))\). Now if instead of

\[
m = n+i, \quad \text{assume that} \quad n = m+i \quad \text{for} \quad i \geq 0.
\]

Let

\[
i = 2j \quad \text{or} \quad 2j+1.
\]

Take \( k = (n, Y) \) if \( i \) is even

and \( k = (n-1, Y) \) if \( i \) is odd where \( Y = X \Theta^j \).

Clearly \((k, n) \in S(n, (m, X))\). Thus axiom III holds.

This proves that \( D \in A^\text{Cat}. \) Now by Theorem 3.38,

\[
S = \mathcal{C}(D) = I \times \Delta \quad \text{with product defined by}
\]

\[
((m, X), n)((m', X'), n') = \begin{cases} ((m, X), n'+2j) & \text{if} \ n = m' = 2j \quad \text{or} \quad 2j+1 \\ ((m+2j, XX'_j), n') & \text{if} \ m' - n = 2j \quad \text{or} \quad 2j-1 \end{cases}
\]
is a combinatorial regular semigroup with biordered set
\[ \mathcal{C}(\mathbb{D}) = \Delta. \] Clearly the mapping
\[ ((m, X), n) \mapsto ((m, \bar{X}), n) \]
where \( \bar{X} \) is the word in \( \langle 0, 1 \rangle \) obtained by replacing \( \Theta \)
by 0 and \( \varphi \) by 1 in \( X \), is a biorder isomorphism
between \( \Delta \) and \( \mathcal{E}(\text{DSP}_4) \). Hence \( S \) is isomorphic to \( \text{DSP}_4 \).

4. IDEMPOTENT GENERATED NATURALLY PARTIALLY ORDERED REGULAR
SEMIGROUPS WITH GREATEST IDEMPOTENTS

A semigroup \( S \) is said to be an ordered semigroup,
if there exists a partial order \( \leq \) on \( S \) such that
\( x \leq y \) implies \( xz \leq yz \) and \( zx \leq zy \) for all \( z \in S \)
(cf. [2], p.15). \( S \) is said to be naturally partially
ordered if the above order is an extension of the natural
partial order on idempotents (cf. [15]).

Let \( S \) be a regular semigroup, which is naturally
partially ordered by \( \leq \) and having a greatest idempotent \( u \).
Then it is easy to see that if \( e, f \in \mathcal{E}(S) \) and \( e \leq f \),
then \( e = efe \) (cf. [15], Proposition 1.5). Further, in this
case it is easy to see that each \( x \in S \), has a greatest inverse
\( x^0 \), such that \( x^0 = ux'u \) for every \( x' \in V(x) \).

Let \( \mathcal{E} \) be a regular idempotent generated (r.i.g.)
naturally partially ordered semigroup, with a greatest
idempotent $u$. The following proposition characterises
Green's relations on $\bar{E}$ and also provides some useful
information about $\bar{E}$.

**PROPOSITION 6.1.** For $\bar{e}, \bar{f} \in \bar{E}$

(i) $\bar{e} R \bar{f} \iff \bar{e}u = \bar{f}u$

(ii) $\bar{e} \mathcal{L} \bar{f} \iff \bar{u}e = u\bar{f}$

(iii) $\bar{e} \mathcal{H} \bar{f} \iff \bar{e} = \bar{f}$

(iv) $\bar{e} \mathcal{D} \bar{f} \iff \bar{u}e\bar{u} = u\bar{f}\bar{u}$

(v) $\bar{e}^0 = \bar{u}e\bar{u}$ is the greatest element of $\bar{D}_\bar{e}$

(vi) $\bar{e} = \bar{u}e\bar{u}$.

**Proof.** (i) follows from the observation that $\bar{e}u = \bar{e}^0\bar{u}$.
For, $\bar{e}$ being a product of idempotents $\bar{e} \leq u$ for
every $\bar{e} \in \bar{E}$. Now

$$\bar{e}^0 \bar{u} = \bar{e}^0\bar{u} \leq \bar{e}^0\bar{u}\bar{u} = \bar{e}^0\bar{u}$$

since $\bar{e}^0 = u\bar{e}'u$ for some $\bar{e}' \in \mathcal{V}(\bar{e})$, and so

$$\bar{e}^0\bar{u} = \bar{e}^0 = u\bar{e}^0. \text{ Thus } \bar{e}u = \bar{e}^0\bar{u}.$$  

(ii) follows similarly from the fact that $\bar{u}e = \bar{e}^0\bar{e}$ and

(iii) is obtained by combining (i) and (ii). To prove (iv)
first note that

$$\bar{e}^0 = \bar{e}^0\bar{e}^0 = u\bar{e}\bar{e}^0 = u\bar{e}u = \bar{e}^0\bar{e}.$$
Since $\overline{E}$ is regular, $\overline{e} \circ \overline{D} \overline{e} = 0$. If $\overline{e} \circ \overline{f}$, then there exists $\overline{g} \in \overline{E}$ such that $\overline{e} \circ \overline{g} \circ \overline{f}$. Now by (i) and (ii),
\[ \overline{e} \circ \overline{0} = \overline{u} \circ \overline{u} = \overline{u} \circ \overline{u} = \overline{u} \circ \overline{u} = \overline{f} \circ \overline{0} \]
Thus (iv) holds. (v) is obtained from the fact that
\[ \overline{0} \leq \overline{e} \circ \overline{0} = \overline{e} \circ \overline{0} = \overline{u} \circ \overline{u}. \]
Finally
\[ \overline{e} = \overline{e} \circ \overline{0} \circ \overline{e} = (\overline{e} \circ \overline{0}) \circ \overline{e} = \overline{e} \circ \overline{u} \]
and so (vi) is obtained \(\square\)

Note that (iii) of Proposition 6.1, shows that $\overline{E}$ is combinatorial. Now we prove that $\overline{E}$ is locally testable.

**Proposition 6.2.** $\overline{E}$ is locally testable.

**Proof.** It is sufficient to prove that for any two $\mathcal{D}$-classes $D, D'$ and $e \in E(D)$, there exists at most one idempotent $e' \in E(D')$ such that $e' \omega e$. Since $\overline{E}$ is combinatorial it is sufficient to prove that for each $\mathcal{D}$-class $D$, there exists only one idempotent $e \in D$ such that $e \omega u$, where $u$ is the greatest idempotent in $\overline{E}$. If $e \in E(D)$ then $u \circ u \in E(D)$ by Proposition 6.1 (v) and (vi). Clearly $u \circ u \omega u$. Suppose $e' \in E(D)$ be such that $e' \omega u$. Then $e' = u \circ e' \circ u$. Now by Proposition 6.1 (iv), $e' = u \circ u$. This proves the result \(\square\)
PROPOSITION 6.3. Let $D$ be a $\mathcal{D}$-class of $\bar{E}$ and $\bar{e} \in D$. If $\bar{f} \mathcal{R} \bar{u} \bar{e} \bar{u}$ then $\bar{f} \in E(D)$.

Proof. Let $\bar{f} \mathcal{R} \bar{u} \bar{e} \bar{u} = g$. Then $g \bar{f} = \bar{f}$ and so

$$\bar{f} = g \bar{f} = (u \bar{e} u) \bar{f} = (u u^{-1}) \bar{f} = \bar{f} \in E(D).$$

Similarly the case for $\bar{f} \mathcal{L} \bar{u} \bar{e} \bar{u}$ may be proved.

It follows from Proposition 6.1 (vi) that the subsemigroups $\bar{E}u$ and $u \bar{E}$ of $\bar{E}$ are bands. Both $\bar{E}u$ and $u \bar{E}$ contain $u$ and the greatest element in each $\mathcal{D}$-class of $\bar{E}$. Further $\bar{E}u$ intersects each $\mathcal{D}$-class $D$ of $\bar{E}$ in the $\mathcal{L}$-class containing the greatest element of $D$ and $u \bar{E}$ meets each $\mathcal{D}$-class $D$ of $E$ in the $\mathcal{R}$-class containing the greatest element of $D$. Therefore it is easy to see that the structure semilattice of $\bar{E}u$ and $u \bar{E}$ are the same.

If $\Gamma$ is this structure semilattice, it is also seen that the partially ordered set $\Gamma$ is isomorphic to the partially ordered set of $\mathcal{D}$-classes of $\bar{E}$ (cf. Section IV.3).

Clearly $\bar{E}u$ is a left normal band and $u \bar{E}$ is a right normal band.

PROPOSITION 6.4. Let $S$ be an r.i.g. semigroup which is naturally partially ordered and having a greatest idempotent $u$, which is a right identity. Then $S$ is
a left normal band. Dually if \( u \) is a left identity, then \( S \) is a right normal band.

**Proof.** From the above discussion \( Su \) is a left normal band and \( uS \) is a right normal band. When \( u \) is a right identity \( Su = S \) and so \( S \) is a left normal band. If \( u \) is a left identity, then \( uS = S \) and hence the result follows. 

Now we proceed to construct a naturally partially ordered r.i.g. semigroup \( E \), with a greatest idempotent, in terms of ordered sets and functors from semilattices into ordered sets.

Let \( \text{OSet}^* \) denote the small category whose objects are partially ordered sets, each with a greatest element and whose morphisms are order preserving mappings which preserve the greatest elements. Let \( \Gamma \) be a semilattice with identity.

**Theorem 6.5.** Let \( P : \Gamma \to \text{OSet}^* \) be a functor which is disjoint and non-empty. On \( E_1 = \bigcup \{ P(\alpha) : \alpha \in \Gamma \} \) define product by

\[
(6.3) \quad ab = aP(\alpha, \alpha\beta)
\]

where \( a \in P(\alpha) \) and \( b \in P(\beta) \). Then \( E_1 \) is a
regular idempotent generated semigroup. Further define a relation \( \leq \) on \( E_1 \) by

\[
(6.4) \quad a \leq b \iff \alpha \leq \beta \quad \text{and} \quad a \leq bP(\beta, \alpha) \quad \text{in} \quad P(\alpha)
\]

where \( a \in P(\alpha) \) and \( b \in P(\beta) \). Then \( \leq \) is a partial order and \( E_1 \) with this order becomes a naturally partially ordered semigroup, having a greatest idempotent which is a right identity. Conversely every naturally partially ordered r.i.g. semigroup with a greatest idempotent which is a right identity is isomorphic to one constructed in this way.

**Proof.** Let \( a \in P(\alpha) \), \( b \in P(\beta) \) and \( c \in P(\gamma) \) for \( \alpha, \beta, \gamma \in \Gamma \). Then

\[
(ab)c = aP(\alpha, \alpha \beta) \cdot c = aP(\alpha, \alpha \beta)P(\alpha \beta, \alpha \gamma \delta) = aP(\alpha, \alpha \beta \gamma \delta)
\]

and

\[
a(bc) = a(bP(\beta, \beta \gamma)) = aP(\alpha, \alpha \beta \gamma)
\]

Therefore \( E_1 \) is a semigroup. Further \( aa = a \) for every \( a \in E_1 \) and so \( E_1 \) is a band. Thus \( E_1 \) is idempotent generated. Now let \( a \in P(\alpha) \), \( b \in P(\beta) \) and \( c \in P(\gamma) \).

If \( a \leq b \) and \( b \leq c \), then \( \alpha \leq \beta \) and \( \beta \leq \gamma \) so that \( \alpha \leq \gamma \). Also \( a \leq bP(\beta, \alpha) \) and \( b \leq cP(\gamma, \beta) \).

Therefore, since morphisms in \( \text{OSet}^* \) are order preserving
\[ a \leq bP(\beta, \alpha) \leq cP(\gamma, \beta)P(\beta, \alpha) = cP(\gamma, \alpha). \]

Thus \( a \leq c \). And since \( \Gamma \) is a partial order, it follows that \( a \leq b \) and \( b \leq a \) implies \( a = b \). Thus \( \leq \) is a partial order on \( E_1 \). Now we show that \( \leq \) is compatible with multiplication in \( E_1 \). Let \( a \leq b \) and \( c \in E_1 \), with \( a \in P(\alpha) \), \( b \in P(\beta) \), \( c \in P(\gamma) \). Then \( \alpha \leq \beta \) and \( a \leq bP(\beta, \alpha) \). Also \( \alpha \leq \beta \) implies \( \alpha \gamma \leq \beta \gamma \) Therefore

\[ ac = aP(\alpha, \alpha \gamma) \]
\[ bP(\beta, \alpha)P(\alpha, \alpha \gamma) = bP(\beta, \alpha \gamma) \]
\[ = bP(\beta, \beta \gamma)P(\beta \gamma, \alpha \gamma) \]
\[ = (bc)P(\beta \gamma, \alpha \gamma) \]

Hence \( ac \leq bc \). Further

\[ ca = cP(\gamma, \alpha \gamma) = cP(\gamma, \beta \gamma)P(\beta \gamma, \alpha \gamma) = (cb)P(\beta \gamma, \alpha \gamma) \]
and so \( ca \leq cb \). Now we show that for \( e, f \in E(E_1) = E_1 \), if \( e \omega f \), then \( e \leq f \). Suppose \( e \omega f \). Then \( e = ef = fe \).

Therefore if \( e \in P(\alpha) \) and \( f \in P(\beta) \), then \( ef \in P(\alpha \beta) \) so that \( \alpha \beta = \alpha \). Thus \( \alpha \leq \beta \) Further \( e = fe = fP(\beta, \alpha) \) and so \( e \leq f \). Hence \( E_1 \) is naturally partially ordered.
Let 1 ∈ Γ be the identity in Γ. Then P(1) has a greatest element u. We show that u is the greatest idempotent in E₁. Clearly u is an idempotent.

Let a ∈ E₁. Then a ∈ P(α) for some α ∈ Γ. Now α ≤ 1 and uP(1, α) = uα is the greatest element in P(α) so that a ≤ uP(1, α). Hence u is the greatest idempotent in E₁. Also for any a ∈ E₁, if a ∈ P(α) then

$$au = aP(α, α1) = aP(α, α) = a$$

Thus u is a right identity for E₁.

Conversely let E be a naturally partially ordered r.i.g. semigroup having a greatest idempotent u, which is a right identity for E. Then by Proposition 6.4, E is a left normal band. Therefore E is a semilattice Γ of left zero semigroups \( \{ Λ_α : α ∈ Γ \} \) where each Λα is ordered with the restriction of the order on E and has a greatest element (see the discussion at the end of Proposition 6.3). Let α_u denote the greatest element in Λα. We first note that if β ≤ α then β_u ≤ α_u.

For, by definition, α_u = ueu for any e ∈ Λα (cf. Proposition 6.1 (v)). Let f ∈ Λ_β. Then ef ∈ Λ_β.
since $\beta \leq \alpha$. Also $f' = efe \omega e$. Further $\beta_u = uf'u$.

Since $\overline{E}$ is a left normal band,

$$(ueu)(uf'u) = u(ef')u = uf'u \quad \text{and} \quad (uf'u)(ueu) = uf'u.$$

Hence $\beta_u \omega \alpha_u$. Since $\overline{E}$ is combinatorial, there exists a combinatorial functor $\varphi$ corresponding to $\overline{E}$.

Define $\mathbf{P} : \Gamma \to \text{OSet}^*$ by

$$\mathbf{P}(\alpha) = \Lambda_\alpha \quad \text{and} \quad \mathbf{P}(\alpha, \beta) = \varphi(\alpha_u, \beta_u)$$

By axiom (C2) of C-functors it follows that $\mathbf{P}$ is a functor, and product in $\overline{E}$ coincides with the product defined by Equation (6.3).

Next we show that the partial order on $\overline{E}$ coincides with the partial order defined by (6.4). Clearly if $a \leq b$ in the partial order defined by (6.4), then $a \leq b$ in the original order. Now let $a \in \mathbf{P}(\alpha)$, $b \in \mathbf{P}(\beta)$, and $a \leq b$ in the original order. Then $\alpha_u = uau$ and $\beta_u = ubu$. Since $a \leq b$, $uau \leq ubu$ and so

$$uau = (uau)^2 \leq (ubu)(uau) \quad \text{and} \quad (ubu)(uau) \in \mathbf{P}(\alpha/\beta).$$

Therefore $\alpha_u \leq (\alpha/\beta)_u$. But $\alpha/\beta \leq \alpha$ and so $(\alpha/\beta)_u \omega \alpha_u$ which implies $(\alpha/\beta)_u \leq \alpha_u$. Thus $\alpha_u = (\alpha/\beta)_u$
so that \( \alpha \leq \beta \). Also

\[
a \leq b \Rightarrow a u a u \leq b u a u = b \varphi ( B_u, \alpha_u ) = b \varphi ( \beta, \alpha )
\]

Hence the partial order on \( \bar{E} \) coincides with the one defined by (6.4). This completes the proof \( \square \)

COROLLARY 6.6. If on \( E_1 \) we define product by

\[
(6.3)' \quad a \ast b = b \varphi ( \beta, \alpha \beta )
\]

for \( a \in P(\alpha) \) and \( b \in P(\beta) \), then \((E_1, \ast)\) becomes a regular idempotent generated semigroup which is naturally partially ordered by (6.4) and having a greatest idempotent which is a right identity \( \square \)

LEMMA 6.7. Let \( \bar{E} \) be a naturally partially ordered r.i.g. semigroup with a greatest idempotent \( u \). Then

\[
x \leq y \iff xu \leq yu \text{ and } ux \leq uy.
\]

Consequently, the order on \( \bar{E} \) is the cartesian order induced by that on \( \bar{E}u \) and \( u\bar{E} \).

Proof. Suppose \( xu \leq yu \) and \( ux \leq uy \). Then by Proposition 6.1.

\[
x = xux \leq yux \leq yuy = y.
\]
By Proposition 6.1, $x \mathcal{R} xu$ and $x \mathcal{R} y$ implies $xu = yu$. Thus $xu$ is the idempotent representative of the $\mathcal{R}$ -class of $x$ and dually $ux$ is the idempotent representative of the $\mathcal{L}$ -class of $x$. Now from the above result and the fact that $\mathcal{E}$ is combinatorial, it follows that, the order on $\mathcal{E}$ is the cartesian order induced by the order on $\mathcal{R}$ -classes and $\mathcal{L}$ -classes of $\mathcal{E}$ which is the same as the order on $\mathcal{E}u$ and $u\mathcal{E}$ respectively $\square$

Let $\Gamma$ be a semilattice with identity and $P_1, P_2 : \Gamma \rightarrow \text{OSet}^*$ be functors each of which are disjoint and non-empty. Let $\Delta : \Gamma \rightarrow \text{OSet}^*$ be a subdirect product of $P_1$ and $P_2$. We denote by $\leq'$ the order on $I = \bigcup P_1(\alpha)$ and $\Lambda = \bigcup P_2(\alpha)$ induced by the functors (cf. Section III.2). We define a new order $\leq$ on $I$ as follows. For $i \in P_1(\alpha)$ and $j \in P_1(\beta)$

$$i \leq j \iff \alpha \leq \beta \quad \text{and} \quad i \leq jP_1(\beta, \alpha) \quad \text{in} \ P_1(\alpha).$$

Clearly $\leq$ is a partial order on $I$ which is an extension of $\leq'$. Dually we define the extended order $\leq$ on $\Lambda$ also. On $\mathcal{\Delta} = \bigcup \Delta(\alpha)$ also we may consider the order $\leq'$ and the extended order $\leq$. We denote the greatest element in $P_1(1)$ and $P_2(1)$ by $1$ itself.
THEOREM 6.8. Let $\Gamma$ be a semilattice with identity and $P_1, P_2 : \Gamma \to \text{OSet}^*$ be functors which are disjoint and non-empty. Let $\Delta$ be a subdirect product of $P_1$ and $P_2$ satisfying the following conditions.

(i) $(1, 1) \in \Delta(1)$

(ii) $(\Delta, \leq')$ is a reflective subcategory of $(I \times \Delta, \leq')$ such that the corresponding left adjoint $F : I \times \Delta \to \bar{\Delta}$ preserves the extended order $\leq$ of $I \times \Delta$

On $S = \bigcup_{\alpha \in \Gamma} P_1 \ast P_2(\alpha)$, define a binary operation as follows. For $(i, \lambda) \in P_1 \ast P_2(\alpha)$ and $(j, \mu) \in P_1 \ast P_2(\beta)$

$$(i, \lambda)(j, \mu) = (iP_1(\alpha, \gamma), \mu P_2(\beta, \gamma))$$

where $F(j, \lambda) \in \Delta(\gamma)$. Then $S$ with the cartesian order of $\leq$, is a naturally partially ordered r.i.g. semigroup with a greatest idempotent. Conversely every naturally partially ordered r.i.g. semigroup with a greatest idempotent is isomorphic to one constructed in this way.

Proof. We first show that $\langle \Gamma, P_1, P_2, \Delta \rangle$ satisfies the conditions (a) and (b) of Theorem 4.18, so that it becomes a P-category. Since $\Gamma$ is a semilattice (a) follows. Let $j \in P_1(\beta)$ and $\lambda \in P_2(\alpha)$ be given. Suppose $F(j, \lambda) = (k, \nu) \in \Delta(\gamma)$. Now $(k, \nu) \leq'(j, \lambda)$ in $I \times \Delta$
and so \( k = jP_1(\beta, \gamma) \) and \( \nu = \lambda P_2(\alpha, \gamma) \).

Since \( F \) is left adjoint to the inclusion 
(\( (\Delta, \leq') \subseteq (I \times \wedge, \leq') \)), (b) holds. Now by Theorem 4.19, 
\( S \) is a combinatorial pseudo-inverse semigroup with product 
defined by (4.7), which by the above observations is same as the 
product in (6.5). Further, since \( \Gamma \) is a semilattice, it 
follows that \( S \) is locally testable also.

We now prove that \( S \) is naturally ordered and that 
\( u = (1, 1) \) is the greatest idempotent in \( S \). Define, for 
\((i, \lambda), (j, \mu) \in S \)

\[(6.6) (i, \lambda) \leq (j, \mu) \text{ in } S \iff i \leq j \text{ and } \lambda \leq \mu\]

Clearly \( \leq \) is a partial order on \( S \). Let 
\( x = (i, \lambda) \in P_1 \ast P_2(\alpha), y = (j, \mu) \in P_1 \ast P_2(\beta) \) and 
z = \((k, \nu) \in P_1 \ast P_2(\gamma) \) with \( x \leq y \). Suppose 
\( F(k, \lambda) \in \Delta(\delta) \) and \( F(k, \mu) \in \Delta(\delta') \). Then 
\[xz = (iP_1(\alpha, \delta), \nu P_2(\gamma, \delta))\]
and 
\[yz = (jP_1(\beta, \delta'), \nu P_2(\gamma, \delta')).\]
Since \( x \leq y \), we have \( i \leq j \), from which \( i \leq jP_1(\beta, \alpha) \)
and \( \lambda \leq \mu \) so that \( F(k, \lambda) \leq F(k, \mu) \), from which \( \delta \leq \delta' \).
Now

\[ i_P(\alpha, \delta) \leq j_P(\beta, \alpha)P_1(\alpha, \delta) = j_P(\beta, \delta) = j_P(\beta, \delta')P_1(\delta', \delta) \]

and

\[ v_P(\chi, \delta) = v_P(\gamma, \delta')P_2(\delta', \delta) \]

so that

\[ i_P(\alpha, \delta) \leq j_P(\beta, \delta') \quad \text{and} \quad v_P(\gamma, \delta) \leq v'P_2(\gamma, \delta') \]

Thus \( xz \leq yz \). Similarly \( zx \leq yz \). Now let \( e \omega f \) in \( S \) with \( e = (i, \lambda) \in \Delta(\alpha) \) and \( f = (j, \mu) \in \Delta(\beta) \).

Then by the definition of \( \omega \) (cf. Lemma 3.8), we get \( \alpha \leq \beta \) and \( (i, \lambda) = (j, \mu) \Delta(\beta, \alpha) = (j_P(\beta, \alpha), \mu_P(\beta, \alpha)). \)

Therefore \( i \leq j \) and \( \lambda \leq \mu \). Thus we see that \( S \) is naturally partially ordered and that \( u = (1, 1) \) is the greatest idempotent in \( S \). Denoting the greatest element in \( P_1(\alpha) \) and \( P_2(\alpha) \) respectively by \( \alpha_u \) and \( \alpha_v \), we get that \( (\alpha_u, \alpha_v) \) is the greatest idempotent in the \( \mathcal{D} \)-class \( P_1 \ast P_2(\alpha) \) of \( S \). If \( eDf \) in \( S \), then \( ueu \omega u \) and \( ufu \omega u \) and since \( S \) is locally testable, this will imply that \( ueu = ufu \). Hence \( e \) and \( f \) may be connected by an \( E \)-chain in \( E(S) \), through \( ueu \).

Therefore \( S \) is idempotent generated.

Conversely, let \( S \) be an r.i.g. semigroup, which is naturally partially ordered and having a greatest idempotent \( u \).
Then by Proposition 6.2, $S$ is locally testable and so $S$ is pseudo-inverse. Therefore by Theorem 4.19, there exists a $P$-category $\mathcal{P} = \langle \Gamma, P_1, P_2, \Delta \rangle$ such that $S$ is isomorphic to $\mathcal{Q}(\mathcal{P})$. Since $S$ is locally testable, $\Gamma$ is a partial order. Condition (b) of Theorem 4.18, implies that $(\bar{\Delta}, \leq')$ is a reflective subcategory of $(I \times \Lambda, \leq')$. Let $F$ be the corresponding left adjoint.

It is now clear that the product defined in $\mathcal{Q}(\mathcal{P})$ by Equation (4.7) coincides with the product defined by Equation (6.5).

Now we show that $P_1(\alpha), P_2(\alpha) \in OSet^*$ for each $\alpha \in \Gamma$. Consider the $\partial$-class $D = P_1(\alpha) \times P_2(\alpha)$. By Proposition 6.1, each $\varnothing$-class of $S$ contains a greatest element which is an idempotent. Let $(\alpha_u, \alpha_v)$ be the greatest element in $D$. By Lemma 6.7, the order in $S$ is the cartesian order, so that $\alpha_u$ is the greatest element in $P_1(\alpha)$ and $\alpha_v$ is greatest in $P_2(\alpha)$. Thus $P_1(\alpha), P_2(\alpha) \in OSet^*$. Now we show that if $\beta \leq \alpha$, then $P_i(\alpha, \beta) : P_i(\alpha) \to P_i(\beta)$ is a morphism in $OSet^*$ for $i = 1, 2$. Since $S$ is idempotent generated, if $\beta \leq \alpha$ and $g_\alpha, g_\beta$ are the
greatest idempotents in $P_1 \times P_2(\alpha)$ and $P_1 \times P_2(\beta)$ respectively, then $g_\beta \omega g_\alpha$, so that $g_\alpha \Delta(\alpha, \beta) = g_\beta$.

Thus $P_i(\alpha, \beta)$ preserves greatest elements. The fact that $P_i(\alpha, \beta)$ is order-preserving, follows from Lemma 6.7 and the proof of Theorem 6.5.

Define a product in $\Gamma$ by $\alpha \beta = \gamma$ for $\alpha, \beta \in \Gamma$ such that $F(\beta U, \alpha, \gamma) \in \Delta(\gamma)$. With this product, $\Gamma$ becomes a semilattice.

Now it remains to prove that $F$ preserves the extended order on $I \times \Delta$. Let $x = (i, \lambda) \in P_1 \times P_2(\alpha)$, $y = (j, \mu) \in P_1 \times P_2(\beta)$ and $F(j, \lambda) \in \Delta(\gamma)$. Then by the definition of product in (6.5) $xy \in P_1 \times P_2(\gamma)$.

Let $j' \leq j$ and $\lambda' \leq \lambda$ in the extended order. Therefore $j' \leq j P_1(\beta')$ and $\lambda' \leq \lambda P_2(\alpha', \lambda')$ where $j' \in P_1(\beta')$ and $\lambda' \in P_2(\alpha')$. Choose $i' \in P_1(\alpha')$, $\mu' \in P_2(\beta')$ such that $i' \leq i$ and $\mu' \leq \mu$. Let $x' = (i', \lambda')$, $y' = (j', \mu')$. Then $x' \leq x$, $y' \leq y$ and so $x'y' \leq xy$. Let $x'y' \in P_1 \times P_2(\gamma')$. Then $F(j', \lambda') \in \Delta(\gamma')$. Also $x'y' \leq xy$ implies by Lemma 6.7 that $\gamma' \leq \gamma$. Now
\[ F(j', \lambda') = (j'P_1(\beta', \gamma'), \lambda'P_2(\alpha', \gamma')) \]

and so
\[ j'P_1(\beta', \gamma') jP_1(\beta, \beta')P_1(\beta', \gamma') = jP_1(\beta, \gamma') \]

and
\[ \lambda'P_2(\alpha', \gamma') \leq P_2(\alpha, \alpha')P_2(\alpha', \gamma') = \lambda P_2(\alpha, \gamma'). \]

Thus \( F \) preserves the extended order. This completes the proof. \( \Box \)