CHAPTER IV

COMBINATORIAL PSEUDO-INVERSE SEMIGROUPS

This chapter is the first of two chapters in which we discuss particular classes of combinatorial regular semigroups, where significant simplification of the general theory presented in Chapter III occurs. Here we deal with combinatorial pseudo-inverse semigroups and some of its important subclasses. The principal simplification that occurs in this case is that the admissible category $\mathfrak{D}(S)$ corresponding to a combinatorial pseudo-inverse semigroup $S$, could be specified completely by the functors $P_1$, $P_2$ and $\Delta$ alone. This further enables us to formulate our results in a form that emphasises the relations between structure of the semigroups and the intrinsic structure of functors $P_1$, $P_2$ and $\Delta$. The first section of the chapter, which is preliminary to the main result of the chapter, gives a characterization of pseudo-semilattices as reflective subcategories of direct products of two partially ordered sets. Section 2 contains the principal structure theorem for combinatorial pseudo-inverse semigroups. The remaining sections deal with locally testable regular semigroups, normal bands and combinatorial inverse semigroups respectively.
1. CONSTRUCTION OF PSEUDO-SEMILATTICES

Schein [27] defined a pseudo-semilattice to be a structure \((E, \omega^l, \omega^r)\) where \(E\) is a set and \(\omega^l, \omega^r\) are quasiorders on \(E\) such that for all \(e, f \in E\) there exists a unique element \(e \wedge f \in E\) such that

\[
(4.1) \quad \omega^l(e) \cap \omega^r(f) = \omega(e \wedge f)
\]

where \(\omega = \omega^l \cap \omega^r\) is a partial order. This defines a binary operation \(\wedge\) on \(E\). Thus \((E, \wedge)\) becomes an algebra which need not be associative or commutative.

When this algebra is associative, Schein [27] showed that, it is a normal band and when it is commutative, it is a semilattice in the usual sense.

It may be noted that if \(E\) is a pseudo-semilattice then

\[
e \wedge f = e \Rightarrow e \omega^r f \quad \text{and} \quad f \wedge e = e \Rightarrow e \omega^l f.
\]

If the converse also holds, we say \(E\) is a regular pseudo-semilattice. Also in any pseudo-semilattice the relations \(\omega^l'\) and \(\omega^r''\) defined by

\[
(4.2) \quad e \omega^l' f \iff f \wedge e = e \quad \text{and} \quad e \omega^r'' f \iff e \wedge f = e
\]

are quasiorders on \(E\) such that \((E, \omega^l', \omega^r'')\) is a
pseudo-semilattice. Further the binary operation $\land$ defined with respect to these new relations coincides with the original one. This new pseudo-semilattice is regular. In what follows all pseudo-semilattices are assumed to be regular unless otherwise stated.

Schein [27] has shown that a pseudo-semilattice may alternatively be defined as an algebra belonging to a variety defined by a set of identities as follows.

**Proposition 4.1.** ([27], Theorem 2). Let $\langle E, \land \rangle$ be an algebra. Then $\langle E, \land \rangle$ is a pseudo-semilattice if and only if it satisfies the following identities and their left-right duals.

1. $\forall x \in E : x \land x = x$
2. $\forall x, y, z \in E : (x \land y) \land (x \land z) = (x \land y) \land z$
3. $\forall x, y, z \in E : x \land ((x \land y) \land z) = (x \land y) \land z$

As observed above an associative pseudo-semilattice is a normal band and a commutative pseudo-semilattice is a semilattice in the usual sense. Thus, when it is either associative or commutative, it is the set of idempotents of a regular semigroup. Therefore it is natural to consider the problem of characterising those pseudo-semilattices which can be embedded as the biordered set of a regular
semigroup. We shall call such pseudo-semilattices to be partially associative pseudo-semilattices. Partially associative pseudo-semilattices can be characterised as a variety of algebras as follows.

PROPOSITION 4.2 (cf. [22]). Let \((E, \land)\) be an algebra. Then \((E, \land)\) is a partially associative pseudo-semilattice if and only if it satisfies the following identities and their duals.

(a) 
\[ x \land x = x \]

(b) 
\[ (x \land y) \land (x \land z) = (x \land y) \land z \]

(d) 
\[ (x \land y) \land ((x \land z) \land (x \land u)) = ((x \land y) \land (x \land z)) \land (x \land u) \]

A useful alternate characterization for partially associative pseudo-semilattices is given by the following proposition.

PROPOSITION 4.3. ([19], Theorem 7.6). Let \(E = (E, \land)\) be a biordered set. Then the following statements are equivalent.

(i) \(E\) is a pseudo-semilattice

(ii) For all \(e \in E\), \(\Omega(e, f)\) contains exactly one element.

(iii) For all \(e \in E\), the biordered subset \(\omega^r(e)\) is right regular and \(\omega^l(e)\) is left regular.
(iv) For all \( e \in E \), \( \omega (e) \) is a semilattice

(Note that a biordered set \( E \) is right [left] regular if and only if \( \omega^l \subseteq \omega^r \) [\( \omega^r \subseteq \omega^l \)].

Recall that a reflective subcategory is a subcategory for which the inclusion functor has a left adjoint. When the category is a partial order, this left adjoint is unique.

Let \( I \) and \( \Lambda \) be partially ordered sets. Let \( \Delta \) be a reflective subcategory of \( I \times \Lambda \). Define \( \omega^+ \) and \( \omega^l \) on \( \Delta \) as follows.

\[
(4.3) \quad (i, \lambda) \omega^+(j, \mu) \ [ (i, \lambda) \omega^l(j, \mu) ] \iff i \leq j [ \lambda \leq \mu ]
\]

Clearly \( \omega^+ \) and \( \omega^l \) are quasiorders on \( \Delta \) such that \( \omega = \omega^+ \cap \omega^l \) is a partial order. The next theorem gives a construction for pseudo-semilattices which need not be regular.

**THEOREM 4.4.** Let \( I \) and \( \Lambda \) be partially ordered sets and \( \Delta \) be a reflective subcategory of \( I \times \Lambda \). Then \( (\Delta, \omega^l, \omega^+) \) where \( \omega^l \) and \( \omega^+ \) are defined by (4.3), is a pseudo-semilattice.

**Proof.** Since \( \Delta \) is a reflective subcategory of the preorder \( I \times \Lambda \) there exists a unique left adjoint \( F : I \times \Lambda \to \Delta \) to the inclusion functor. For \( e = (i, \lambda) \) and
f = (j, \mu) \in \Delta$, define

\[(4.4) \quad e \land f = F(j, \lambda).\]

Clearly $F(j, \lambda) \leq (j, \lambda)$ in $I \times \Delta$ and if $(k, \nu) \in \Delta$ with $k \leq j$ and $\nu \leq \lambda$ in $I$ and $\Delta$ respectively, then $(k, \nu) \leq F(j, \lambda)$. Thus $\omega(e) \cap \omega(f) = \omega(e \land f)$. Therefore $(\Delta, \omega^\ell, \omega^+)$ is a pseudo-semilattice.

It may be noted that the assertion that $\Delta$ is a reflective subcategory of $I \times \Delta$ is equivalent to saying that $\Delta$ is a residuated subset of the partially ordered set $I \times \Delta$. Also the left adjoint $F$ of the inclusion functor is then the residual of the inclusion mapping (cf. [2]). In [22] Nambooripad has described the construction of pseudo-semilattices in terms of residuated sets. Another construction of pseudo-semilattices from partially ordered sets, has been given by Meakin and Pastijn [17].

Let $A$ be a graph with vertex set $I$. For each vertex $i \in A$, $A_i^* = \bigcup \{ [i, j]_A : j \in I \}$ of all edges with source $i$ is called the star of $i$ (cf. [9], p. 97). If $\Theta : A \to B$ is a graph map, then $\Theta_i^* : A_i^* \to B(\Theta)^*$ is also a graph map which we call the star of $\Theta$. $\Theta$ is said to be star-bijective if $\Theta_i^*$ is
a bijection for each vertex \( i \). The construction of pseudo-semilattices described in the next theorem is similar to the one given by Nambooripad [22] in terms of residuated sets.

If \( P_I : I \times \Lambda \to I \) and \( P_\Lambda : I \times \Lambda \to \Lambda \) are projections, then for any \( E \subseteq I \times \Lambda \), we denote by \( P_I : E \to I \) and \( P_\Lambda : E \to \Lambda \) the composites \( \subseteq \circ P_I \) and \( \subseteq \circ P_\Lambda \) respectively.

**THEOREM 4.5.** Let \( I \) and \( \Lambda \) be partially ordered sets and \( \Delta \subseteq I \times \Lambda \) be a reflective subcategory of the preorder \( I \times \Lambda \) such that

(P) \( P_I : \Delta \to I \) and \( P_\Lambda : \Delta \to \Lambda \) are star-bijections of the corresponding preorders. For each \( e, f \in \Delta \) define,

\[
e \wedge f = F(P_I f, P_\Lambda e)
\]

(4.5) where \( F : I \times \Delta \to \Delta \) is the left adjoint to the inclusion functor \( \subseteq : \Delta \to I \times \Lambda \). Then Equation (4.5) defines a binary operation on \( \Delta \) such that \( (\Delta, \wedge) \) is a partially associative pseudo-semilattice. Conversely every partially associative pseudo-semilattice can be constructed in this way.

The following two lemmas will be useful in proving the theorem. In the following \( I, \Lambda, \Delta \) etc. are as
in Theorem 4.5.

**LEMMA 4.6.** If \((i, \lambda), (i, \mu) \in \Delta\) with \(\lambda \leq \mu\) then \(\lambda = \mu\)

**Proof.** Since \(P_\Delta : \Delta \rightarrow I\) is a star-bijection, the restriction of \(P_\Delta\) (denoted by \(P_\Delta\) itself) to the principal ideal \([\leftarrow, (i, \mu)]_\Delta\) is an order isomorphism,

\[P_\Delta : [\leftarrow, (i, \mu)]_\Delta \rightarrow [\leftarrow, i]_I.\]

Therefore \((i, \lambda) \preceq (i, \mu)\) implies \((i, \lambda) = (i, \mu)\). Hence \(\lambda = \mu\) \(\square\)

**LEMMA 4.7.** If \((i, \lambda), (j, \mu) \in \Delta\), \(i \leq j\) and \(\lambda, \mu \leq v\) for some \(v \in \Delta\) then \((i, \lambda) \preceq (j, \mu)\).

**Proof.** Let \(x = (j, v)\) and \(F(j, v) = (1, m)\). Then \(1 \leq j\) and \(m \leq v\). Now \((j, \mu) \in \Delta\) and \(\mu \leq v\) so that \((j, \mu) \preceq F(j, v) = (1, m)\). Thus \(j \leq 1\) and so \(j = 1\). Further \(\mu \leq m\). Now \(F(j, v) = (j, m)\) and \((j, \mu), (j, m) \in \Delta\) with \(\mu \leq m\). Therefore by Lemma 4.6, \(\mu = m\). Thus \(F(j, v) = (j, \mu)\). Also \((i, \lambda) \in \Delta\) with \(i \leq j, \lambda \leq v\). Therefore \((i, \lambda) \preceq F(j, v) = (j, \mu)\).

Hence the lemma \(\square\)

**Proof of Theorem 4.5.** We prove that the identities (a), (b) and (d) of Proposition 4.2 hold. Let \(x = (i, \lambda)\), \(y = (j, \mu)\), \(z = (k, v)\) and \(u = (l, m) \in \Delta\). (a) follows since for every \(x \in \Delta\), \(Fx = x\).
(b) Let
\[ x \land y = F(j, \lambda) = (j', \lambda') \]
\[ x \land z = F(k, \lambda) = (k', \lambda'') \]
\[ (x \land y) \land (x \land z) = F(k', \lambda') = (p, q) \]
and
\[ (x \land y) \land z = F(k, \lambda') = (r, s). \]

Since \( k' \leq k \), we have \( F(k', \lambda') \leq F(k, \lambda') \) and so \( (p, q) \leq (r, s) \). Also since \( (r, s) = F(k, \lambda') \) we get \( r \leq k \) and \( s \leq \lambda' \leq \lambda \) and \( (r, s) \in \Delta \). Therefore \( (r, s) \leq (k, \lambda) \) in \( I \times \Lambda \) so that \( (r, s) \leq F(k, \lambda) = (k', \lambda'') \). Thus \( r \leq k' \) and \( s \leq \lambda' \). Therefore \( (r, s) \leq F(k', \lambda'') = (p, q) \) and so \( (p, q) = (r, s) \). Hence (b).

(d) Let,
\[ x \land y = F(j, \lambda) = (j', \lambda') \]
\[ x \land z = F(k, \lambda) = (k', \lambda'') \]
\[ x \land u = F(l, \lambda) = (l', \lambda'''') \]
\[ (x \land z) \land (x \land u) = F(l', \lambda''') = (l'', \lambda_2) \]
\[ (x \land y) \land ((x \land z) \land (x \land u)) = F(l'', \lambda') = (p, q) \]
\[ (x \land y) \land (x \land z) = F(k', \lambda') = (k'', \lambda_1) \]
and
We will prove that \((r, s) = (p, q)\). Since \(k'' \leq k' \leq k\) and \(\lambda_1 \leq \lambda' \leq \lambda\) we get \((k'', \lambda_1) \leq (k, \lambda)\) so that 
\((k'', \lambda_1) \leq F(k, \lambda) = (k', \lambda'')\). Thus \(\lambda_1 \leq \lambda''\). Now \(r \leq l', s \leq \lambda_1 \leq \lambda''\) and so 
\((r, s) \leq F(l', \lambda'') = (l'', \lambda_2)\).
Thus \(r \leq l''\) and \(s \leq \lambda'\) so that 
\((r, s) \leq F(l'', \lambda') = (p, q)\). Now we proceed to prove the reverse inequality. Since \((p, q), (l'', \lambda_2) \in \Delta, p \leq l''\) and \(q, \lambda_2 \leq \lambda\) it follows from Lemma 4.7 that 
\((p, q) \leq (l'', \lambda_2)\). Therefore \(q \leq \lambda_2 \leq \lambda''\). Also 
\((k', \lambda'') \in \Delta\). Hence by condition (P) of the theorem, 
there exists \(p' \in I\) such that \((p', q) \leq (k', \lambda'')\) in \(\Delta\).
Then \(p' \leq k'\). Already \(q \leq \lambda'.\) Therefore 
\((p', q) \leq F(k', \lambda') = (k'', \lambda_1)\). Thus \(q \leq \lambda_1\). Already \(p \leq l'' \leq l'\). Hence \((p, q) \leq F(l', \lambda_1) = (r, s)\). Thus 
(d) holds.

Conversely let \((E, \wedge)\) be a partially associative 
pseudo-semilattice. Let \(\omega^l, \omega^+\) be quasiorders on \(E\) 
induced by the binary operation \(\wedge\) as in (4.2). Let 
\(\omega = \omega^l \cap \omega^+, \mathcal{R} = \omega^+ \cap (\omega^+)^{-1}\) and 
\(\mathcal{L} = \omega^l \cap (\omega^l)^{-1}\).
Let $I = E/R$ and $\Lambda = E/L$. Define $\leq$ on $I$ and $\Lambda$ as follows.

$$R_e \leq R_f \iff e \omega^r f \quad \text{and} \quad L_e \leq L_f \iff e \omega^l f.$$  

Clearly these relations are well-defined and $I$ and $\Lambda$ are partially ordered by these relations. Let

$$\Delta = \{(R_e, L_e) : e \in E\}.$$  

Then $\Delta$ is a partially ordered subset of $I \times \Lambda$ such that the mapping

$$\Theta : E \rightarrow \Delta \text{ defined by } e \mapsto (R_e, L_e)$$

is an order-isomorphism between the partially ordered sets $(E, \omega)$ and $\Delta$. Because of this isomorphism we can treat $E$ itself to be a partially ordered subset of $I \times \Lambda$ and $\Theta : E \rightarrow I \times \Lambda$ may be treated as an inclusion functor.

To prove that $E$ is a reflective subcategory of the preorder $I \times \Lambda$ it is sufficient to produce a left adjoint to the functor $\Theta$. Define $F : I \times \Lambda \rightarrow E$ by

$$F(R_e, L_f) = f \wedge e.$$  

We show that $F$ is a left adjoint to $\Theta$. Let $g \in E$ with $g \Theta = (R_g, L_g) \leq (R_e, L_f)$ in $I \times \Lambda$. Then $g \in \omega^r(e) \cap \omega^l(f) = \omega(f \wedge e)$. Therefore $g \omega f \wedge e$ in $E$. And if $g \omega F(R_e, L_f)$ in $E$, then $g \Theta = (R_g, L_g) \leq (R_e, L_f)$ in $I \times \Lambda$. 
Theorem 4.5, is equivalent to the existence of a cross-
such subcategories, reflective subdirect
products of \( I \) and \( \Lambda \). Thus there exists a one-to-one correspondence between partially associative pseudo-semilattices \( E \), with
\( E/\mathcal{R} \cong I \), \( E/\mathcal{L} \cong \Lambda \) and reflective subdirect products of \( I \) and \( \Lambda \) satisfying condition (P) of Theorem 4.5. It may be noted that the existence of a left adjoint \( F \) as in
Theorem 4.5, is equivalent to the existence of a cross-
connection between \( I \) and \( \Lambda \) which produces a pseudo-
semilattice (cf. [21]).
2. STRUCTURE OF COMBINATORIAL PSEUDO-INVERSE SEMIGROUPS

A regular semigroup $S$, is said to be a \textit{pseudo-inverse} semigroup if $\mathcal{G}(e, f)$ contains exactly one element for every $e, f \in E(S)$. Proposition 4.3 provides the following equivalent characterizations of this class of semigroups (cf. [22]).

**Proposition 4.8.** Let $S$ be a regular semigroup. Then the following are equivalent.

1. $S$ is pseudo-inverse
2. $E(S)$ is a pseudo-semilattice
3. For every $e \in E(S)$, $eSe$ is an inverse semigroup

Let $\mathcal{D}$ be a category and $K : \mathcal{D} \rightarrow \text{Set}$ be a functor. Let $Y : \mathcal{D}^\star \rightarrow \text{Set}^{\mathcal{D}}$ be the Yoneda functor (cf. Proposition 1.1). Consider the comma category $(Y \downarrow K)$ (cf. [12], p.46). From the definition of comma category, objects of $(Y \downarrow K)$ are natural transformations $m : H^\alpha \rightarrow K$ for $\alpha \in \mathcal{D}$ and a morphism $\sigma : m \rightarrow m'$ for $m' : H^\beta \rightarrow K$ is a natural transformation $H^\theta : H^\alpha \rightarrow H^\beta$ for some $\theta : \beta \rightarrow \alpha$ in $\mathcal{D}$ ($\theta : \alpha \rightarrow \beta$ in $\mathcal{D}^\star$) such that $m = H^\theta m'$. That is, the following diagram is commutative
We use the notation \((\mathcal{C} \downarrow K)\) to denote the comma category \((Y \downarrow K)\) so as to indicate the domain category explicitly.

Let \(\mathcal{C}\) be a small category, \(P_1, P_2 : \mathcal{C} \to \text{Set}\) be functors and \(\Delta\) be a subfunctor of \(P_1 \times P_2\), such that for each \(\alpha \in \mathcal{C}\), \(\Delta(\alpha)\) is a subdirect product of \(P_1(\alpha)\) and \(P_2(\alpha)\). Let \(p_1 : \Delta \to P_1\) and \(p_2 : \Delta \to P_2\) be natural transformations whose components are the corresponding projections. Now \((\mathcal{C} \downarrow \Delta)\), \((\mathcal{C} \downarrow P_1)\) and \((\mathcal{C} \downarrow P_2)\) are comma categories as described above. Let

\[ \delta : (\mathcal{C} \downarrow \Delta) \to (\mathcal{C} \downarrow P_1) \times (\mathcal{C} \downarrow P_2) \]

be defined by:

\[ m \mapsto (mP_1, mP_2) \quad \text{and} \quad H^\Theta \mapsto (H^\Theta, H^\Theta). \]

It is clear that by this definition \(\delta\) is a functor.

As already observed in the introduction, the functors \(P_1, P_2\) and \(\Delta\) completely determine the admissible category of a combinatorial pseudo-inverse semigroup. For this reason,
we shall reformulate the axioms in Definition 3.2 in terms of these functors above. Since the resulting object is technically different from an admissible category (even though this object belongs to ACat) we shall use a different name to indicate the same.

**DEFINITION 4.9.** Let $\mathcal{O}$ be a strictly skeletal small category, $P_1, P_2 : \mathcal{O} \to \text{Set}$ be functors which are non-empty and disjoint, and $\Delta$ be a subdirect product of $P_1$ and $P_2$. We call $\mathcal{P} = \langle \mathcal{O}, P_1, P_2, \Delta \rangle$ a $P$-category if the following axioms hold.

I. For each $\alpha \in \mathcal{O}$ every natural transformation from $H^\alpha$ to $P_1$ and $H^\alpha$ to $P_2$ is injective.

II. The functor $\mathcal{S} : (\mathcal{O} \downarrow \Delta) \to ((\mathcal{O} \downarrow P_1) \times (\mathcal{O} \downarrow P_2)$ has a right adjoint $\bar{\mathcal{S}} : (\mathcal{O} \downarrow P_1) \times (\mathcal{O} \downarrow P_2) \to (\mathcal{O} \downarrow \Delta) \square$

It may be noted that we have not assumed in this definition, that $P_1 \ast P_2$ is faithful (as in Definition 3.2).

However this is a consequence of axioms I and II (see Lemma 4.17 below). In what follows we assume that $\mathcal{P} = \langle \mathcal{O}, P_1, P_2, \Delta \rangle$ is a fixed $P$-category.

**LEMMA 4.10.** (i) Every natural transformation $m : H^\alpha \to \Delta$ for $\alpha \in \mathcal{O}$ is injective.
(ii) For each object \((x, y) \in (\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2)\)

\[ \mathcal{E}_{(x, y)} : \delta(\delta(x, y)) \to (x, y) \]

is a universal arrow from \(\delta\) to \((x, y)\), where \(\mathcal{E} : \delta \delta \to 1(\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2)\) is the co-unit of the adjunction given by axiom II of Definition 4.9 (cf. Proposition 1.3).

Proof. (i) follows from axiom I and the fact that \(\Delta\) is a subdirect product of \(P_1\) and \(P_2\).

(ii) Let \(\mathcal{E} : \delta \delta \to 1(\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2)\) be the co-unit of the adjunction. Then by Proposition 1.3, for each \((x, y) \in (\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2)\), \(\mathcal{E}_{(x, y)} : \delta(\delta(x, y)) \to (x, y)\) is a universal arrow from \(\delta\) to \((x, y)\). Hence (ii)

REMARK. If \(\mathcal{E}_{(x, y)} = (H^{\theta}, H^\varphi)\) and

\[ \delta(x, y) = m \in (\mathcal{D} \downarrow \Delta) \]

then we can represent the universal arrow \((\delta(x, y), \mathcal{E}_{(x, y)})\) by \((m, H^{\theta}, H^\varphi)\).

Sometimes we write just \((H^{\theta}, H^\varphi)\) to denote this universal arrow. In what follows we shall use this notation without comment. For \(P\)-categories defined above, we shall follow a convention similar to the one introduced for admissible categories. Thus we shall denote \(P\)-categories by symbols \(\mathcal{P}, \mathcal{P}'\) etc. which have components.
\[ (\mathcal{D}, P_1, P_2, \Delta), (\mathcal{D}', P'_1, P'_2, \Delta') \] etc. respectively.

Let \( K : \mathcal{D} \rightarrow \text{Set} \) be a functor. Then we denote by \( [\mathcal{D}, K] \), the preorder of the dual category \( (\mathcal{D} \downarrow K)^* \). That is, \( [\mathcal{D}, K] \) consists of all natural transformations \( n : H^\alpha \rightarrow K \) for \( \alpha \in \mathcal{D} \), in which \( \leq \) is defined as follows.

For \( m : H^\alpha \rightarrow K \) and \( n : H^\beta \rightarrow K \)

\[(4.6) \quad m \leq n \iff \text{there exists a natural transformation } \theta : H^\alpha \rightarrow H^\beta \text{ such that } m = H^\theta n.\]

**Lemma 4.11.** For any functor \( K : \mathcal{D} \rightarrow \text{Set} \), the preorder \( [\mathcal{D}, K] \) is order isomorphic with \( \bar{K} \), the preorder associated with \( K \) (cf. Equation (3.2)).

**Proof.** The mapping from \( [\mathcal{D}, K] \rightarrow \bar{K} \) defined by

\[ m \mapsto (1_\alpha)m_\alpha \text{ where } m : H^\alpha \rightarrow K \text{, is a bijection, by } \]

Yoneda Lemma (cf. Proposition 1.1). Suppose \( m : H^\alpha \rightarrow K \) and \( n : H^\beta \rightarrow K \) with \( m \leq n \). Then there exists \( \theta : \beta \rightarrow \alpha \) such that \( m = H^\theta n \). Therefore

\[ (1_\alpha)m_\alpha = (\theta)n_\alpha = (1_\beta)n_\beta \quad K(\theta) \]

Hence \( (1_\alpha)m_\alpha \leq (1_\beta)n_\beta \) in \( \bar{K} \). Conversely if \( a, b \in \bar{K} \) with \( a \leq b \), then there exists \( \theta : \beta \rightarrow \alpha \) such that

\[ a = bK(\theta) \quad \text{where } a \in K(\alpha) \text{ and } b \in K(\beta). \]

Then by Yoneda Lemma again, there exists a unique \( m : H^\alpha \rightarrow K \)
such that \((1_\alpha)m_\alpha = a\) and a unique \(n : H^\beta \to K\)
such that \((1_\beta)n_\beta = b\). Now
\[
(1_\alpha)m_\alpha = (1_\beta)n_\beta \quad K(\theta) = (1_\alpha)(H^\theta n)_\alpha \quad \text{and so} \quad m = H^\theta n.
\]
Thus \(m \leq n\). Hence the lemma \(\square\)

In particular, it now follows that the quasiordered
set \(I, \Lambda\) and \(\bar{\Delta}\) associated with the functors \(P_1, P_2\)
and \(\Delta\) respectively are order isomorphic with \([D, P_1]\)
\([D, P_2]\) and \([D, \Delta]\).

**Lemma 4.12.** The quasiorders on \([D, \Delta]\), \([D, P_1]\) and
\([D, P_2]\) defined by \((4.6)\), are partial orders.

**Proof.** Let \(m : H^\alpha \to \Delta\) and \(n : H^\beta \to \Delta\) be such
that \(m \leq n\) and \(n \leq m\). Then \(m = H^\theta n\) and \(n = H^\varphi m\)
for some \(\theta : \beta \to \alpha\) and \(\varphi : \alpha \to \beta\) in \(D\). Now
\[
m = H^\theta H^\varphi m = H^\varphi H^\theta m \quad \text{and} \quad n = H^\theta \varphi n.
\]

Then
\[
(1_\alpha)m_\alpha = (\varphi \theta)m_\alpha
\]
and so \(\varphi \theta = 1\), since \(m_\alpha\) is injective. Similarly
\(\theta \varphi = 1\). Since \(D\) is strictly skeletal, it follows that
\(\theta = \varphi = 1\) and so \(m = n\). The results for \([D, P_1]\)
and \([D, P_2]\) follows similarly \(\square\)
LEMMA 4.13. $\Delta$ is a reflective subcategory of the preorder $I \times \Delta$.

Proof. It is enough to prove the following. If $(j, \lambda) \in I \times \Delta$, then there exists $(k, \nu) \in \Delta$ such that $(k, \nu) \leq (j, \lambda)$ in $I \times \Delta$ and if $(k', \nu') \in \Delta$ with $(k', \nu') \leq (j, \lambda)$ in $I \times \Delta$, then $(k', \nu') \leq (k, \nu)$.

Let $(j, \lambda) \in I \times \Delta$ be given. By Yoneda Lemma there exists $x : H^\alpha \to P_1$ and $y : H^\beta \to P_2$ such that

$$(1_\alpha)x_\alpha = j \quad \text{and} \quad (1_\beta)y_\beta = \lambda,$$

if $j \in P_1(\alpha)$ and $\lambda \in P_2(\beta)$. Let $u = \delta(x, y)$ and

$$(H^\theta, H^\varphi) : \delta(u) \to (x, y)$$

be the universal arrow described in Lemma 4.10. Suppose that $u : H^\gamma \to \Delta$.

Let $(k, \nu) = (1_\gamma)u_\gamma$. Then

$$k = (1_\gamma)u_\gamma p_{1_\gamma} = (1_\gamma)(up_{1_\gamma})_\gamma = (1_\gamma)(Hx)_\gamma = (\theta)x_\gamma = (1_\alpha)x_\alpha \theta_1 = j \theta_1.$$ Thus $k \leq j$ and similarly

$\nu = \lambda \varphi_2 \leq \lambda$. Now let $(k', \nu') \in \Delta(\gamma')$ with $k' \leq j$ and $\nu' \leq \lambda$. Then there exists $\theta' : \alpha \to \gamma'$ and

$\varphi' : \beta \to \gamma'$ such that

$$k' = j \theta'_1 \quad \text{and} \quad \nu' = \lambda \varphi'_2.$$ Let $u' : H'^{\gamma'} \to \Delta$ be such that $(1_{\gamma'})u_\gamma' = (k', \nu')$. 

Then as before \( u'p_1 = H\theta'x \) and \( u'p_2 = H\varphi'y \). Therefore \((H\theta', H\varphi') : \delta(u') \rightarrow (x, y)\) is a morphism in 
\((\mathcal{X}_1 \downarrow P_1) \times (\mathcal{X}_1 \downarrow P_2)\). Hence by the universality of 
\((u, H\theta', H\varphi')\), there exists \( \psi : \gamma \rightarrow \gamma' \) in \( \mathcal{X}_1 \) such that 
\( \delta(H\psi)(H\theta', H\varphi') = (H\theta', H\varphi') \). Therefore 
\[ H\psi H\theta = H\theta' \varphi = H\theta' \quad \text{and} \quad H\varphi \psi = H\varphi'. \]
Hence \( \theta \psi = \theta' \) and \( \varphi \psi = \varphi' \). Now 
\[ k' = j\theta_1' = j\theta_1 \psi_1 = k \psi_1 \leq k \]
and 
\[ \nu' = \lambda \varphi_2' = \lambda \varphi_2 \psi_2 = \nu \psi_2 \leq \nu \]
This proves the theorem \( \square \)

We proceed to show that a combinatorial pseudo-
inverse semigroup can be constructed from a \( P \)-category. We first prove a few lemmas. In what follows we use the notations introduced above. In particular 
\( \mathcal{P} = \langle \mathcal{X}_1, P_1, P_2, \Delta \rangle \) is a \( P \)-category.

**Lemma 4.14.** If \( \varphi : \lambda \rightarrow \gamma \) and \( \theta : \gamma \rightarrow \lambda \) in \( \mathcal{X}_1 \) with 
\( \varphi \theta = 1_{\lambda} \), then \( \theta = \varphi = 1 \).

**Proof.** Let \( m : H\lambda \rightarrow \Delta \) and \( n = H\varphi m \). Then
Now by Lemma 4.10, \( n \gamma \) is injective and so \( \Theta \varphi = 1 \).

Already \( \varphi \Theta = 1 \). Since \( \mathcal{D} \) is strictly skeletal, the only isomorphism in \( \mathcal{D} \) is identity and so \( \Theta = \varphi = 1 \) \( \Box \)

**Lemma 4.15.** For every \( m \in (\mathcal{D} \downarrow \Delta), \bar{\delta}(\delta(m)) = m \) and \((m, 1, 1)\) is a universal arrow from \( \delta \) to \( \delta(m) \).

**Proof.** By Lemma 4.10, there exists a universal arrow (\( u, H^\delta, H^\varphi \)) from \( \delta \) to \( \delta(m) \), where \( u = \bar{\delta}(\delta(m)) \).

Then \( u_1 = H^\delta m_1 \) and \( u_2 = H^\varphi m_2 \). Now \((1, 1) : \delta(m) \to \delta(m)\) is an arrow in \((\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2)\).

Since \((u, H^\delta, H^\varphi)\) is universal, there exists \( \psi \) such that \( H^\psi H^\delta = 1 = H^\psi H^\varphi \). Hence by Yoneda Lemma \( \Theta \psi = 1 \) and \( \varphi \psi = 1 \), and so by Lemma 4.14, it follows that \( \Theta = \varphi = 1 \) and \( \psi = 1 \). Hence \( u = m \) and \((m, 1, 1)\) is universal \( \Box \)

**Lemma 4.16.** If \((i, \lambda), (i, \lambda \Theta_2) \in \Delta(\lambda)\), then \( \Theta = 1 \).

**Proof.** Let \( m : H^\lambda \to \Delta \) be such that \( (1_\lambda)m_\lambda = (i, \lambda) \)

Now \((m, 1, 1)\) is a universal arrow from \( \delta \) to \( \delta(m) \).

Let \( n : H^\lambda \to \Delta \) be defined by \( (1_\lambda)n_\lambda = (i, \lambda \Theta_2) \).
Then by Yoneda Lemma we get that \( mp_1 = np_1 \). Also \( \Theta \) is such that \( \Theta : \alpha \rightarrow \alpha \) and \((1_\alpha)n_\alpha p_2 \alpha = \lambda \Theta_2 = \)
\[
(1_\alpha)m_\alpha p_2(\Theta) = (1_\alpha)(H^\Theta mp_2)\alpha \quad \text{and so} \quad np_2 = H^\Theta mp_2.
\]
Thus \((1, H^\Theta) : \delta(n) \rightarrow \delta(m)\) is a morphism in \((\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2)\). Hence by universality of \((m, 1, 1)\), there exists \( \psi : \alpha \rightarrow \alpha \) such that \( H^\psi = 1 \) and \( H^\psi = H^\Theta \). Therefore \( \Theta = 1 \).

**Lemma 4.17.** If \((i \theta_1, \mu), (i \varphi_1, \mu) \in \bar{\Delta} \) then \( \Theta = \varphi \).

**Proof.** Suppose that \((i \theta_1, \mu), (i \varphi_1, \mu) \in \Delta(\beta)\), \((i, \lambda) \in \Delta(\alpha) \) and \( \theta, \varphi : \alpha \rightarrow \beta \) in \( \mathcal{D} \). Let \( m : H^\alpha \rightarrow \Delta \) be defined by \((1_\alpha)m_\alpha = (i, \lambda) \) and \( n, s : H^\beta \rightarrow \Delta \) be defined by \((1_\beta)n_\beta = (i \theta_1, \mu) \) and \((1_\beta)s_\beta = (i \varphi_1, \mu) \) respectively. Let \((u, H^f, H^g)\) be universal from \( \delta \) to \((mp_1, np_2)\). Now \((H^\Theta, 1) : \delta(s) \rightarrow (mp_1, np_2) \) in \((\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2)\).

Hence there exists a morphism \( \psi \) in \( \mathcal{D} \) such that \( H^\psi H^f = H^\Theta \) and \( H^\psi H^g = 1 \). Hence by Lemma 4.14, \( \psi = 1 \) and so \( f = \Theta \). Similarly \((H \varphi, 1) : \delta(s) \rightarrow (mp_1, np_2) \) is in \((\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2) \) and so \( f = \varphi \). Thus \( \Theta = \varphi \).
We now give an alternate characterization of P-categories as follows.

**THEOREM 4.18.** \( \mathcal{P} = \langle \mathcal{D}, P_1, P_2, \Delta \rangle \) is a P-category if and only if the following conditions hold:

(a) If \( \theta, \varphi : \alpha \to \beta \) in \( \mathcal{D} \) and if

\[
(i \theta_1, \mu), (i \varphi_1, \mu) \in \Delta \quad [ (j, \lambda \theta_2), (j, \lambda \varphi_2) \in \Delta ]
\]

for some \( i \in P_1(\alpha) \) and \( \mu \in P_2(\beta) \) [ \( \lambda \in P_2(\alpha) \) and \( j \in P_1(\beta) \)], then \( \theta = \varphi \).

(b) Given \( \lambda \in P_2(\alpha) \) and \( j \in P_1(\beta) \), there exists \( \gamma \in \mathcal{D} \) and morphisms \( \theta = \theta \lambda j : \alpha \to \gamma \) and

\( \varphi = \varphi \lambda j : \beta \to \gamma \) such that

(i) \( (j \varphi_1, \lambda \theta_2) \in \Delta(\gamma) \)

(ii) If \( \theta' : \alpha \to \gamma' \) and \( \varphi' : \beta \to \gamma' \) are such that \( (j \varphi_1', \lambda \theta_2') \in \Delta(\gamma') \) then there exists a unique morphism \( \psi : \gamma \to \gamma' \) such that

\[ \theta' = \theta \lambda j \psi \quad \text{and} \quad \varphi' = \varphi \lambda j \psi \]

**Proof.** Suppose that \( \mathcal{P} \) is a P-category. Then (a) follows from Lemma 4.17. To prove (b), let \( \lambda \in P_2(\alpha) \)}
and \( j \in P_1(\beta) \). Define \( x : H^\beta \rightarrow P_1 \) by

\[(1_\beta)x_\beta = j \quad \text{and} \quad y : H^\alpha \rightarrow P_2 \quad \text{by} \quad (1_\alpha)y_\alpha = \lambda.\]

Then by Lemma 4.10, there exists a universal arrow

\( (m, H^\varphi, H^\theta) \) from \( \delta \) to \( (x, y) \). Now if \( m : H^f \rightarrow \Delta \)
then \( \varphi : \beta \rightarrow \gamma \) and \( \theta : \alpha \rightarrow \gamma \) are such that

\[mp_1 = H^\varphi x \quad \text{and} \quad mp_2 = H^\theta y. \]

Therefore

\[(j \varphi_1, \lambda \theta_2) = (1_\gamma)m_\gamma \in \Delta. \]

Now if \( \theta' : \alpha \rightarrow \gamma' \)
and \( \varphi' : \beta \rightarrow \gamma' \) are such that \( (j \varphi'_1, \lambda \theta'_2) \in \Delta(\gamma') \)
then define \( m' : H^f \rightarrow \Delta \) by

\[(1_{\gamma'})m'_{\gamma'} = (j \varphi'_1, \lambda \theta'_2). \]

Then \( (H^\varphi', H^\theta') : \delta(m') \rightarrow (x, y) \)
is a morphism in \( (\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2) \). Hence by universality
of \( (m, H^\varphi, H^\theta) \) there exists a unique morphism

\[\psi : \gamma \rightarrow \gamma' \] such that \( H^\varphi' = H^\psi H^\varphi \)
and

\[H^\theta' = H^\varphi H^\theta \]
so that \( \varphi' = \varphi \psi \) and \( \theta' = \theta \psi \).

Thus (b) holds.

Conversely suppose that \( < \mathcal{D}, P_1, P_2, \Delta > \) satisfies
conditions (a) and (b) above. Let \( x : H^\alpha \rightarrow P_1 \) be a
natural transformation. Let \( \theta, \varphi : \alpha \rightarrow \beta \) in \( \mathcal{D} \) and

\[(\theta)x_\beta = (\varphi)x_\beta \quad \text{in} \quad P_1(\beta). \]

If \( (1_\alpha)x_\alpha = i \),
then \( (\theta)x_\beta = i \theta_1 = (\varphi)x_\beta = i \varphi_1. \)
Now by (a) above,
\( \theta = \varphi \). Hence \( x \) is injective. Similarly every natural transformation from \( H^\alpha \) to \( P_2 \), for \( \alpha \in \mathcal{D} \) is also injective. Thus axiom (I) of Definition 4.9, holds.

Now we show that \( \delta : (\mathcal{D} \downarrow \Delta) \longrightarrow (\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2) \) has a right adjoint \( \delta^\ast \). Define

\[
\delta : (\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2) \longrightarrow (\mathcal{D} \downarrow \Delta)
\]
as follows. If \( x : H^\beta \longrightarrow P_1 \) and \( y : H^\lambda \longrightarrow P_2 \) are given, then let

\[
j = (1_\beta)(x) \in P_1(\beta) \quad \text{and} \quad \lambda = (1_\lambda)(y) \in P_2(\lambda).
\]

Then by (b) above, there exists \( \gamma \in \mathcal{D} \) and \( \alpha : \gamma \longrightarrow \alpha \) and \( \varphi : \beta \longrightarrow \gamma \) such that \((j \varphi_1, \lambda \theta_2) \in \Delta(\gamma)\). Define \( m : H^\gamma \longrightarrow \Delta \) by \((1_\gamma)m = (j \varphi_1, \lambda \theta_2)\).

Further \((H^\varphi, H^\theta) : \delta(m) \longrightarrow (x, y)\) is an arrow in \((\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2)\). Also condition (ii) of (b) above shows that \((m, H^\varphi, H^\theta)\) is universal from \( \delta \) to \((x, y)\). Denoting \( m = \delta_0(x, y)\), we see from Proposition 1.3 (iv) that there exists a right adjoint for \( \delta \). Hence \( \langle \mathcal{D}, P_1, P_2, \Delta \rangle \) is a \( P \)-category \( \square \).

**THEOREM 4.19.** Let \( \mathcal{P} = \langle \mathcal{D}, P_1, P_2, \Delta \rangle \) be a \( P \)-category.

On \( S = \mathcal{C}(\mathcal{P}) = \bigcup \{ P_1 \times P_2(\alpha) : \alpha \in \mathcal{D} \} \) define product by

\[
(i, \lambda)(j, \mu) = (i(\theta \lambda j)_1, \mu(\varphi \lambda j)_2)
\]
where \((i, \lambda) \in P_1 \times P_2(\alpha), (j, \mu) \in P_1 \times P_2(\beta)\) and

\[ \Theta_{\lambda j}, \phi_{\lambda j} \]

are as in (b) of Theorem 4.18. Then S with this product is a combinatorial pseudo-inverse semigroup. Conversely every combinatorial pseudo-inverse semigroup can be constructed in this way.

First we prove a few lemmas, which we use in the proof of the theorem.

**Lemma 4.20.** Let S be a combinatorial pseudo-inverse semigroup and \(\varphi = \Phi(S)\) be the \(\sigma\)-functor such that

\[ \Sigma(\varphi) = S \] (cf. Theorem 2.12). Let \(\mathcal{A} = \text{im } \varphi\) and \(P_1, P_2, \Delta\) be as in Theorem 3.21. Then

\[ \mathcal{P}(S) = \langle \mathcal{A}, P_1, P_2, \Delta \rangle \]

is a \(\mathcal{P}\)-category.

**Proof.** It suffices to prove (a) and (b) of Theorem 4.18.

(a) follows from the fact that \(\omega^+(e)\) is right regular and \(\omega^-(e)\) is left regular for each \(e \in E(S)\) (cf. Proposition 4.3).

(b) Let \(\lambda \in P_2(\alpha)\) and \(j \in P_1(\beta)\) be given. Let \(e, f \in E(S)\) be such that \(e = (i, \lambda)\) and \(f = (j, \mu)\) for some \(i \in P_1(\alpha)\) and \(\mu \in P_2(\beta)\). Let

\[ \mathcal{S}(e, f) = \{h\} \]

Take \(\Theta_{\lambda j} = \varphi(e, eh)\) and
\[ \varphi_{\lambda j} = \varphi(f, hf). \] Clearly if \( e \leq e' \) and \( f \not\equiv f' \), then \( \emptyset(e, f) = \emptyset(e', f') = \{ h \} \), and
\[ \varphi(e, eh) = \varphi(e', e'h) \] and \[ \varphi(f, hf) = \varphi(f', hf'). \] Hence \( \Theta_{\lambda j} \) and \( \varphi_{\lambda j} \) are independent of the choice of \( i \) and \( \mu \). Further
\[ (j(\varphi_{\lambda j}), \lambda(\Theta_{\lambda j})) \in R_{hf} \cap L_{eh} = \{ h \}. \]

Therefore \( (j(\varphi_{\lambda j}), \lambda(\Theta_{\lambda j})) \in \Delta. \) Now suppose \( \Theta', \varphi' \) are such that \( h' = (j \varphi'_1, \lambda \theta'_2) \in \Delta \). Then
\[ h' \in \omega^e(e) \cap \omega^f(f) = \omega(h). \] Let \( \psi = \varphi(h, h'). \)
\[ \text{Then } \varphi(e, eh) \psi(h, h') = \varphi(e, eh'). \] Also
\[ \theta'_2 = \varphi_2(e, eh'). \] Therefore by (a) of Theorem 4.18, \( \Theta' = \varphi(e, eh') \). Thus \( \Theta_{\lambda j} \psi = \Theta' \). Similarly
\[ \varphi_{\lambda j} \psi = \varphi'. \] Hence (b) holds.

Now let \( \mathcal{P} \) be a \( \mathcal{P} \)-category. As in Chapter III we define \( \Lambda_j, \mathcal{I}_\mu, H_j \) and \( K_\mu \). Let \( \lambda \in \Lambda_j \).
Then \( (j \Theta_1, \lambda) \in \Delta \) for some \( \Theta \) in \( \mathcal{D} \). By condition (a) of Theorem 4.18, \( \Theta \) is unique. Hence for \( j \in \mathcal{P}_1(\beta) \) we define the cone \( F(j; -) : \mathcal{P}_2(\beta) \to H_j \) by setting \( F(j; \lambda) = \Theta_2 \), such that \( (j \Theta_1, \lambda) \in \Delta \).
Also for \( i \in I \), define \( G(\mu ; i) = \theta_1 \), such that \( (i, \mu \theta_2) \in \Delta \). It easily follows that \( F(j; -) \)
and \( G(\mu ; -) \) are cones. Now we prove that 
\[
\langle \mathcal{D}, P_1, P_2, \Delta, F, G \rangle \in A'\text{Cat}.
\]

**Lemma 4.21.** Let \( \mathcal{D} = \langle \mathcal{D}, P_1, P_2, \Delta, F, G \rangle \) be as described above. Then \( \mathcal{D} \in A'\text{Cat} \).

**Proof.** We prove that axioms I and III of Definition 3.2 and II' of Proposition 3.31 and their duals hold.

(I) Let \( j \in P_1(\beta) \) and \( \lambda \in \Lambda_j \). Let \( (j \theta_1, \lambda) \in \Delta \)

Then \( F(j; \lambda) = \theta_2 \) and so \( (\theta_1, F(j; \lambda)) = (\theta_1, \theta_2) \)
is in \( \Delta(\mathcal{D}) \). Conversely suppose \( (\theta_1, F(j; \lambda)) \) in \( \Delta(\mathcal{D}) \). Since \( \lambda \in \Lambda_j \), \( (j \varphi_1, \lambda) \in \Delta \) for some \( \varphi \)
in \( \mathcal{D} \). Then by the definition of \( F(j; -) \), \( F(j; \lambda) = \varphi_2 \)

Therefore \( (\theta_1, \varphi_2) \) in \( \Delta(\mathcal{D}) \). Then by (a) of
Theorem 4.18, \( \theta = \varphi \). Further if \( (j, \mu) \in \Delta \)
then \( (j \theta_1, \mu) \in \Delta \) implies \( \theta = 1 \), and so
\( F(j; \mu) = 1 \). Thus (I) holds.

(II) Let \( k \in I \mu \), \( \mu \in \Lambda_i \), \( (i, \lambda) \in \Delta(\mathcal{A}) \) and \( k \leq i \). Then there exists \( \theta, \varphi, \psi \) such that
(k, \mu \theta_2), (i \varphi_1, \mu) \in \Delta and k = i \psi_1. Now
\( G(\mu; k) = \theta_1, F(i; \mu) = \varphi_2 \) and \( G(\lambda; k) = \psi_1 \)
Further \((k, \mu \theta_2) = (i \psi_1, \mu \theta_2) \in \Delta\). Also
\((i \varphi_1 \theta_1, \mu \theta_2) \in \Delta\). Hence by Lemma 4.17, \( \psi = \varphi \theta \)
Let \((i', \lambda') \in \Delta(\alpha)\). Then \( i'G(\lambda; k) = i' \psi_1 \)
and \( \lambda' F(i; \mu) = \lambda' \varphi_2 \). Also
\((i' \psi_1, \lambda' \psi_2) = (i'G(\lambda; k), \lambda' \varphi_2 \theta_2) =
= (i'G(\lambda; k), \lambda' F(i; \mu) \theta_2) \in \Delta\).
Hence
\( G(\lambda' F(i; \mu); i'G(\lambda; k)) = \theta_1 = G(\mu; k) \).

Now instead of \( k \leq i \) in the hypothesis, assume that \( \mu \leq \lambda \).

Let \( \mu = \lambda \chi_2 \). Then \((k, \mu \theta_2) = (k, \lambda \chi_2 \theta_2) \in \Delta\)
and so \( G(\lambda; k) = \chi_1 \theta_1 \). Further \((i \varphi_1, \lambda \varphi_2) \)
and \((i \varphi_1, \mu) = (i \varphi_1, \lambda \chi_2)\) are in \( \Delta \). Therefore
\( \chi = \varphi \) and
\( G(\lambda' F(i; \mu); i'G(\lambda; k)) = G(\lambda' \varphi_2; i' \chi_1 \theta_1) \)
\( = G(\lambda' \varphi_2; i' \varphi_1 \theta_1) \)
\( = \theta_1 \) since \((i' \varphi_1 \theta_1, \lambda' \varphi_2 \theta_2) \in \Delta\)
\( = G(\mu; k) \).
Thus (II') holds.

(III) Given $j \in I$ and $\lambda \in \Lambda$ let

$$(k, \nu) = (j(\varphi_{\lambda j})_1, \lambda (\varphi_{\lambda j})_2) \in \Delta(\gamma)''$. Suppose

$$(k', \nu') \in \Delta(\gamma'')$$

with $k' \leq j$ and $\nu' \leq \lambda$. Then there exists $\theta'$, $\varphi'$ in $\mathcal{D}$ such that $k' = j \varphi_1'$ and $\nu' = \lambda \theta_2'$. Now by (b) of Theorem 4.18, there exists a unique morphism $\psi : \gamma \rightarrow \gamma'$ such that $\theta' = \theta_{\lambda j} \psi$ and $\varphi' = \varphi_{\lambda j} \psi$. Hence

$$G(\lambda; k') = \Theta_1' = (\theta_{\lambda j} \psi)_1 = G(\lambda; k) \psi_1$$

and

$$F(j; \nu') = \varphi_2' = (\varphi_{\lambda j} \psi)_2 = F(j; \nu) \psi_2$$

Hence III holds. The duals may be proved similarly.

Thus $\mathcal{D} \in \mathcal{A}'\mathcal{C}at$. $\Box$

**Lemma 4.22.** Let $\mathcal{C}(\mathcal{D}) = \bar{\Lambda}$ be the biordered set given by $\mathcal{D}$, above (cf. Theorem 3.11). Then for $e, f \in \mathcal{C}(\mathcal{D})$

$\mathcal{S}(e, f)$ contains exactly one element.

**Proof.** Let $e = (i, \lambda)$ and $f = (j, \mu)$. Then by Lemma 3.17, $\mathcal{S}(e, f) = S(\lambda, j)$. Let $h, g \in \mathcal{S}(e, f)$, with $h = (k, \nu) = (j \varphi_1', \lambda \theta_2) \in \Delta(\gamma)$ and $g = (k', \nu') = (j \varphi_1', \lambda \theta_2') \in \Delta(\gamma')$ for some $\theta, \varphi$ in $\mathcal{D}$. 
Since \( h \in S(\lambda, j) \), by axiom III of Definition 3.2 and (a) of Theorem 4.18 it follows that there exists

\[ \psi' : i \mapsto \psi \] such that \( \Theta' = \Theta \psi \) and \( \varphi = \varphi \psi \).

Similarly since \( g \in S(\lambda, j) \) there exists \( \psi' : i \mapsto i' \) such that \( \Theta = \Theta' \psi' \) and \( \varphi = \varphi \psi' \). Now

\[ \Theta = \Theta' \psi = \Theta \psi \psi' = \Theta' \psi' \psi' \] and \( \Theta' = \Theta' \psi' \psi \). Hence by (a) of Theorem 4.18, \( \varphi \psi' = 1 \) and \( \varphi' \psi = 1 \). Since \( \Theta \) is strictly skeletal we get \( \psi = \psi' = 1 \). Thus \( \Theta = \Theta' \) and \( \varphi = \varphi' \), so that \( h = g \). Hence \( S(e, f) \) is singleton.

Proof of Theorem 4.19. By Lemma 4.21 and Lemma 4.22 it follows that \( S \) with product defined by Equation (3.10) is a combinatorial pseudo-inverse semigroup (cf. Theorem 3.38). Now it remains to prove that the product defined by Equation (4.7) coincides with the product defined by Equation (3.10). Let \( (i, \lambda), (j, \mu) \in S \). Then by Equation (3.10) \( (i, \lambda)(j, \mu) = (i G(\lambda; k), \mu F(j; \nu)) \) where \( (k, \nu) \in S(\lambda, j) \). From the proof of Lemma 4.21 it follows that \( k = j(\varphi_{\lambda, j})_1 \) and \( \nu = \lambda(\Theta_{\lambda, j})_2 \).

Then \( F(j; \nu) = (\varphi_{\lambda, j})_2 \) and \( G(\lambda; k) = (\Theta_{\lambda, j})_1 \).

Hence the product defined by Equation (3.10) coincides with the product given by Equation (4.7). Conversely if \( S \) is any combinatorial pseudo-inverse semigroup then by
Lemma 4.20 there exists a \( P \)-category \( \mathcal{P}(S) \). It follows from Theorem 3.39 and what is proved above that
\[
S = \mathcal{C}(\mathcal{P}(S)).
\]
This completes the proof. We now give an alternate formulation of Theorem 4.19, in terms of categories \((\mathcal{D} \downarrow \Delta)\), \((\mathcal{D} \downarrow P_1)\) and \((\mathcal{D} \downarrow P_2)\).

**Theorem 4.23.** Let \( \mathcal{P} = \langle \mathcal{D}, P_1, P_2, \Delta \rangle \) be a \( P \)-category.

Let
\[ S = \bigcup_{\lambda \in \mathcal{D}} \left[ H^\lambda, P_1 \right] \times \left[ H^\lambda, P_2 \right] \]
where \( \left[ H^\lambda, P_i \right] \) denotes the set of all natural transformations of \( H^\lambda \) into \( P_i \) for \( i = 1, 2 \).

Define product in \( S \) by
\[
(4.7)' \quad (x, y)(u, v) = (H^\theta x, H^\varphi v)
\]
where \( (H^\varphi, H^\theta) \) is universal for the pair \((u, y)\).
Then \( S \) with this product is a combinatorial pseudo-inverse semigroup. Conversely every combinatorial pseudo-inverse semigroup is isomorphic to one constructed in this way.

**Proof.** Let \( S' \) denote the semigroup \( \mathcal{C}(\mathcal{P}) \) defined in Theorem 4.19. Define \( \psi : S \rightarrow S' \) as follows.
\[(x, y) \psi = ((1_\alpha)x_\alpha, (1_\alpha)y_\alpha)\]

where \(x : H^\alpha \to P_1\) and \(y : H^\alpha \to P_2\). Now it is sufficient to prove that \(\psi\) is a bijection such that if \((x, y), (u, v) \in S\) then

\[
(H^\theta x, H^\varphi v) \psi = (i(\theta_{\lambda j})_1, \mu(\varphi_{\lambda j})_2)
\]

where \((H^\varphi, H^\theta)\) is universal for the pair \((u, y)\), \(i = (1_\alpha)x_\alpha, \lambda = (1_\alpha)y_\alpha, j = (1_\beta)u_\beta, \mu = (1_\beta)v_\beta\) and \(x : H^\alpha \to P_1, y : H^\alpha \to P_2, u : H^\beta \to P_1\) and \(v : H^\beta \to P_2\). By Yoneda Lemma, every natural transformation \(z : H^\alpha \to P_i\) for \(i = 1, 2\), is uniquely determined by an element \(j \in P_i(\alpha)\), such that \((1_\alpha)z_\alpha = j\) and conversely. Hence \(\psi\) defined above is a bijection. In the proof of Theorem 4.18, we have seen that if \(j = (1_\beta)u_\beta\) and \(\lambda = (1_\alpha)y_\alpha\), then \(\varphi = \varphi_{\lambda j}\) and \(\Theta = \Theta_{\lambda j}\) given by (b) of Theorem 4.18 is such that \((H^\varphi, H^\theta)\) is universal for the pair \((u, y)\). Since \(\mathcal{O}\) is strictly skeletal, the universal arrow is uniquely determined. Therefore if \(\Theta : \gamma \to \alpha\) and \(\varphi : \gamma \to \beta\) then

\[
(H^\theta x, H^\varphi v) \psi = ((1_\gamma)(H^\theta x)_\gamma, (1_\gamma)(H^\varphi v)_\gamma).
\]
Now

\[(1 \gamma)(H^\theta x)_\gamma = (\Theta )x_\gamma = (1_\alpha )x_\alpha \Theta_1 = i\Theta_1 = i(\Theta_{\lambda j})_1\]
and

\[(1 \gamma)(H^\varphi v)_\gamma = (\varphi )v_\gamma = (1_\beta )v_\beta \varphi_2 = \mu \varphi_2 = \mu (\varphi_{\lambda j})_2.\]

Therefore

\[(H^\theta x, H^\varphi v)_\Psi = (i(\Theta_{\lambda j})_1, \mu (\varphi_{\lambda j})_2).\]

Hence the theorem follows from Theorem 4.19 \(\Box\)

REMARK. It is clear from the proof that \(S\) defined in this theorem is isomorphic to \(\mathcal{C}(\mathcal{P})\) defined in Theorem 4.19. Hence we will denote this \(S\) also by \(\mathcal{C}(\mathcal{P})\). Further the biordered set of \(S\) will be denoted by \([\mathcal{O}, \Delta]\) by identifying the element \(m \in [\mathcal{O}, \Delta]\) with \(\mathcal{C}(m) \in S\).

Now we proceed to describe the homomorphisms of combinatorial pseudo-inverse semigroups in terms of transformations of the corresponding \(P\)-categories. First of all we derive a result on the morphisms between the comma categories \((\mathcal{O} \downarrow K)\) and \((\mathcal{O} \downarrow K')\).

Note that the class of all pairs \((\mathcal{O}, K)\) where \(\mathcal{O}\) is a small category and \(K : \mathcal{O} \rightarrow \text{Set}\) is a functor, forms a category with morphisms defined as follows. If
(Ω, K) and (Ω', K') are two objects, a morphism from (Ω, K) to (Ω', K') is a pair (T, σ)
where T : Ω → Ω' is a functor and σ : K → TK'
is a natural transformation.

LEMMA 4.24. If (T, σ) : (Ω, K) → (Ω', K') is a
morphism as described above, then define
(T ↓ σ) : (Ω ↓ K) → (Ω' ↓ K') as follows. For objects
m : H^α → K and morphisms H^θ,

(T ↓ σ)(m) = m* : H^T^α → K'
defined by

(1^T^α)m*^T^α = (1^α)m*^σ^α and (T ↓ σ)(H^θ) = H^T(θ)

Then (T ↓ σ) is a functor. Further the assignment
(Ω, K) → (Ω ↓ K) and (T, σ) → (T ↓ σ) is functorial.

Proof. Let m : H^α → K and n : H^β → K be
objects in (Ω ↓ K). Let θ : β → α in Ω be such
that m = H^θ n. Now we show that H^T^θ : m* → n* is a
morphism in (Ω' ↓ K'), that is, m* = H^T^θ n*, where
m* = (T ↓ σ)(m) and n* = (T ↓ σ)(n). By the definition
of m* and n*, we have

(1^T^α)(H^T^θ(n*))^T^α = (T(θ))n*^T^α = (1^T^β)n*^T^β K'(T(θ))

= (1^β)n^σ^β K'(T(θ)) = (1^β)n^K(θ)σ^α = (θ)n^σ^α =
\[(1_{\alpha})(H^\theta n)_\alpha \sigma_\alpha = (1_{\alpha})m_\alpha \sigma_\alpha = (1_{\alpha})m^*T_\alpha \]

Hence \(H^T(\theta) : m^* \rightarrow n^*\) is a morphism in \((\mathcal{D}' \downarrow \mathcal{K}')\).

Since \(T\) is a functor it follows that \((T \downarrow \sigma)\) is a functor from \((\mathcal{D} \downarrow \mathcal{K})\) to \((\mathcal{D}' \downarrow \mathcal{K}')\). The last statement follows by routine verification.

**DEFINITION 4.25.** A morphism of \(\mathcal{P}\)-categories from \(\mathcal{P}\) to \(\mathcal{P}'\) is a functor \((T \downarrow \sigma) : (\mathcal{D} \downarrow \Delta) \rightarrow (\mathcal{D}' \downarrow \Delta')\) where \(T : \mathcal{D} \rightarrow \mathcal{D}'\) is a functor and \(\sigma : \Delta \rightarrow T \Delta'\) is a natural transformation satisfying the following axioms.

(i) \(\sigma\) is a subdirect product of two natural transformations \(\sigma^1 : \mathcal{P}_1 \rightarrow TP'_1\) and \(\sigma^2 : \mathcal{P}_2 \rightarrow TP'_2\).

(ii) The pair \(((T \downarrow \sigma^1) \times (T \downarrow \sigma^2), (T \downarrow \sigma))\) is a transformation of adjunctions from \((\mathcal{D}, \mathcal{D}', \varphi)\) to \((\mathcal{D}', \mathcal{D}', \varphi')\) (cf. Section I.1), where \((\mathcal{D}, \mathcal{D}', \varphi)\) and \((\mathcal{D}', \mathcal{D}', \varphi')\) are the adjunctions corresponding to \(\mathcal{P}\) and \(\mathcal{P}'\) respectively given by axiom II of Definition 4.9.

**THEOREM 4.26.** Let \((T \downarrow \sigma) : (\mathcal{D} \downarrow \Delta) \rightarrow (\mathcal{D}' \downarrow \Delta')\) be a morphism of \(\mathcal{P}\)-categories from \(\mathcal{P}\) to \(\mathcal{P}'\). Define \(\mathcal{C}(T \downarrow \sigma) : S = \mathcal{C}(\mathcal{P}) \rightarrow S' = \mathcal{C}(\mathcal{P}')\) by
(4.8) \((i, \lambda) \circ (T, \sigma) = (i \sigma^1, \lambda \sigma^2)\).

where \((i, \lambda) \in P_1 \times P_2(\alpha)\). Then \(C(T, \sigma)\) is a homomorphism of \(C(\mathcal{P})\) into \(C(\mathcal{P}')\). Conversely if \(h : S \rightarrow S'\) is a homomorphism of combinatorial pseudo-inverse semigroups, then there exists a unique morphism \((T, \sigma) : \mathcal{P}(S) \rightarrow \mathcal{P}(S')\) of \(P\)-categories such that \(C(T, \sigma) = h\).

Proof. To prove that \(C(T, \sigma)\) is a homomorphism, it is sufficient to show that \((T, \sigma)\) satisfies the axioms (a), (b) and (c) of Definition 3.23. Here (a) is the same as axiom (i) above.

To show that (b) holds, it is sufficient to prove that

\[T^2(F(j; \lambda)) = F'(j \sigma^1, \lambda \sigma^2)\]

and

\[T^1(G(\mu ; i)) = G'(\mu \sigma^2, i \sigma^1)\]

where \(T^1, T^2\) are as in Definition 3.23 and \(F, G\) and \(F', G'\) are the cones corresponding to \(\mathcal{P}\) and \(\mathcal{P}'\) respectively (cf. Lemma 4.21). Let \((j \theta_1, \lambda) \in \Delta(\alpha)\), and \(\theta : \beta \rightarrow \alpha\). Then

\[(j \theta_1, \lambda) \sigma_\alpha = (j \theta_1 \sigma^1, \lambda \sigma^2) = (j \sigma^1(\theta \theta), \lambda \sigma^2) \in \Delta(T'\alpha)\]
Similarly
\[ T^2(F(j; \lambda)) = T^2(\theta_2) = (T\theta)_2 = F'(j\sigma_\beta^{-1}; \lambda\sigma_\alpha^{-2}). \]
Thus axiom (b) holds.

Now if \( j \in \Pi_1(\beta) \) and \( \lambda \in \Pi_2(\alpha) \) then there exists \( \theta = \theta_{\lambda j} : \alpha \to \gamma \) and \( \varphi = \varphi_{\lambda j} : \beta \to \gamma \)
such that, \( \{(j \varphi_1, \lambda \theta_2)\} = S(\lambda, j) \), and \( (H\varphi, H\theta) \)
is universal from \( \delta \) to \( (x, y) \) where \( x : H\beta \to \Pi_1 \) and \( y : H\alpha \to \Pi_2 \) are defined by \( (1_\beta)x_\beta = j \) and \( (1_\alpha)y_\alpha = \lambda \) (see the proof of Theorem 4.18 and Lemma 4.22).
Let us write \( T^* = (T \downarrow \sigma) \) and \( T^{**} = (T \downarrow \sigma^1) \times (T \downarrow \sigma^2) \).
Since \( (T^{**}, T^*) \) is a transformation of adjunctions, it follows that \( T^{**}(H\varphi, H\theta) \) is universal from \( \delta' \) to \( T^{**}(x, y) \) (cf. Section I.1). Now
\[ T^{**}(H\varphi, H\theta) = (H^T\varphi, H^T\theta) \] and \( T^{**}(x, y) = (x^*, y^*) \)
such that \( x^* : H^T\beta \to \Pi_1' \) and \( y^* : H^T\alpha \to \Pi_2' \) with
\[ (1_{T\beta})x^*_{T\beta} = (1_\beta)x_\beta \sigma_\beta^{-1} = j \sigma_\beta^{-1} \]
and

\[(1_{\mathcal{A}})^*_{\mathcal{A}} = (1_{\mathcal{A}})^*_{\mathcal{A}} \sigma_\alpha^2 = \lambda \sigma_\alpha^2.\]

Therefore as proved in the case of $S(\lambda, j)$, we see that

\[S(\lambda \sigma_\alpha^2, j \sigma_\beta^1) = \{(j \sigma_\beta^1(\theta \varphi), (\lambda \sigma_\alpha^2)(\theta \psi), z)\} \]

Moreover

\[\begin{align*}
\theta_1 \sigma_\gamma^{-1} &= j \sigma_\beta^1(\theta \varphi), \\
\lambda \theta_2 \sigma_\gamma^{-2} &= \lambda \sigma_\alpha^2(\theta \psi),
\end{align*}\]

that is

\[(j \varphi_1, \lambda \theta_2) \sigma_\gamma \in S(\lambda \sigma_\alpha^2, j \sigma_\beta^1).\]

Hence (c) holds. Now the result follows from Theorem 3.43

3. LOCALLY TESTABLE SEMIGROUPS

Locally testable regular semigroups form an important subclass of the class of pseudo-inverse semigroups. In this section we shall obtain a structure theorem for the class of locally testable semigroups.

DEFINITION 4.27 (Zalcstein, [29]). Let $S$ be a semigroup. $S^+$ denotes the set of all finite sequences of elements of $S$. Let $k$ be a positive integer. Two elements $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n) \in S^+$ with
m, n \geq k, are said to have the same k-test vectors if
\[(x_1, \ldots, x_k) = (y_1, \ldots, y_k),\]
\[(x_{m-k+1}, \ldots, x_m) = (y_{n-k+1}, \ldots, y_n)\]
and
\[\left\{(x_i, \ldots, x_{i+k-1}) : 2 \leq i \leq m-k \right\} = \left\{(y_j, \ldots, y_{j+k-1}) : 2 \leq j \leq n-k \right\}\]
S is said to be k-testable if, \(u = (x_1, \ldots, x_m)\) and \(v = (y_1, \ldots, y_n) \in S^+\), \(m, n \geq k\) and if \(u\) and \(v\) have the same k-test vectors, then \(x_1x_2\cdots x_m = y_1y_2\cdots y_n\).
S is said to be locally testable if S is k-testable for some \(k > 0\).

Note that if S is k-testable then S is m-testable for any \(m \geq k\).

The class of locally testable semigroups was introduced as a simultaneous generalization of normal bands and nilpotent semigroups. The concept of a pseudo-semilattice is another generalization of the concept of normal band. The following proposition makes explicit the relation between the two concepts.

PROPOSITION 4.28 ([29], Theorem 4). Let S be a regular periodic semigroup. Then the following are equivalent.
Statement (2) shows that every regular locally testable periodic semigroup is pseudo-inverse. Thus the biordered set of idempotents of such a semigroup is a pseudo-semilattice. The next proposition shows that the condition that $S$ is periodic may be omitted.

**PROPOSITION 4.29** (cf. [22]). Let $S$ be a regular semigroup. Then the following are equivalent.

1. $S$ is locally testable.
2. $S$ is combinatorial and satisfies the following condition: If $D$ and $D'$ are two $\mathcal{D}$-classes of $S$, then for $x \in D$ there exists at most one $y \in D'$ such that $y \leq x$ in the natural partial order on $S$ (cf. Definition 1.19).
3. For every $e \in E(S)$, $eSe$ is a semilattice

Now we can characterise the biordered set of a locally testable semigroup. Let $E$ be a biordered set and let $\delta_0 = (R U L)^*$ be the transitive closure of $R U L$.
PROPOSITION 4.30 (cf. [22]). Let \( E \) be a biordered set and \( \mathcal{S}_0 = (\mathcal{R} \cup \mathcal{L})^\dagger \). Then \( \mathcal{S}_0 \) is the biordered set of a locally testable semigroup if and only if for every \( e \in \mathcal{S}_0 \), \( \delta \cap \omega(e) \) contains at most one element for every \( \delta \in \mathcal{S}_0 / \mathcal{S}_0 \).

Now taking \( \mathcal{D} \) to be a partial order, Theorem 4.23 gives a structure theorem for locally testable semigroups as follows.

THEOREM 4.31. Let \( \mathcal{D} \) be a small category which is a partial order and \( P_1, P_2, \Delta : \mathcal{D} \rightarrow \text{Set} \) be functors as in Definition 4.9; such that

(L) The functor \( \delta : (\mathcal{D} \downarrow \Delta) \rightarrow (\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2) \) has a right adjoint \( \delta^\dagger \). Then

\[
S = \bigcup_{\alpha \in \mathcal{D}} [H^\alpha, P_1] \times [H^\alpha, P_2]
\]

with product defined by (4.7)' is a locally testable regular semigroup. Conversely every locally testable regular semigroup can be constructed in this way.

Proof. Since \( \mathcal{D} \) is a partial order axiom I of Definition 4.9 holds, and axiom II is the same as (L) above. Therefore \( \mathcal{P} = \langle \mathcal{D}, P_1, P_2, \Delta \rangle \) is a \( P \)-category and so by Theorem 4.23, \( S \) is a combinatorial pseudo-inverse.
semigroup with $E(S) = \mathcal{S}([D, \Delta])$ and whose $\mathcal{D}$-classes are $[H^\alpha, P_1] \times [H^\alpha, P_2]$. Therefore to prove that $S$ is locally testable, it suffices, by Proposition 4.29, to prove that for each $(x, y) \in [H^\alpha, P_1] \times [H^\alpha, P_2]$ there exists at most one $(x', y') \in [H^\beta, P_1] \times [H^\beta, P_2]$ such that $(x', y') \leq (x, y)$ in the natural partial order on $S$. Suppose $(x', y'), (x'', y'') \in [H^\beta, P_1] \times [H^\beta, P_2]$ be such that $(x', y') \leq (x, y)$ and $(x'', y'') \leq (x, y)$ in $S$. Let $f = \mathcal{S}(m) \in E(R(x, y))$. Since $(x', y') \leq (x, y)$, by Proposition 1.20, there exists $e \in \mathcal{S}(n) \in E(R(x', y'))$ such that $(x', y') = \mathcal{S}(n)(x, y)$ and $e \in f$. Similarly, since $(x'', y'') \leq (x, y)$ there exists $e' \in \mathcal{S}(n') \in E(R(x'', y''))$ such that $(x'', y'') = \mathcal{S}(n')(x, y)$ and $e' \in f$. Now by the definition of $\leq$ in $[D, \Delta]$ there exist $\Theta, \varphi$ such that $n = H^\Theta m$ and $n' = H^\varphi m$. Further $\Theta, \varphi : \alpha \rightarrow \beta$ where $n, n' : H^\beta \rightarrow \Delta$ and $m : H^\alpha \rightarrow \Delta$. Since $\mathcal{D}$ is a preorder, this will imply that $\Theta = \varphi$. Thus $n = n'$ and so $(x', y') = (x'', y'')$. Hence $S$ is locally testable.
Conversely let $S$ be a locally testable regular semigroup. Then, by Proposition 4.29, $S$ is combinatorial and pseudo-inverse. Therefore by Theorem 4.23 $S$ is isomorphic to $\mathcal{C}(\mathcal{P})$ for a $\mathcal{P}$-category $\mathcal{P}$. Let $\mathcal{P} = \langle \emptyset, P_1, P_2, \Delta \rangle$. Since $S$ is locally testable by Proposition 4.29 (2) we get that $\emptyset$ is a partial order. Hence the theorem.

4. NORMAL BANDS

A band $B$ satisfying the permutation identity $efgh = egfh$ for every $e, f, g, h \in B$ is called a normal band. Equivalently the identity may be given as $efge = egfe$ for every $e, f, g \in B$. Zalcstein [29] has shown that a semigroup $S$ is a normal band if and only if it is a 1-testable semigroup. The following proposition shows the relation between normal bands and locally testable semigroups.

PROPOSITION 4.22 ([29], Theorem 5). Let $S$ be a semigroup. Then the following are equivalent.

1. $S$ is a normal band
2. $S$ is a band and for all $x \in S$, $xSx$ is a semilattice.
3. $S$ is a subdirect product of semigroups $S_i$ where each $S_i$ is a rectangular band or a rectangular band with zero.
(4) $S$ is 1-testable

It now follows from Proposition 4.29, that normal bands are precisely those bands which are locally testable. Therefore Theorem 4.31 may be modified as follows to yield a structure theorem for normal bands. Note that this is equivalent to the well-known structure theorem for normal bands (cf. [10], Proposition 5.14, p. 120).

THEOREM 4.33. Let $\mathcal{D}$ be a semilattice and $P_1, P_2 : \mathcal{D} \to \text{Set}$ be functors which are disjoint and non-empty. Then

$$S = \bigcup_{\alpha \in \mathcal{D}} \left[ H^\alpha, P_1 \right] \times \left[ H^\alpha, P_2 \right]$$

with product defined by Equation (4.7)' is a normal band. Conversely every normal band is isomorphic to one constructed in this way.

**Proof.** In view of Proposition 4.32, it suffices to prove that $S$ is locally testable and that $S$ is a band. Both these will be accomplished if we show that with

$$\Delta = P_1 * P_2, \langle \mathcal{D}, P_1, P_2, \Delta \rangle$$

satisfies axiom II of Definition 4.9. Define

$$\overline{\delta} : (\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2) \to (\mathcal{D} \downarrow P_1 * P_2)$$

as follows.

Let $(x, y) \in (\mathcal{D} \downarrow P_1) \times (\mathcal{D} \downarrow P_2)$ with $x : H^\alpha \to P_1$
and \( y : H^\beta \to P_2 \). Let \( \alpha \beta \) be the coproduct of \( \alpha \) and \( \beta \) in \( \mathfrak{D} \). We denote the morphism from \( \alpha \) to \( \beta \) in \( \mathfrak{D} \) by \( (\alpha, \beta) \). Then

\[
\delta(x, y) = u : H^{\alpha\beta} \to P_1 \ast P_2
\]
such that

\[
(1_{\alpha\beta})u_{\alpha\beta} P_1^{\alpha\beta} = (1_{\alpha})x_{\alpha} P_1(\alpha, \alpha\beta)
\]
and

\[
(1_{\alpha\beta})u_{\alpha\beta} P_2^{\alpha\beta} = (1_{\beta})y_{\beta} P_2(\beta, \alpha\beta)
\]

Since \( \alpha\beta \leq \alpha \) and \( \alpha\beta \leq \beta \) in the semilattice \( \mathfrak{D} \), we get that

\[
(H(\alpha, \alpha\beta), H(\beta, \alpha\beta)) : \delta(u) \to (x, y)
\]
in \((\mathfrak{D} \downarrow P_1) \times (\mathfrak{D} \downarrow P_2)\). Further, \( \mathfrak{D} \) being a partial order this is a universal arrow. Hence by Proposition 1.3, \( \delta \) extends to a functor which is a right adjoint to

Hence the theorem follows from Theorem 4.31 \( \Box \)

5. COMBINATORIAL INVERSE SEMIGROUPS

Inverse semigroups are regular semigroups \( S \) for which \( E(S) \) is a semilattice (cf. [10], Theorem 1.2, p.130). Also in an inverse semigroup every \( \mathcal{R} \) -class and every \( \mathcal{L} \) -class contains exactly one idempotent. Therefore in
Theorem 4.23 if we take \( P_1 = P_2 \) and \( \Delta \) to be the diagonal of \( P_1 \times P_2 \), the resulting semigroup will be a combinatorial inverse semigroup.

Recall that if a choice of the product \( A \times B \) is given for each pair \( A, B \) of objects in a category then the product functor \( \Pi : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) is a right adjoint to the diagonal functor \( \delta \) (cf. [12] p.85). When \( \mathcal{C} \) is strictly skeletal product of every pair of objects, if it exists, is unique and so the diagonal \( \delta \) has a right adjoint which is the product functor. Now we state the structure theorem for combinatorial inverse semigroups.

**THEOREM 4.34.** Let \( \mathcal{D} \) be a strictly skeletal small category and \( P : \mathcal{D} \to \text{Set} \) be a functor which is disjoint and non-empty, such that the following conditions hold.

(i) For each \( \alpha \in \mathcal{D} \) every natural transformation from \( H^\alpha \) to \( P \) is injective.

(ii) Every pair of objects \( (x, y) \in (\mathcal{D} \downarrow P) \times (\mathcal{D} \downarrow P) \) has a product in \( (\mathcal{D} \downarrow P) \). On \( S = \bigcup_{\alpha \in \mathcal{D}} [H^\alpha, P] \times [H^\alpha, P] \) define a binary operation by

\[
(x, y)(u, v) = (H^\beta x, H^\gamma v)
\]

where \( H^\beta : y \times u \to y \) and \( H^\gamma : y \times u \to u \) are the projections in \( (\mathcal{D} \downarrow P) \). Then \( S \), with this binary
operation, is a combinatorial inverse semigroup, whose semilattice of idempotents is isomorphic to \( [\mathcal{D}, P] \). Conversely every combinatorial inverse semigroup may be constructed in this way.

**Proof.** Define \( \Delta : \mathcal{D} \rightarrow \text{Set} \) by setting

\[
\Delta(\alpha) = \{(i, i) : i \in P(\alpha)\}
\]

and for \( \theta : \alpha \rightarrow \beta \) in \( \mathcal{D} \), \( \Delta(\theta) = P(\theta) \times P(\theta) \mid \Delta(\alpha) \). Then \( \Delta \) is a subdirect product of \( P \) and \( P \). Now by Yoneda Lemma, every element of \( (\mathcal{D} \downarrow P) \) uniquely determines, and is determined by, an element of \( (\mathcal{D} \downarrow \Delta) \). Hence \( (\mathcal{D} \downarrow \Delta) \) may be identified with \( (\mathcal{D} \downarrow P) \) and then the functor

\[
\xi : (\mathcal{D} \downarrow \Delta) \rightarrow (\mathcal{D} \downarrow P) \times (\mathcal{D} \downarrow P)
\]

reduces to the diagonal functor

\[
\xi : (\mathcal{D} \downarrow P) \rightarrow (\mathcal{D} \downarrow P) \times (\mathcal{D} \downarrow P).
\]

Since \( \mathcal{D} \) is strictly skeletal, (i) implies that \( (\mathcal{D} \downarrow P) \) is also strictly skeletal. Hence the product given by condition (ii) uniquely determines a right adjoint to the diagonal functor \( \xi \). Thus \( \langle \mathcal{D}, P, P, \Delta \rangle \) becomes a \( P \)-category and so by Theorem 4.23, \( S \) with product defined by (4.7)' is a combinatorial pseudo-inverse semigroup, whose biordered set is \( [\mathcal{D}, \Delta] = [\mathcal{D}, P] \). Since \( H^\theta \) and \( H^\varphi \) are projections, \( (H^\varphi, H^\theta) \) is universal for the pair \( (u, y) \) as required in Equation (4.7)'. Hence the product
defined by Equation (4.9) coincides with that in (4.7)'.
Now we prove that \([\mathcal{O}, P]\) is a semilattice. Let
\(x, y \in [\mathcal{O}, P]\). By the remark at the end of Theorem 4.23
we identify the element \(x \in [\mathcal{O}, P]\) with the
idempotent \((x, x) \in S\). Thus \((x, x), (y, y) \in S\). Since
\((\mathcal{O} \downarrow P)\) is strictly skeletal the products \(x \times y\) and
\(y \times x\) coincide in \((\mathcal{O} \downarrow P)\). Therefore if
\(H^\Theta : x \times y = y \times x \rightarrow x\) and \(H^\varphi : x \times y = y \times x \rightarrow y\) are
the projections then \(H^\Theta x = H^\varphi y = x \times y = y \times x\) and so
\((x, x)(y, y) = (H^\Theta x, H^\varphi y) = (x \times y, x \times y) = (H^\varphi y, H^\Theta x)\)
\((y, y)(x, x)\). Thus the set of idempotents \([\mathcal{O}, P]\) is a
semilattice and so \(S\) is a combinatorial inverse semigroup.

Conversely let \(S\) be an inverse semigroup. Then \(S\)
is clearly pseudo-inverse and so there exists a \(P\)-category
\(\mathcal{C}(\mathcal{P})\) such that \(S\) is isomorphic to
\(\mathcal{C}(\mathcal{P})\) and whose \(\mathcal{O}\) -classes are \(P_1(\alpha) \times P_2(\alpha)\)
(cf. Theorem 4.19). Since \(S\) is an inverse semigroup
\(P_1(\alpha)\) may be identified with \(P_2(\alpha)\) for all \(\alpha \in \mathcal{O}\)
Let \(P = P_1\). Then \(P\) satisfies condition (i). Also each
\(\mathcal{R}\) -class and each \(\mathcal{L}\) -class of \(S\) contains exactly one
idempotent, so that by properly renaming the elements we can get
\(\Delta(\alpha) = \{(i, i) : i \in P(\alpha)\}\). Hence \(\Delta\) may be
identified with \( P \). So by axiom II of Definition 4.9, the diagonal functor \( \delta : (\mathcal{D} \downarrow P) \to (\mathcal{D} \downarrow P) \times (\mathcal{D} \downarrow P) \) has a right adjoint \( \delta^\sim \). Clearly for \( (x, y) \in (\mathcal{D} \downarrow P) \), \( \delta^\sim (x, y) \) is the product of \( x \) and \( y \) in \( (\mathcal{D} \downarrow P) \). Thus condition (ii) holds. This completes the proof \( \square \)