Chapter III

FREDHOLM OPERATORS
ON A LOCALLY CONVEX SPACE

In the final section of the last chapter, we showed that the operators with finite dimensional ranges on a locally convex space $X$ form a regular semigroup. Dually, we can consider operators whose ranges are closed and of finite codimension in $X$ as a possible class of regular operators on a locally convex space. But whereas the finite dimensionality of $R(t)$ forces $t$ to be a topological homomorphism and $N(t)$ to be of finite codimension in $X$, the assumption that $R(t)$ is closed and is of finite codimension in $X$ does not in general make $t$ a topological homomorphism; nor does it imply that $N(t)$ is of finite dimension. So we explicitly assume these conditions also.

The resulting operators are called Fredholm operators. It is to this class of operators that we turn our attention in the rest of this work. Throughout this chapter, $X$ denotes a locally convex space, unless indicated otherwise. Subsequent chapters deal with the same class of operators on Banach spaces and Hilbert spaces.

III.1. Definition and simple properties

We start with the definition of Fredholm operators and go on to prove that the set $F(X)$ of Fredholm operators on $X$ is a regular semigroup. Further, we show that $F(X)$ is a unitary subsemigroup of $CL(X)$ and that $F(X)$ is disjoint with every non-trivial ideal in the algebra $CL(X)$. We also characterize the Green's relations and the natural partial order on $F(X)$ in this section.
Definition 1.1. An operator $t$ on $X$ is said to be a Fredholm operator if
the following conditions are satisfied:

1. $t$ is a topological homomorphism;
2. $R(t)$ is closed;
3. $R(t)$ is of finite codimension in $X$ and $N(t)$ is of finite dimension.

The set of all Fredholm operators on $X$ is denoted by $F(X)$.

Remark 1.2. The above definition can be simplified when $X$ is assumed
to have additional structure. Thus if $X$ is an $F$-space, then as noted in
II.1.5, (1) and (2) of 1.1 are equivalent. Again it is noted in 15.1.7 of [43]
that if $X$ and $Y$ are Banach spaces and $t$ a continuous linear transformation
from $X$ to $Y$ with $R(t)$ of finite codimension in $Y$, then $R(t)$ is closed. The
proof suggested there works equally well for Frechet spaces also. Thus an
operator $t$ on a Frechet space $X$ is a Fredholm operator if and only if
$N(t)$ is of finite dimension and $R(t)$ is of finite codimension in $X$.

As in the case of finite rank operators (cf. II.5.2) it can be seen that
a Fredholm operator $t$ is regular and every inverse of $t$ in $CL(X)$ is also a
Fredholm operator.

Proposition 1.3. If $t \in F(X)$, then $t$ is a regular element of $CL(X)$. Also
if $t'$ is an inverse of $t$ in $CL(X)$, then $t' \in F(X)$.

Proof. Let $t \in F(X)$. Then by 1.1(3), $R(t)$ is a closed subspace of finite
codimension in $X$ and $N(t)$ is of finite dimension, so that $R(t)$ and $N(t)$
have topological complements in the locally convex space $X$ (I.3.14, I.3.23).
This together with 1.1(1) shows that $t$ is regular (II.1.3).
Let $t'$ be an inverse of $t$ in $CL(X)$. Then $R(t')$ is a topological complement of $N(t)$ and $N(t')$ is a topological complement of $R(t)$ (II.1.4). Hence $R(t')$ is closed with $X/R(t) \cong N(t)$ and $N(t') \cong X/R(t)$ (I.3.12) so that $R(t')$ is of finite codimension in $X$ and $N(t')$ is of finite dimension. Also $t'$ is a regular operator on $X$, since $t$ is an inverse of $t'$ in $CL(X)$. Hence by II.1.3, $t'$ is a topological homomorphism. Thus $t' \in F(X)$. 

Unlike the case of finite rank operators, it is not immediately apparent that the product of two Fredholm operators is also a Fredholm operator. This fact is proved in [2] for the case when $X$ is a Banach space (see also [19], [29], [40]). We give below a proof based on II.4.3.

**Theorem 1.4.** $F(X)$ is a regular semigroup.

**Proof.** Let $s, t \in F(X)$. We first show that $st$ is a regular element of $CL(X)$. Since $R(t)$ is of finite codimension in $X$ and $N(s) + R(t) \supseteq R(t)$, it follows that $N(s) + R(t)$ is of finite codimension in $X$. Also, since $R(t)$ is closed and $N(s)$ is of finite dimension in $X$, we have $N(s) + R(t)$ closed (I.3.15). Hence $N(s) + R(t)$ has a topological complement $B$ in $X$ (I.3.14). Let $A = B + N(s)$. Then

$$A + (N(s) + R(t)) = B + (N(s) + R(t)) = X$$

and using the modularity of $L(X)$,

$$A \cap (N(s) + R(t)) = (B + N(s)) \cap (N(s) + R(t))$$

$$= (B \cap (N(s) + R(t))) + N(s)$$

$$= N(s).$$

So $A$ is a complement of $N(s) + R(t)$ in $[N(s), X]$. 

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Again, $N(s) \cap R(t) \subseteq N(s)$ and $N(s)$ is of finite dimension so that $N(s) \cap R(t)$ is a finite dimensional subspace of $R(t)$. Since $R(t)$ is a locally convex space (I.3.19), it follows that $N(s) \cap R(t)$ has a topological complement $A'$ in $R(t)$ (I.3.23). In other words, $A'$ is a complement of $N(s) \cap R(t)$ in $[0, R(t)]$.

It now follows from the first part of II.4.3 that $X = A + A'$. From this we have $\text{codim } A' = \dim A$ ([26], 7.7(1)). Moreover, $B$ is of finite dimension, since $B \cong X/(N(s) + R(t))$ (II.2.12). Hence $A = B + N(s)$ is of finite dimension, since $N(s)$ is of finite dimension. Thus $A'$ is of finite codimension in $X$; also, $A'$ is closed in $R(t)$ (I.3.12) and $R(t)$ is closed in $X$ so that $A'$ is closed in $X$. Together with $X = A + A'$, this gives $X = A \oplus A'$.

By the second part of II.4.3, it now follows that $st$ is regular.

The regularity of $st$ implies that $st$ is a topological homomorphism and that $R(st)$ is closed. Now

$$R(s) = s(X) = s(B + N(s) + R(t)) = s(B) + R(st)$$

and since $B$ is of finite dimension, so is $s(B)$. Hence $R(st)$ is of finite codimension in $R(s)$ and since $R(s)$ itself is of finite codimension in $X$, it follows that $R(st)$ is of finite codimension in $X$. Again, $N(t) \subseteq N(st)$ so that $N(t)$ has an algebraic complement $D$ in $N(st)$. Then

$$t(D) = t(N(t) + D) = t(N(st)) = tt^{-1}(N(s)) \subseteq N(s)$$

and so $t(D)$ is of finite dimension. Also, since $D \cap N(t) = \{0\}$, $t$ is injective on $D$ and hence $\dim D = \dim t(D)$. So

$$\dim N(st) = \dim N(t) + \dim D = \dim N(t) + \dim t(D)$$

which is finite. Thus $st \in F(X)$ and hence $F(X)$ is a semigroup. By 1.3, it is regular.
Henceforth, by an inverse of a Fredholm operator $t$, we shall mean an inverse of $t$ which is also a Fredholm operator. (By 1.3, this only means a continuous inverse).

Recall that $G(X)$ denotes the group of all topological isomorphisms of $X$ onto itself and that $G(X)$ is the group of units of $CL(X)$. Now if $u \in G(X)$, then $R(u) = X$ and $N(u) = \{0\}$, so that $u \in F(X)$. Thus $G(X) \subseteq F(X)$ and it follows that $F(X)$ is a monoid with $G(X)$ as the group of units. However, a left or right unit of $CL(X)$ may not be in $F(X)$. The following result characterizes the left and right units of $F(X)$.

**Proposition 1.5.** $F(X)$ is a monoid with $G(X)$ as the group of units. Also $u \in F(X)$ is a right unit in $F(X)$ if and only if it is a surjection and is a left unit in $F(X)$ if and only if it is an injection.

**Proof.** We have seen that $F(X)$ is a monoid and $G(X)$ is the group of units in $F(X)$. Let $u \in F(X)$. If $u$ is a right unit in $F(X)$ then $u$ is a right unit in $CL(X)$ so that $u$ is a surjection and if $u$ is a left unit in $F(X)$, then $u$ is a left unit in $CL(X)$ so that $u$ is an injection (II.1.9).

To prove the converses, first let $u$ be a surjection in $F(X)$. Then $u$ is a topological homomorphism and $N(u)$ has a topological complement in $X$, so that from II.1.9 it follows that $u$ is a right unit in $CL(X)$. So there exists $u' \in CL(X)$ with $uu' = 1$. Evidently, $u'$ is an inverse of $u$ in $CL(X)$ and hence $u' \in F(X)$ by 1.3. Thus $u$ is a right unit in $F(X)$. Similarly, it can be proved that if $u$ is an injection in $F(X)$ then $u$ is a left unit in $CL(X)$ and hence a left unit in $F(X)$.

We next consider the relation between $F(X)$ and $K(X)$, starting with the following simple relation between their idempotents.
Proposition 1.6. $E(F(X)) = \{1 - e : e \in E(K(X))\}$.

Proof. Let $e \in E(K(X))$. Then $f = 1 - e \in E(CL(X))$ so that $f$ is a topological homomorphism (I.3.13) with $R(f) = N(1 - f) = N(e)$ closed. Also, since $X = R(f) \oplus N(f)$, we have

$$X/R(f) \cong N(f) = R(1 - f) = R(e),$$

so that $X/R(f)$ is of finite dimension and hence $R(f)$ is of finite codimension in $X$. Again, $N(f) = R(1 - f) = R(e)$ is of finite dimension. Hence

$$1 - e = f \in E(F(X)).$$

Conversely, let $f \in E(F(X))$ and $e = 1 - f$. Then $e$ is in $E(CL(X))$ and $R(e) = N(1 - e) = N(f)$ is of finite dimension, so that $e \in E(K(X))$; and $f = 1 - e$. 

We now give a characterization of Fredholm operators in terms of finite rank operators. First we note that since $K(X)$ is an ideal in the algebra $CL(X)$ (II.5.4), the quotient $CL(X)/K(X)$ is also an algebra with multiplication defined by $(s + K(X))(t + K(X)) = st + K(X)$; and $1 + K(X)$ is the unity of this algebra. Now an element $a$ of an algebra $A$ with unity 1 is said to be invertible in $A$ if there exists $a'$ in $A$ with $aa' = a'a = 1$. In this case $a'$ (which is necessarily unique) is called the inverse of $a$ and we write $a' = a^{-1}$.

The following result which equates $F(X)$ with the invertible elements of $CL(X)/K(X)$ was first proved for the case when $X$ is a Banach space in [2] (see also [19], [29] and [40]). The proof given below is modelled on that given for a normed space in [19], 6.4.3.

Theorem 1.7. Let $t \in CL(X)$. Then $t \in F(X)$ if and only if $t + K(X)$ is invertible in the algebra $CL(X)/K(X)$. 
Proof. Let \( t \in F(X) \). Then by 1.3, there exists an inverse \( t' \) of \( t \) in \( F(X) \), so that \( e = t't \) and \( f = tt' \) are idempotents in \( CL(X) \) with \( N(e) = N(t) \) and \( N(f) = N(t') \) (see the proof of II.1.1). Hence \( 1-e \) and \( 1-f \) are idempotents in \( CL(X) \) with \( R(1-e) = N(e) = N(t) \) and \( R(1-f) = N(f) = N(t') \). Thus \( 1-t't = 1-e \) and \( 1-tt' = 1-f \) are in \( K(X) \) so that

\[
(t + K(X))(t' + K(X)) = t't + K(X) = 1 + K(X)
\]

and

\[
(t' + K(X))(t + K(X)) = t't + K(X) = 1 + K(X).
\]

Conversely, let \( t \in CL(X) \) and suppose that \( t + K(X) \) is invertible in \( CL(X)/K(X) \). Let \((t + K(X))^{-1} = s + K(X) \). Then \( ts + K(X) = 1 + K(X) = st + K(X) \) so that \( 1-st \) and \( 1-ts \) are in \( K(X) \). Let \( u = 1-ts \) and \( v = 1-st \). Then \( t - tst = ut \in K(X) \), since \( u \in K(X) \) and \( K(X) \) is an ideal. So by II.5.2, \( t - tst \) is a regular element of \( CL(X) \) and hence it follows from II.5.5 that \( t \) itself is a regular element of \( CL(X) \). Thus \( t \) is a topological homomorphism and \( R(t) \) is closed (II.1.3). Also

\[
N(t) \subseteq N(st) = N(1-v) \subseteq R(v)
\]

so that \( N(t) \) is of finite dimension. Again,

\[
R(t) \supseteq R(ts) = R(1-u) \supseteq N(u)
\]

and \( N(u) \) is of finite codimension in \( X \) (see the proof of II.5.2) so that \( R(t) \) is of finite codimension in \( X \). Hence \( t \in F(X) \). 

Using this characterization of \( F(X) \), it can be easily seen that \( F(X) \) is invariant under finite rank perturbations.
Corollary 1.8. If \( t \in F(X) \) and \( s \in K(X) \), then \( t + s \in F(X) \).

Proof. Let \( t \in F(X) \). Then by 1.7, \( t + K(X) \) is invertible in \( CL(X)/K(X) \). If now \( s \in K(X) \), then \( t + s + K(X) = t + K(X) \) so that \( (t + s) + K(X) \) is invertible in \( CL(X)/K(X) \). Hence by 1.7, \( t + s \in F(X) \).

Remark 1.9. Note that the proof of 1.7 given above does not make use of 1.4. An alternate proof of 1.4 can be based on 1.7, for if \( s, t \in CL(X) \) and \( s + K(X) \) and \( t + K(X) \) are both invertible in \( CL(X)/K(X) \) then so is \( st + K(X) = (s + K(X))(t + K(X)) \).

In connexion with 1.7, we introduce the following terminology.

Definition 1.10. An operator \( s \) on \( X \) is said to be a \( K \)-regularizer of an operator \( t \) on \( X \) if \( 1 - st \) and \( 1 - ts \) are finite rank operators.

The following result is mostly a reformulation of 1.7.

Proposition 1.11. Let \( t \in CL(X) \). Then \( t \in F(X) \) if and only if \( t \) has a \( K \)-regularizer. In particular, if \( t \in F(X) \) then any inverse of \( t \) is a \( K \)-regularizer of \( t \). Also if \( t \in F(X) \) and \( s \) is a \( K \)-regularizer of \( t \), then \( s \in F(X) \).

Proof. Let \( t \in F(X) \) and \( t' \) an inverse of \( t \). Then \( tt' \) and \( t't \) are in \( E(F(X)) \) and hence by 1.6, \( 1 - tt' \) and \( 1 - t't \) are in \( K(X) \). Thus \( t' \) is a \( K \)-regularizer of \( t \).

Conversely, let \( t \in CL(X) \) and let \( s \) be a \( K \)-regularizer of \( t \). Then \( (t + K(X))(s + K(X)) = ts + K(X) = 1 + K(X) \), since \( 1 - ts \in K(X) \). Hence \( t + K(X) \) is invertible in \( CL(X)/K(X) \) and so \( t \in F(X) \), by 1.7.

If \( t \in F(X) \) and \( s \) is a \( K \)-regularizer of \( t \), then \( t \) is a \( K \)-regularizer of \( s \) and hence \( s \in F(X) \).
Now if \( t \in F(X) \) and \( s_1, s_2 \) are \( K \)-regularizers of \( t \), then as in the above proof \( s_1 + K(X) = (t + K(X)^{-1}) = s_2 + K(X) \). This gives the following

**Corollary 1.12.** Let \( t \in F(X) \). If \( s_1 \) and \( s_2 \) are \( K \)-regularizers of \( t \), then \( s_1 - s_2 \in K(X) \). In particular, if \( t' \) and \( t'' \) are inverses of \( t \), then \( t' - t'' \in K(X) \).

To describe another property of the semigroup \( F(X) \), we require the following definition.

**Definition 1.13.** A subsemigroup \( T \) of a semigroup \( S \) is said to be unitary if whenever \( x \in S \) and \( y \in T \) with \( xy \in T \), we have \( x \in T \) and whenever \( x \in S \) and \( y \in T \) with \( yx \in T \), we have \( x \in T \).

The following result is a reformulation of Theorem 3.4 in Chapter V of [40].

**Proposition 1.14.** \( F(X) \) is a unitary subsemigroup of \( CL(X) \).

*Proof.* Let \( s \in CL(X) \), \( t \in F(X) \) and \( st \in F(X) \). Let \( t' \) be an inverse of \( t \). Then by 1.11, \( 1 - tt' \in K(X) \). Now \( s = s(1 - tt') + (st)t' \) and \( (st)t' \in F(X) \), since \( st \) and \( t' \) are in \( F(X) \); also \( s(1 - tt') \in K(X) \), since \( 1 - tt' \in K(X) \) and \( K(X) \) is an ideal. Hence by 1.8, \( s \in F(X) \). Similarly, it can be proved that if \( s \in CL(X) \), \( t \in F(X) \) and \( ts \in F(X) \), then \( s \in F(X) \).

We have seen in II.5.4 that \( K(X) \) is contained in any non-zero ideal of the algebra \( CL(X) \). Dually, we now show that \( F(X) \) is disjoint with any ideal of the algebra \( CL(X) \), other than \( CL(X) \) itself.
**Proposition 1.15.** If $I$ is an ideal in the algebra $CL(X)$ and $I \neq CL(X)$ then $F(X)$ is disjoint with $I$.

**Proof.** We will show that if $I$ is an ideal in the algebra $CL(X)$ with $F(X) \cap I$ nonempty, then $I = CL(X)$. Let $t \in F(X) \cap I$, where $I$ is an ideal in the algebra $CL(X)$ and let $t'$ be an inverse of $t$. Then by 1.11, $1 - tt' \in K(X)$. Since by II.5.4, $K(X) \subseteq I$, we have $1 - tt' \in I$. Also $tt' \in I$, since $t \in I$ and $I$ is an ideal. Hence $1 = tt' + (1 - tt') \in I$ and so for any $s \in CL(X)$, $s = 1s \in I$. Thus $I = CL(X)$.

The next result characterizes the Green's relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{D}$ on $F(X)$.

**Theorem 1.16.** Let $s, t \in F(X)$. Then in $F(X)$,

1. $e \mathcal{R} t \iff R(s) = R(t)$;
2. $s \mathcal{L} t \iff N(s) = N(t)$;
3. $s \mathcal{D} t \iff R(s) \simeq R(t)$.

**Proof.** Since $F(X)$ is a regular semigroup, (1) and (2) follow from II.3.3. Also if $s \mathcal{D} t$ in $F(X)$ then by II.3.2, $R(s) \simeq R(t)$. Conversely suppose that $R(s) \simeq R(t)$. Let $u_0$ be a topological isomorphism of $R(s)$ onto $R(t)$ and let $u = u_0s$. Then $u$ is a topological homomorphism and $R(u) = u_0(R(s)) = R(t)$ so that $R(u)$ is a closed subspace of finite codimension in $X$. Also $N(u) = N(s)$, since $u_0$ is an injection. Hence $N(u)$ is of finite dimension. Thus $u \in F(X)$. Again since $R(u) = R(t)$ and $N(u) = N(s)$, we have $s \mathcal{L} u \mathcal{R} t$. Hence $s \mathcal{D} t$ in $F(X)$. 

We now consider the natural partial order on the regular semigroup $F(X)$ (I.1.26). In the following, if $s, t$ are operators on $X$ with $R(s + t) = R(s) \oplus R(t)$, then we write $s \oplus t$ for $s + t$. 
Theorem 1.17. Let the natural partial order on the regular semigroup $F(X)$ be denoted by $\leq$. Then for $s,t \in F(X)$, the following are equivalent:

1. $s \leq t$;
2. $N(t) \subseteq N(s)$ and there exists a topological complement $N'$ of $N(s)$ such that $s = t$ on $N'$;
3. there exists $u \in K(X)$ such that $t = s \oplus u$.

Proof. We will prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Assume (1). Then by I.1.27(2), $L_s \leq L_t$ and there exists $e \in E(L_s)$ with $s = te$. Since $L_s \leq L_t$, we have $N(t) \subseteq N(s)$. Again since $e \mathcal{L} s$, we have $N(s) = N(e)$, by II.3.1. Let $N' = R(e)$. Then $N'$ is a topological complement of $N(e) = N(s)$. Also for $x \in N' = R(e)$, $sx = tex = tx$.

Next assume (2) and let $u = t - s$. Then,

$$R(u) = u(X) = (t - s)(N(s) + N')$$

$$= (t - s)(N(s)) + (t - s)(N')$$

$$= t(N(s)) + s(N(s)) = t(N(s))$$

using the fact that $s = t$ on $N'$. Since $N(s)$ is of finite dimension so is $t(N(s))$ and hence $u \in K(X)$. Also

$$R(s) + R(u) = s(X) + t(N(s))$$

$$= s(N' + N(s)) + t(N(s))$$

$$= s(N') + t(N(s))$$

$$= t(N') + t(N(s))$$

$$= t(N') + t(N(s)) = t(X) = R(t),$$

again since $s = t$ on $N'$; and
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\[ y \in R(s) \cap R(u) \Rightarrow y \in t(N') \cap t(N(s)) \]

\[ \Rightarrow tx = y = tx' \text{ for some } x \in N(s) \text{ and } x' \in N' \]

\[ \Rightarrow x - x' \in N(t) \subseteq N(s) \]

\[ \Rightarrow x' = x - (x - x') \in N(s) \]

\[ \Rightarrow x' \in N(s) \cap N' = \{0\} \]

\[ \Rightarrow y = tx' = 0 \]

so that \( R(t) = R(s) \oplus R(u) \). Since \( R(s) \) is closed and is of finite codimension in \( X \) and hence in \( R(t) \), it follows from I.3.14 that \( R(t) = R(s) \oplus R(u) \).

Finally assume (3). Since \( R(t) \) is closed and is of finite codimension in \( X \), it has a topological complement \( A \) in \( X \). Let \( B = R(u) + A \). Then

\[ X = R(t) + A = (R(s) + R(u)) + A = R(s) + B \]

Also

\[ x \in R(s) \cap B \Rightarrow x \in R(s) \text{ and } x = y + z \text{ for some } y \in R(u), z \in A \]

\[ \Rightarrow z = x - y \in (R(s) + R(u)) \cap A \]

\[ \Rightarrow x - y \in R(t) \cap A = \{0\} \]

\[ \Rightarrow x = y \in R(s) \cap R(u) = \{0\} \]

Hence \( X = R(s) + B \). Since \( R(s) \) is closed and is of finite codimension in \( X \), it follows from I.3.14 that \( X = R(s) \oplus B \). So there exists \( e \in E(CL(X)) \) with \( R(e) = R(s) \) and \( N(e) = B \). Moreover, \( e \in E(F(X)) \) by 1.6, since \( R(1 - e) = N(e) = R(u) + A \) is of finite dimension; and \( e \in E(R_s) \) by 1.16, since \( R(e) = R(s) \). Now \( t = s + u \) by (3) so that \( et = es + eu = s \), since \( R(s) = R(e) \) and \( R(u) \subseteq N(e) \). Finally since \( R(s) \subseteq R(t) \), we have \( R_s \leq R_t \) by II.3.3. Hence \( s \leq t \), using I.1.26. \[ \square \]
In [4], a partially ordered set $P$ is called a **Noetherian poset** if each nonempty subset of $P$ has a maximal element.

**Proposition 1.18.** $F(X)$ with the natural partial order is a Noetherian poset.

**Proof.** Let $A$ be a nonempty subset of $F(X)$. Then $\{\dim N(t) : t \in A\}$ is a nonempty subset of non-negative integers and hence has a minimum. Let

$$m = \min\{\dim N(t) : t \in A\}.$$

Then there exists $t \in A$ with $\dim N(t) = m$. Suppose there exists $s \in A$ with $t \leq s$. Then by 1.17(2), $N(s) \subseteq N(t)$ and hence $\dim N(s) \leq \dim N(t) = m$. So by definition of $m$, $\dim N(s) = m$. Since $N(s) \subseteq N(t)$ and $\dim N(s) = \dim N(t) = m$, we have $N(s) = N(t)$ ([21], IV.2.13(ii)). Hence $s \preceq t$ by 1.16. Since $s \preceq t$ and $s \preceq t$, it follows that $s = t$ (cf. [32], Corollary 1.3(b)). Thus $t$ is a maximal element of $A$. \[Q.E.D.\]

The natural partial order on $F(X)$ can be used to give a characterization of the $J$-relation on $F(X)$. We make use of the fact that if $x$ and $y$ are elements of a regular semigroup $S$ with $J_x \leq J_y$, then there exists $u \in S$ with $x \mathcal{D} u \leq y$ ([32], Lemma 2.2).

**Theorem 1.19.** Let $s, t \in F(X)$. Then $J_s \leq J_t$ in $F(X)$ if and only if there exists a closed subspace $A$ of finite codimension in $X$ with $R(s) \cong A \subseteq R(t)$.

**Proof.** Suppose $J_s \leq J_t$ in $F(X)$. Then by the result cited above, there exists $u \in F(X)$ with $s \mathcal{D} u \leq t$. Let $A = R(u)$. Then $A$ is a closed subspace of finite codimension in $X$. Since $s \mathcal{D} u$, we have $R(s) \cong R(u) = A$ by 1.16. Also since $u \leq t$, we have $A = R(u) \subseteq R(t)$ by 1.17(3).
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Conversely, suppose there exists a closed subspace $A$ of finite codimension in $X$ with $R(s) \cong A \subseteq R(t)$. Then there exists a subspace $A'$ of $X$ with $X = A \oplus A'$ and hence there exists $e \in E(\mathcal{CL}(X))$ with $R(e) = A$ and $N(e) = A'$. Since $R(1 - e) = N(e) = A' \cong X/A$, we have $1 - e \in E(K(X))$ and so $e \in E(F(X))$ by 1.5. Also, $R(s) \cong A = R(e)$ so that by 1.16, $sD e$ in $F(X)$. Hence $sJ e$ in $F(X)$, since $D \subseteq J$ in any semigroup (I.1.11). Again $R(e) = A \subseteq R(t)$ so that $R_e \leq R_t$ in $F(X)$ by II.3.3. So $J_e \leq J_t$ by definition of the partial orders on the set of $R$-classes and $J$-classes. Thus $J_s = J_e \leq J_t$.

**Proof.** Let $s \leq t$ in $F(X)$. Then by 1.17, $N(t) \subseteq N(s)$ so that $N(s)$ has finite codimension, and hence $\dim N(s) = \dim N(t) = \dim A + \dim A'$. Moreover, $R(t) = R(e) = R(s)$ and $A = A' \cong X/A$, so $\alpha(t) = \alpha(s)$ and $\beta(t) = \beta(s)$. Thus $sD e$ in $F(X)$, since $D \subseteq J$ in any semigroup (I.1.11). Again $R(e) = A \subseteq R(t)$ so that $R_e \leq R_t$ in $F(X)$ by II.3.3. So $J_e \leq J_t$ by definition of the partial orders on the set of $R$-classes and $J$-classes. Thus $J_s = J_e \leq J_t$. 

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If $t$ is a Fredholm operator, then $\dim N(t)$ and $\text{codim } N(t)$ are non-negative integers. The difference of these two numbers, called the index of $t$, plays an important role in the theory of Fredholm operators. In this section, we see how the index is related to the structure of the semigroup $F(X)$. It is shown that the index is invariant under the natural partial order and this is used to deduce Atkinson’s well known theorem on the additivity of the index. $F(X)$ is proved to be a simple semigroup if and only if there exists a Fredholm operator on $X$ with nonzero index. Using properties of the index, we also show that $F(X)$ is bisimple if and only if all closed subspaces of finite codimension are topologically isomorphic with $X$; and $F(X)$ is completely semisimple and unit regular if and only if no proper closed subspace of finite codimension is topologically isomorphic with $X$.

**Definition 2.1.** Let $t$ be a Fredholm operator. Then the dimension of $N(t)$ is denoted by $\alpha(t)$ and the codimension of $R(t)$ by $\beta(t)$. The index of
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$t$, denoted by $i(t)$ is defined by

$$i(t) = \alpha(t) - \beta(t)$$

Our first result shows that the natural partial order on $F(X)$ (see 1.17) preserves the index. Throughout the sequel, the symbol $\leq$ used between elements of $F(X)$, denotes the natural partial order.

**Proposition 2.2.** If $s, t \in F(X)$ with $s \leq t$, then $i(s) = i(t)$.

**Proof.** Let $s \leq t$ in $F(X)$. Then by 1.17(2), $N(t) \subseteq N(s)$ so that $N(s)$ has a subspace $A$ with $N(s) = N(t) \oplus A$. Since all these spaces are of finite dimension, we have $\dim N(s) = \dim N(t) + \dim A - \dim (N(t) \cap A)$ (see [21], 2.15). Since $N(t) \cap A = 0$, we have

$$\alpha(s) = \alpha(t) + \dim A. \tag{1}$$

Now if $u = u - s = t - u$, then by 1.17(3), $u \in K(X)$ and $R(t) = R(s) \oplus R(u)$. So if $B$ is a topological complement of $R(t)$ in $X$, then $X = R(t) \oplus B = R(s) + (R(u) \oplus B)$, as in the last part of the proof of 1.17. Hence $X/R(s) \cong B \oplus R(u)$, so that $\dim R(s) = \dim (X/R(s)) = \dim R(s) + \dim B$. Also, $B \cong X/R(t)$ so that $\dim B = \dim (X/R(t)) = \dim R(t)$. Thus we have

$$\beta(s) = \beta(t) + \dim R(u). \tag{2}$$

Again, as seen in the second part of the proof of 1.17, we have $R(u) = t(N(s))$ so that $R(u) = t(N(t) + A) = t(A)$. Also since $A \cap N(t) = \{0\}$, it
follows that \( t \) is injective on \( A \) and so \( \dim t(A) = \dim A \). Hence \( \dim R(u) = \dim A \) and so from (2), we get

\[
\beta(s) = \beta(t) + \dim A.
\]

From (1) and (3) we have by the definition of the index,

\[
i(s) = \alpha(s) - \beta(s) = \alpha(t) - \beta(t) = i(t).
\]

This completes the proof. \( \square \)

The next result gives another relation between the index and the natural partial order.

**Proposition 2.3.** Let \( t \in F(X) \). Then,

1. \( i(t) \geq 0 \) if and only if there exists a right unit \( u \in F(X) \) with \( t \leq u \);
2. \( i(t) \leq 0 \) if and only if there exists a left unit \( u \in F(X) \) with \( t \leq u \);
3. \( i(t) = 0 \) if and only if there exists a unit \( U \in F(X) \) with \( t \preceq u \).

**Proof.** Let \( t \in F(X) \) and \( t' \) an inverse of \( t \). Then \( e = t't \) and \( f = tt' \) are in \( E(F(X)) \) with \( R(t) = R(f) \) and \( N(t) = N(e) \), so that

\[
i(t) = \alpha(t) - \beta(t) = \alpha(e) - \beta(f)
\]

using the fact that \( X/R(f) \cong N(f) \) which gives \( \beta(f) = \alpha(f) \).

First suppose \( i(t) \geq 0 \). Then, \( \dim N(e) \geq \dim N(f) \) and so there exist linear maps \( u_0 : N(e) \to N(f) \) and \( u_0' : N(f) \to N(e) \) such that \( u_0u_0' \) is the identity map on \( N(f) \) ([19], 6.2.5). Also, these maps are continuous, since \( N(e) \) and \( N(f) \) are of finite dimension (I.3.16). Define \( u, u' \) by

\[
(*) \quad u = t + u_0(1-e) \quad \text{and} \quad u' = t' + u_0'(1-f).
\]
Then \( u, u' \in \text{CL}(X) \). Also,
\[
R(u_0(1 - e)) = u_0(R(1 - e)) = u_0(N(e)) \subseteq N(f)
\]
and
\[
R(u'_0(1 - f)) = u'_0(R(1 - f)) = u'_0(N(f)) \subseteq N(e)
\]
so that \( u_0(1 - e) \) and \( u'_0(1 - f) \) are in \( K(X) \). Since \( t \) and \( t' \) are in \( F(X) \), it follows from 1.8 that \( u \) and \( u' \) are in \( F(X) \). Moreover,
\[
uu' = tt' + tu_0(1 - f) + u_0(1 - e)t' + u_0(1 - e)u'_0(1 - f)
\]
\[
= tt' + u_0(1 - e)t' + u_0u'_0(1 - f)
\]
\[
= tt' + u_0u'_0(1 - f)
\]
\[
= f + (1 - f) = 1
\]
since \( R(u') \subseteq N(e) = N(t) \) so that \( tu'_0 = 0 \); and \( R(t') = R(e) = N(1 - e) \) so that \( (1 - e)t' = 0 \). Thus \( u \) is a right unit in \( F(X) \). Now since \( u_0u'_0 \) is the identity on \( N(f) \), we have
\[
R(u_0) \subseteq N(f) = R(u_0u'_0) \subseteq R(u_0)
\]
so that
\[
N(f) = R(u_0) = u_0(N(e)) = u_0(R(1 - e)) = R(u_0(1 - e)).
\]
Also \( R(u) \supseteq R(uu') = X \). Hence
\[
R(u) = X = R(f) \oplus N(f) = R(t) \oplus R(u_0(1 - e)).
\]
From 1.17(3), it now follows that \( t \leq u \). Conversely, if \( t \leq u \), where \( u \) is a right unit in \( F(X) \), then \( i(t) = i(u) \), by 2.2; also, \( R(u) = X \) by 1.5, so that \( \beta(u) = 0 \) and hence \( i(u) = \alpha(u) \geq 0 \).
Next, let $i(t) \leq 0$. Then $\dim N(f) \geq \dim N(e)$, so that there exist continuous linear maps $u'_0 : N(f) \to N(e)$ and $u_0 : N(e) \to N(f)$ such that $u'_0 u_0$ is the identity map on $N(e)$. Define $u, u'$ by (*) as before. Then $u, u' \in F(X)$ and

$$u'u = t't + t'u_0(1 - e) + u'_0(1 - f)t + u'_0(1 - f)u_0(1 - e)$$

$$= t't + u'_0(1 - f)t + u'_0u_0(1 - e)$$

$$= t't + u'_0u_0(1 - e)$$

$$= e + (1 - e) = 1$$

since $R(u_0) \subset N(f) = N(t')$ so that $t'u_0 = 0$; and $R(t) = R(f) = N(1 - f)$, so that $(1 - f)t = 0$. Hence $u$ is a left unit in $F(X)$. Further $N(u) \subset N(u'u) = \{0\} \subset N(t)$; and $R(e)$ is a topological complement of $N(e) = N(t)$ such that for each $x \in R(e) = N(1 - e)$, $ux = tx + u_0(1 - e)x = tx$. So $t \leq u$ by 1.17(2). Conversely if $t \leq u$ with $u$ a left unit in $F(X)$, then $i(t) = i(u)$ by 2.2; also $N(u) = \{0\}$ by 1.5, so that $\alpha(u) = 0$ and hence $i(u) = -\beta(u) \leq 0$.

Finally, if $i(t) = 0$, then $\dim N(e) = \dim N(f)$, so that there exist linear maps $u_0 : N(e) \to N(f)$ and $u'_0 : N(f) \to N(e)$ such that $u_0u'_0$ is the identity map on $N(f)$ and $N(e)$ ([19], 6.2.5). If now $u, u'$ are defined by (*), then $u, u' \in F(X)$. Also in this case, the computations in the first two paragraphs give $uu' = u'u = 1$ and $t \leq u$. Conversely, if $t \leq u$ where $u$ is a unit in $F(X)$ then $i(t) = i(u)$ by 2.2, and since $R(u) = X$ and $N(u) = \{0\}$ we have $\alpha(u) = \beta(u) = 0$. This proves (3).

An important property of the index is that $i(st) = i(s) + i(t)$ for $s, t$ in $F(X)$. This was first proved in the case when $X$ is a Banach space in [2]. Alternate proofs can be found in [19], [29] and [40]. These (especially
the one given in [40]) can be easily extended to the case of a locally convex
space. However, in order to highlight the relation between the index and
the structure of the semigroup $F(X)$, we provide a proof of this result using
2.2 and other semigroup theoretic properties of $F(X)$.

**Theorem 2.4.** Let $s, t \in F(X)$. Then $i(st) = i(s) + i(t)$.

**Proof.** Since $F(X)$ is a regular semigroup, there exist $u, v \in F(X)$ with
$u \leq s, v \leq t$ and $st = u * v$ (I.1.28). So by 2.2,

\begin{equation}
(1) \quad i(s) = i(u) \quad \text{and} \quad i(t) = i(v).
\end{equation}

Also since $st = uv$, we have

\begin{equation}
(2) \quad i(st) = i(uv).
\end{equation}

Now by definition of the trace product, $uv \in R_u \cap L_v$, so that by 1.16,
$R(uv) = R(u)$ and $N(uv) = N(v)$. Hence $\alpha(uv) = \alpha(v)$ and $\beta(uv) = \beta(u)$.
This gives by definition of the index,

\begin{equation}
(3) \quad i(uv) = \alpha(v) - \beta(u).
\end{equation}

Again since $uv = u * v$, we have $X = N(u) \oplus R(v)$ by II.3.4, so that
$N(u) \cong X/R(v)$ and hence $\dim N(u) = \text{codim} R(v)$. In other words,

\begin{equation}
(4) \quad \alpha(u) = \beta(v).
\end{equation}

Thus we have,

\[ i(s) + i(t) = i(u) + i(v) \]
\[ = \alpha(u) - \beta(u) + \alpha(v) - \beta(v) \]
\[ = \alpha(v) - \beta(u) \]
\[ = i(uv) = i(st) \]

using (1), (2), (3) and (4). This concludes the proof. \qed
From this theorem, we can deduce the following simple facts (which can also be proved directly using the definition of the index and I.3.12, II.1.4).

**Corollary 2.5.** If $e \in E(F(X))$, then $i(e) = 0$. Also if $t \in F(X)$ and $t'$ an inverse of $t$, then $i(t') = -i(t)$.

*Proof.* Let $e \in E(F(X))$. Then $e^2 = e$, so that $i(e) = i(e^2) = i(e) + i(e)$ by 2.4, so that $i(e) = 0$. Now let $t \in F(X)$ and $t'$ an inverse of $t$. Then $tt' \in E(F(X))$ so that $i(t) + i(t') = i(tt') = 0$. Hence $i(t') = -i(t)$.

Theorem 2.4 shows that $i$ is a homomorphism from the semigroup $F(X)$ to the semigroup $\mathbb{Z}$ of integers under addition. So, the range $i(F(X))$ of $i$ is a subsemigroup of $\mathbb{Z}$. Also if $n \in i(F(X))$, then there exists $t \in F(X)$ with $i(t) = n$, so that for any inverse $t'$ of $t$ we have $i(t') = -i(t) = -n$, by 2.5 and hence $-n \in i(F(X))$. Thus $i(F(X))$ is a subgroup of the additive group $\mathbb{Z}$. Now any subgroup of $\mathbb{Z}$ is of the form $m\mathbb{Z}$ for some non-negative integer $m$ (cf. [21], I.3.1). This gives the following

**Proposition 2.6.** $i(F(X)) = m\mathbb{Z}$ for some non-negative integer $m$.

Thus to every locally convex space $X$, there corresponds a non-negative integer $m$ given by $i(F(X)) = m\mathbb{Z}$. An alternate description of this number can be given in terms of the following

**Definition 2.7.** Let $X$ be a locally convex space. We define a non-negative integer $m(X)$ as follows: If there exists no proper, closed subspace $Y$ of finite codimension in $X$ such that $Y \cong X$, then $m(X) = 0$. If there exists a proper, closed subspace $Y$ of finite codimension in $X$ with $Y \cong X$, then $m(X)$ is the minimum of the codimension of such spaces.
Remark 2.8. If $X$ is (a locally convex space) of finite dimension, then evidently $m(X) = 0$. It seems unknown whether there exists a locally convex space of infinite dimension with $m(X) \neq 1$ (cf. Problem 1 in the list of unsolved problems given in [27]). Most of the classical Banach spaces do have $m(X) = 1$. For example, it is proved in [7] that if $X = l^p, p \geq 1$ and $Y$ a complemented subspace of infinite dimension, then $Y \cong X$. Also, it is shown that if $X = L^p[0,1]$ with $1 < p < \infty$ or if $X = C[0,1]$ and $X = Y \oplus Z$ then either $Y \cong X$ or $Z \cong X$. It follows that $m(X) = 1$ for all these spaces. Again, we will see in Chapter V that $m(X) = 1$ for any Hilbert space of infinite dimension.

To show that the integers occurring in 2.6 and 2.7 are equal, we first prove the following result.

**Proposition 2.9.** Let $n$ be a positive integer. Then the following are equivalent:

1. there exists a closed subspace $Y$ of $X$ with codim $Y = n$ and $Y \cong X$;
2. there exists $t \in F(X)$ with $i(t) = n$;
3. there exists $t \in F(X)$ with $i(t) = -n$.

**Proof.** We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Assume (1) and let $u$ be topological isomorphism of $X$ onto $Y$. Then $u \in F(X)$, since $R(u) = Y$ is closed and is of finite codimension in $X$ and $N(u) = \{0\}$. Also $i(u) = -\beta(u) = -\text{codim } R(u) = -\text{codim } Y = -n$. Let $t$ be an inverse of $u$. Then by 2.5, $i(t) = -i(u) = n$. Thus we have (2). If (2) holds and $t'$ is an inverse of $t$, then $i(t') = -i(t) = n$, which gives (3).

Finally, assume (3). Then $i(t) = -n \leq 0$ so that by 2.3, there exists a left unit $u \in F(X)$ with $t \leq u$. Let $Y = R(u)$. Then $Y$ is a closed subspace
of finite codimension in $X$. Also $u$, being a left unit, is an injection by 1.5; and so is a topological isomorphism of $X$ onto $Y$. Again since $u$ is an injection, we have $\alpha(u) = 0$ and so $\text{codim} Y = \beta(u) = -i(u) = -i(t) = n$. Thus we have (1).

**Proposition 2.10.** Let $m$ be a non-negative integer. Then $m(X) = m$ if and only if $i(F(X)) = m\mathbb{Z}$.

**Proof.** By definition of $m(X)$, we have $m(X) = 0$ if and only if the only subspace of finite codimension in $X$ which is topologically isomorphic to $X$ is itself. The latter condition is equivalent to $i(F(X)) = 0$ by 2.9, since the only subspace of $X$ with codimension 0 is itself. Thus $m(X) = 0$ if and only if $i(F(X)) = 0$. Again by definition of $m(X)$ and 2.9, $m(X) = m > 0$ if and only if $i(F(X)) \neq 0$ and $m$ is the least positive integer in $i(F(X))$. This latter condition is equivalent to $i(F(X)) = m\mathbb{Z}$ with $m > 0$, since $i(F(X))$ is a subgroup of $\mathbb{Z}$ ([21], I.3.1).

From 2.9 and 2.10, it can be easily seen that a closed subspace $Y$ of finite codimension in $X$ with $\text{codim} Y = n$ is topologically isomorphic with $X$ if and only if $n \in m\mathbb{Z}$, where $m = m(X)$. In fact, we can prove a general result which gives necessary and sufficient conditions for any two closed subspaces of finite codimension in $X$ to be topologically isomorphic to each other. This also generalizes II.5.9.

**Proposition 2.11.** Let $m(X) = m$ and let $A$, $B$ be closed subspaces of finite codimension in $X$ with $\text{codim} A - \text{codim} B = n$. Then $A \cong B$ if and only if $n \in m\mathbb{Z}$. In particular, a proper closed subspace $Y$ of finite codimension in $X$ is topologically isomorphic with $X$ if and only if $\text{codim} Y$ is a multiple of $m$. 

From 1.10(1) and 1.10(2), we now have $N(i) = N(c)$ and...
Proof. First suppose $n \geq 0$. Let $n \in m\mathbb{Z}$. Then by 2.10 we have $n$ in $i(F(X))$ and so by 2.9, there exists a closed subspace $Y$ of $X$ with $\text{codim} Y = n$ and a topological isomorphism $u$ of $Y$ onto $X$. Let $\text{codim} B = k$, so that $\text{codim} A = n + k$. Now $k = \text{codim} B \leq \dim X$ (cf. [26], 7.7(2)) and since $X \cong Y$, we have $\dim X = \dim Y$ ([26], 7.5(1)). So $k \leq \dim Y$ and hence $Y$ has a subspace $C$ of dimension $k$. Since $Y$ is locally convex, there exists a subspace $C'$ of $Y$ with $Y = C \oplus C'$. Also $Y$ being a closed subspace of finite codimension in $X$, there exists a subspace $Y'$ of $X$ with $X = Y \oplus Y'$. Thus $X = (C' \oplus C) \oplus Y' = C' \oplus (C \oplus Y')$, as in the proof of the last part of 1.17, since $C'$ is closed and is of finite codimension in $Y$ and hence in $X$. It follows that $\text{codim} C' = \dim C + \text{codim} Y$ (see the proof of equation (2) in 2.2). So $\text{codim} C' = k + n = \text{codim} A$ and hence $C' \cong A$ by II.5.9. Since $u$ is an isomorphism, we also have $C' \cong u(C')$. Thus $A \cong u(C')$. Now $X = u(Y) = u(C + C') = u(C) + u(C')$; and $u(C) \cap u(C') = u(C \cap C') = \{0\}$, since $u$ is an injection and $C \cap C' = \{0\}$.

Also if $e$ is the idempotent operator on $Y$ with $R(e) = C$ and $N(e) = C'$, then $f = ueu^{-1}$ is easily seen to be an idempotent operator on $X$ with $R(f) = ueu^{-1}(X) = u(e(Y)) = u(C)$ and $N(f) = uei^{-1}u^{-1}(0) = u^{-1}(0) = u(N(e)) = u(C')$. Thus $X = u(C) \oplus u(C')$ and hence $X/u(C') \cong u(C)$ (I.3.12(4)). So $\text{codim} u(C') = \dim u(C) = \dim C$, since $u$ is an isomorphism. Again, $\dim C = k = \text{codim} B$, so that $\text{codim} u(C') = \text{codim} B$ and hence $u(C') \cong B$, by II.5.9. Thus $A \cong u(C') \cong B$.

Conversely, let $A \cong B$. Since $A$ and $B$ are closed subspaces of finite codimension in $X$, there exist $e, f \in E(F(X))$ with $R(e) = A$ and $R(f) = B$. Then it follows from 1.16(3) that $eDf$ in $F(X)$. So, there exists $t \in F(X)$ with $eL(t)Rf$. From 1.16(1) and 1.16(2), we now have $N(t) = N(e)$ and
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$R(t) = R(f)$ so that $i(t) = \alpha(t) - \beta(t) = \alpha(e) - \beta(f)$. Also $i(e) = 0$ by 2.5, from which we have $\alpha(e) = \beta(e)$. Thus $i(t) = \beta(e) - \beta(f) = \text{codim } A - \text{codim } B = n$. So $n \in i(F(X)) = m\mathbb{Z}$ by 2.10.

Now if $n < 0$, then $\text{codim } B - \text{codim } A = -n > 0$. Also, it is easily seen that $-n \in m\mathbb{Z}$ if and only if $n \in m\mathbb{Z}$. So we again have $A \cong B$ if and only if $n \in m\mathbb{Z}$.

The second statement of the theorem follows from the first by taking $A = Y$ and $B = X$.

Corollary 2.12. Let $m(X) = m$ and let $p, q$ be non-negative integers less than or equal to $\dim X$. Then there exists $t \in F(X)$ with $\alpha(t) = p$ and $\beta(t) = q$ if and only if $p - q \in m\mathbb{Z}$.

Proof. If there exists $t \in F(X)$ with $\alpha(t) = p$ and $\beta(t) = q$ then $p - q = i(t) \in i(F(X)) = m\mathbb{Z}$, by 2.10. Conversely suppose $p - q \in m\mathbb{Z}$. Since $p, q \leq \dim X$, there exist subspaces $A, B$ of $X$ with $\dim A = p$ and $\dim B = q$. Then $A$ and $B$ have topological complements $A'$ and $B'$ in $X$. Since $A \cong X/A'$ and $B \cong X/B'$, we have $\text{codim } A' = \dim A = p$ and $\text{codim } B' = \dim B = q$. Hence $\text{codim } A' - \text{codim } B' = p - q \in m\mathbb{Z}$ and so $A' \cong B'$, by 2.11. Also there exist $e, f \in \mathcal{CL}(X)$ with $R(e) = A'$, $N(e) = A$ and $R(f) = B'$, $N(f) = B$. Since $A' \cong B'$, we have $e\mathcal{D}f$ and so there exists $t \in F(X)$ with $e\mathcal{L}t \mathcal{R}f$. By 1.16, $N(t) = N(e)$ and $R(t) = R(f)$, so that $\alpha(t) = \alpha(e) = p$ and $\beta(t) = \beta(f) = q$.

Using 2.10 and 2.11, we have the following description of the $\mathcal{D}$ relation in $F(X)$.

Proposition 2.13. Let $m(X) = m$. Then for $s, t \in F(X)$, the following are equivalent:
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(1) $sDt$;
(2) $\alpha(s) - \alpha(t) \in mZ$;
(3) $\beta(s) - \beta(t) \in mZ$.

Proof. By 1.16, $sDt$ if and only if $R(s) \cong R(t)$ and by 2.11, $R(s) \cong R(t)$ if and only if $\beta(s) - \beta(t) \in mZ$. Thus (1) and (3) are equivalent. Again, $i(s)$ and $i(t)$ are in $mZ$ by 2.10, so that

$$(\alpha(s) - \alpha(t)) - (\beta(s) - \beta(t)) = i(s) - i(t) \in mZ.$$  

From this, it follows that (2) and (3) are equivalent.

We can now give a complete description of the $D$-class structure of $F(X)$.

Theorem 2.14. Let $m(X) = m$. If $m = 0$, then there exists a bijection between the set of all non-negative integers $n \leq \dim X$ and the set of $D$-classes of $F(X)$ defined by

$$D_n = \{ t \in F(X) : \alpha(t) = \beta(t) = n \}.$$  

If $m \neq 0$, then there exists a bijection between the set of all non-negative integers $n \leq m - 1$ and the set of all $D$-classes of $F(X)$ defined by

$$D_n = \{ t \in F(X) : \alpha(t) \equiv \beta(t) \equiv n \mod m \}.$$  

Proof. Let $m = 0$. Then by 2.10, $i(F(X)) = 0$ so that $i(t) = 0$ and hence $\alpha(t) = \beta(t)$ for all $t \in F(X)$. Let $n$ be a non-negative integer with $n \leq \dim X$ and $D_n = \{ t \in F(X) : \alpha(t) = \beta(t) = n \}$. Since $n \leq \dim X$, there exists a subspace $Y$ of $X$ with $\dim Y = n$. Also since $X$ is locally convex, $Y$
has a topological complement $Y'$ in $X$. Hence there exists $e \in E(CL(X))$ with $R(e) = Y$ and $N(e) = Y'$. Evidently $e \in E(F(X))$ with $\beta(e) = \alpha(e) = n$. Let $t \in F(X)$. Then by 2.13, $t \mathcal{D} e$ if and only if $\alpha(t) = \alpha(e) = n$ and $\beta(t) = \beta(e) = n$, since $m = 0$. Thus $D_n$ is the $\mathcal{D}$-class of $e$. Conversely, let $D$ be a $\mathcal{D}$-class of $F(X)$ and $s \in D$. Then $\alpha(s) = \beta(s)$ is equal to a non-negative integer $n \leq \dim X$. Let $t \in F(X)$. Then $t \in D$ if and only if $t \mathcal{D} s$ and again by 2.13 and the fact that $m = 0$, $t \mathcal{D} s$ if and only if $\alpha(t) = \alpha(s) = n$ and $\beta(t) = \beta(s) = n$. Hence $D = \{ t \in F(X) : \alpha(t) = \beta(t) = n \}$.

Suppose that $m > 0$. Then by 2.10, $i(F(X)) = m\mathbb{Z}$ so that $\alpha(t) - \beta(t) = i(t) \in m\mathbb{Z}$ and hence $\alpha(t) \equiv \beta(t) \mod m$ for all $t \in F(X)$. Let $n$ be an integer with $0 \leq n \leq m - 1$ and $D_n = \{ t \in F(X) : \alpha(t) \equiv \beta(t) \equiv n \mod m \}$. Now since $m \neq 0$, it follows that $X$ is of infinite dimension and hence has a subspace $Y$ of dimension $n$. So, as in the first paragraph, there exists $e \in E(F(X))$ with $\beta(e) = \alpha(e) = n$. Let $t \in F(X)$. Then by 2.13, $t \mathcal{D} e$ if and only if $\alpha(t) - \alpha(e)$ and $\beta(t) - \beta(e)$ are in $m\mathbb{Z}$; and this is clearly equivalent to $\alpha(t) \equiv \beta(t) \equiv n \mod m$. Thus $D_n$ is the $\mathcal{D}$-class of $e$. Conversely let $D$ be a $\mathcal{D}$-class of $F(X)$ and $s \in D$. Then $\alpha(s) \equiv \beta(s) \mod m$ and so there exists a non-negative integer $n \leq m - 1$ with $\alpha(s) \equiv \beta(s) \equiv n \mod m$. Let $t \in F(X)$.

By 2.13, $t \mathcal{D} s$ if and only if $\alpha(t) \equiv \alpha(s) \mod m$ and $\beta(t) \equiv \beta(s) \mod m$; and this is equivalent to $\alpha(t) \equiv n \mod m$ and $\beta(t) \equiv n \mod m$. Hence $D = \{ t \in F(X) : \alpha(t) \equiv \beta(t) \equiv n \mod m \}$.

We now characterize each of the conditions $m(X) = 0$, $m(X) = 1$ and $m(X) > 1$ in terms of the properties of the semigroup $F(X)$.

First, recall that a semigroup $S$ is simple if and only if it has no ideals other than itself (I.1.33). It is not difficult to see that this is equivalent to the condition that $x \mathcal{J} y$ for all $x, y \in S$ ([8], [20]).
Proposition 2.15. \( F(X) \) is simple if and only if \( m(X) \neq 0 \).

Proof. Let \( m(X) = m > 0 \) and let \( s, t \in F(X) \). Let \( \beta(s) = k \) and \( \beta(t) = l \). Then there exist \( q, r \in \mathbb{Z} \) with \( 0 \leq r < m \) such that \( k - l = mq + r \). Let \( R(s) = A \) and \( R(t) = B \). Since \( m(X) \neq 0 \), it follows that \( X \) is of infinite dimension. Also \( B \) is of finite codimension in \( X \) and \( \dim X = \dim B + \text{codim} B \) ([26], 7.7(2)) so that \( B \) is of infinite dimension. Hence \( B \) has a subspace \( C \) of dimension \( r \); and since \( B \) is locally convex, \( C \) has a topological complement \( C' \) in \( B \). Then \( \text{codim} C' = \text{codim} B + \dim C \) (cf. the proof of equation (2) in 2.2). So \( \text{codim} A - \text{codim} C' = k - (l + r) = mq \in m\mathbb{Z} \). Hence by 2.11, \( A \cong C' \). Thus \( R(s) = A \cong C' \subseteq B = R(t) \) and \( C' \) is a closed subspace of finite codimension in \( X \), so that \( J_s \leq J_t \) by 1.19. Again, if \( r_1 = m - r \), then \( r_1 \geq 0 \) so that as in the case of \( B \) above, \( A \) has subspaces \( D \) and \( D' \) with \( \dim D = r_1 \), and \( A = D \oplus D' \). Then \( \text{codim} D' = \text{codim} A + \dim D = k + r_1 \) so that \( \text{codim} D' - \text{codim} B = (k + r_1) - l = m(q+1) \in m\mathbb{Z} \). So \( B \cong D' \) by 2.11. Thus \( R(t) = B \cong D' \subseteq A = R(s) \), so that \( J_t \leq J_s \) by 1.19. Hence \( J_s = J_t \) and so \( s \notin t \). This proves \( F(X) \) is simple.

Conversely, let \( F(X) \) be simple. Let \( a \in X, a \neq 0 \) and let \( A \) be the one dimensional subspace of \( X \) generated by \( a \). Then \( A \) has a topological complement \( A' \) in \( X \) and so there exists \( e \in E(F(X)) \) with \( R(e) = A' \) and \( N(e) = A \). Since \( F(X) \) is simple, we have \( e \notin 1 \) so that by 1.19, \( A' = R(e) \) contains a closed subspace \( Y \) of finite codimension in \( X \) with \( Y \cong R(1) = X \). Also since \( N(e) = A \neq \{0\} \), we have \( e 
eq 1 \) so that \( R(e) \) and hence \( Y \) is a proper subspace of \( X \). So by definition, \( m(X) \neq 0 \).
Theorem 2.16. The following statements are equivalent:

(1) there exists no proper, closed subspace \( Y \) of finite codimension in \( X \) with \( Y \cong X \);
(2) \( i(t) = 0 \) for each \( t \in F(X) \);
(3) \( F(X) \) is completely semisimple;
(4) \( F(X) \) is unit regular;
(5) \( F(X) \) is not simple.

Proof. Since (1) is equivalent to \( m(X) = 0 \), we have (1) equivalent to (2) by 2.10 and (1) is equivalent to (5) by 2.15. Again, by 2.3(3), condition (2) above holds if and only if to each \( t \in F(X) \) there exists a unit in \( F(X) \) with \( t \leq u \); and this latter condition is equivalent to (4) ([41], 2.1). Thus (1), (2), (4) and (5) are equivalent.

We now show that (1) is equivalent to (3). Assume (1) and let \( e, f \) be in \( E(F(X)) \) with \( eDf \) and \( ewf \). Since \( m(X) = 0 \) by (1) and \( eDf \), we have \( \dim N(e) = \alpha(e) = \alpha(f) = \dim N(f) \) by 2.13. Also from \( ewf \) we have \( N(f) \subseteq N(e) \) by II.2.1. Hence we have \( N(e) = N(f) \) ([21], IV.2.13 (ii)) so that \( eLf \) by 1.16. From \( ewf \) and \( eLf \), it follows that \( e = f \) (cf. [32], 1.3(b)). Since \( F(X) \) is regular, this gives (3) (I.1.36). Next assume (3) and let \( Y \) be a proper, closed subspace of finite codimension in \( X \). Then there exists \( e \in E(F(X)) \) with \( R(e) = Y \). Since \( Y \) is a proper subspace, we have \( e \neq 1 \); also \( e1 = 1e = e \) so that \( ew1 \) (I.1.23). Hence it follows from (3) that \( e \) is not \( D \) related to 1. So by 1.16, \( Y = R(e) \) is not topologically isomorphic with \( R(1) = X \). Since \( Y \) is arbitrary, we have (1).

We next consider the case when \( m(X) = 1 \). Recall that a semigroup \( S \) is said to be bisimple if any two elements of \( S \) are \( D \) related (I.1.37).
Theorem 2.17. The following statements are equivalent:

1. there exists a closed subspace Y of X with \( \text{codim} \ Y = 1 \) and \( Y \cong X \);
2. \( Y \cong X \) for every closed subspace of finite codimension in X;
3. for every \( n \in \mathbb{Z} \), there exists \( t \in F(X) \) with \( i(t) = n \);
4. for every pair of non-negative integers \( p, q \) there exists \( t \in F(X) \) with \( \alpha(t) = p \) and \( \beta(t) = q \);
5. \( F(X) \) is bisimple.

Proof. We will prove that (1) \( \Rightarrow \) (4) \( \Rightarrow \) (3) \( \Rightarrow \) (5) \( \Rightarrow \) (2) \( \Rightarrow \) (1). Assume (1). Then by definition \( m(X) = 1 \). Hence X is of infinite dimension so that if \( p, q \) are any two non-negative integers, then \( p, q \leq \text{dim} \ X \). Also \( p - q \in \mathbb{Z} = m(X)\mathbb{Z} \). So by 2.12, there exists \( t \in F(X) \) with \( \alpha(t) = p \) and \( \beta(t) = q \). Thus we have (4). Next assume (4). If \( n \in \mathbb{Z} \), then we can choose non-negative integers \( p \) and \( q \) with \( p - q = n \). By (4) there exists \( t \in F(X) \) with \( \alpha(t) = p \) and \( \beta(t) = q \) so that \( i(t) = p - q = n \), giving (3). Now if (3) holds, then \( i(F(X)) = \mathbb{Z} \) so that by 2.10, \( m(X) = 1 \); and so for any \( s, t \in F(X) \), \( \alpha(s) - \alpha(t) \in \mathbb{Z} = m(X)\mathbb{Z} \), which gives \( s \Delta t \) by 2.13. Thus \( F(X) \) is bisimple (1.1.37). Suppose that (5) holds and let \( Y \) be a closed subspace of finite codimension in X. Then there exists \( e \in E(F(X)) \) with \( R(e) = Y \) and by (5), we have \( e \Delta 1 \). So \( Y = R(e) \cong R(1) = X \), by 1.16. This gives (2). That (2) \( \Rightarrow \) (1) is obvious.

Finally, we take up the case \( m(X) > 1 \).

Theorem 2.18. The following statements are equivalent:

1. \( \text{codim} \ Y > 1 \) for every proper, closed subspace \( Y \) of finite codimension in X with \( Y \cong X \);
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(2) $i(F(X))$ is a proper subgroup of $\mathbb{Z}$;

(3) $F(X)$ is simple, but not bisimple.

Proof. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). If (1) holds, then $m = m(X) > 1$ so that by 2.10, $i(F(X)) = m\mathbb{Z}$ with $m > 1$ and hence $i(F(X))$ is a subgroup of $\mathbb{Z}$ which is not equal to 0 or $\mathbb{Z}$. This gives (2). Next assume (2). Then $i(F(X)) \neq 0$ so that $F(X)$ is simple by the equivalence of (2) and (5) in 2.16 and $i(F(X)) \neq \mathbb{Z}$ so that $F(X)$ is not bisimple by the equivalence of (3) and (5) in 2.17. Hence we have (3). Finally if (3) holds and $m(X) = m$, then $m \neq 0$ by 2.15, and $m \neq 1$ by the equivalence of (1) and (5) in 2.17. Hence $m > 1$ and this gives (3).

Remark 2.19. If $X$ is of finite dimension, then it is easily seen that $F(X) = CL(X) = L(X)$ (see I.3.16 and I.3.17). By 2.16, this semigroup is unit regular and completely semisimple, but not simple. If $X$ is any of the Banach spaces $l^p$ with $p \geq 1$, $L^p[0,1]$ with $1 < p < \infty$ or $C[0,1]$, then by 2.8 and 2.17, $F(X)$ is bisimple. We will see in chapter V that $F(X)$ is bisimple for any Hilbert space $X$ of infinite dimension.

III.3. Group congruences on $F(X)$

One important class of congruences on a semigroup consists of those for which the quotients are groups. In this section, we characterize the smallest such congruence on $F(X)$. This gives an alternate proof of the well known fact that the index is invariant under finite rank perturbations. We also show that, as in the case of congruences on a group (cf. I.1.7), all group congruences on $F(X)$ can be described in terms of what we call normal subsemigroups of $F(X)$.
Recall that if $\rho$ is a congruence on a semigroup $S$ (I.1.6), then $S/\rho$ denotes the semigroup of all $\rho$-classes under the operation defined by

$$\rho(x)\rho(y) = \rho(xy)$$

for $x, y \in S$.

**Definition 3.1.** A congruence $\rho$ on a semigroup $S$ is said to be a group congruence if $S/\rho$ is a group.

A group congruence on a regular semigroup can be given a simple characterization. First note that if $S$ is a regular semigroup with a unique idempotent, then $S$ is a group; for if $e$ is the idempotent in $S$ and $x \in S$, then for any inverse $x'$ of $x$, we have $xx' = x'x = e$, so that $xe = ex = x$.

Now we have the following result (cf. [20], V.3)

**Proposition 3.2.** Let $S$ be a regular semigroup and let $\rho$ be a congruence on $S$. Then $\rho$ is a group congruence on $S$ if and only if $(e, f) \in \rho$ for all $e, f \in E(S)$.

**Proof.** Let $\rho$ be a group congruence. Then there exists $a \in S$ such that $\rho(a)$ is the identity of the group $S/\rho$. Let $e \in E(S)$. Then $\rho(e)\rho(e) = \rho(e^2) = \rho(e) = \rho(e)\rho(a)$, so that $\rho(e) = \rho(a)$, since $S/\rho$ is a group. So for all $e, f \in E(S)$, $\rho(e) = \rho(a) = \rho(f)$ and hence $(e, f) \in \rho$.

Conversely, let $\rho$ be a congruence on $S$ such that $(e, f) \in \rho$ for all $e, f \in E(S)$. Let $A$ denote the $\rho$-class containing all the idempotents in $S$. Then $A$ is clearly an idempotent element of $S/\rho$. Moreover, $S/\rho$ is easily seen to be regular (see [20], II.4). So if $B$ is an idempotent in $S/\rho$, then there exists $e \in E(S)$ with $B = \rho(e)$ ([20], II.4.6). Hence $e \in A \cap B$ and from
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this it follows that $B = A$. Thus $S/\rho$ is a regular semigroup with a unique idempotent and so is a group.

From the above result, it follows that the intersection of all group congruences on a regular semigroup $S$ is again a group congruence on $S$. (Note that on $S$, there exists at least one group congruence, namely $S \times S$.) This is the minimum group congruence in $S$, in the sense that it is contained in every group congruence on $S$. We next characterize this congruence on $F(X)$.

**Theorem 3.3.** The minimum group congruence on $F(X)$ is given by

$$
\sigma = \{(s,t) \in F(X) \times F(X) : s - t \in K(X)\}.
$$

Also $F(X)/\sigma$ is equal to the group of invertible elements of $CL(X)/K(X)$.

**Proof.** Using the fact that $K(X)$ is an ideal in the algebra $CL(X)$, it is easily seen that $\sigma$ is a congruence on $F(X)$. Also, if $e, f \in E(F(X))$, then $e - f = (1 - f) - (1 - e) \in K(X)$ by 1.6 and II.5.4. Hence by 3.2, $\sigma$ is a group congruence.

We next show that $\sigma$ is the smallest group congruence which contains the natural partial order on $F(X)$. Let $\rho$ denote the latter. If $s, t \in F(X)$ with $s \leq t$, then by 1.17(3), $s - t \in K(X)$ and so $(s, t) \in \sigma$. It follows that $\rho \subseteq \sigma$.

Conversely, let $(t_1, t_2) \in \sigma$. We will show that there exists $s \in F(X)$ with $s \leq t_1$ and $s \leq t_2$. Since $F(X)$ is regular there exist $e_1, e_2 \in E(F(X))$ with $t_1 \mathcal{R} e_1$ and $t_2 \mathcal{R} e_2$ (I.1.17). Then $R(e_1) \cap R(e_2)$ is closed, since $R(e_1)$ and $R(e_2)$ are closed; and $R(e_1) \cap R(e_2)$ is of finite codimension in $X$ since $R(e_1)$ and $R(e_2)$ are such spaces (cf. [26], 7.7(6)). So, there exists $e$ in
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$E(F(X))$ with $R(e) = R(e_1) \cap R(e_2)$. Again, if $u = t_1 - t_2$, then $R(u)$ is of finite dimension and hence there exists $f \in E(F(X))$ with $N(f) = R(u)$. Since $ef$ is regular, there exists $h \in S_1(f, e)$. Then $h \omega f$, so that $N(h) \supseteq N(f) = R(u)$ and hence $hu = 0$. So $ht_1 = h(t_2 + u) = ht_2$. Let $s = ht_1 = ht_2$. Now $h \omega e$, since $h \in S_1(e, f)$; and $ew e_1$, since $R(e) \subseteq R(e_1)$ (II.2.1). Consequently, we have $h \omega e_1$ from which it easily follows that $he_1 \in E(F(X))$ with $h \mathcal{R} he_1$; also since $e_1 \mathcal{R} t_1$, we have $he_1 \mathcal{R} ht_1 = s$. So $h \in E(R_s)$. Moreover, $R(s) = R(h) \subseteq R(e_1) = R(t_1)$, so that $R_s \subseteq R_t_1$, by II.3.3. It now follows from I.1.26 that $s \leq t_1$. Similarly $s \leq t_2$. Hence $(s, t_1)$ and $(s, t_2)$ are in $\mathcal{R}$ and since $\mathcal{R}$ is symmetric and transitive, we have $(t_1, t_2) \in \rho$. Thus $\sigma \subseteq \rho$ also.

The above characterization of $\sigma$, in the light of Theorem 4.4 of [32], shows that $\sigma$ is the smallest congruence on $F(X)$ such that $F(X)/\sigma$ is completely simple (see I.1.34). If $\sigma'$ is a group congruence on $F(X)$, then in particular $F(X)/\sigma'$ is a completely simple semigroup and so $\sigma \subseteq \sigma'$. Thus $\sigma$ is the minimum group congruence on $F(X)$.

Let $t \in F(X)$. If $s \in \sigma(t)$, then $s - t \in K(X)$ and so $s \in t + K(X)$. Conversely, if $s \in t + K(X)$, then $s \in F(X)$ by 1.8; and $s - t \in K(X)$, so that $s \in \sigma(t)$. Hence $\sigma(t) = t + K(X)$ for each $t \in F(X)$. From 1.7, it now follows that $F(X)/\sigma$ is equal to the group of invertible elements in $CL(X)/K(X)$.

By I.1.7, there is a one to one correspondence between congruences on a group $G$ and normal subgroups of $G$. To see how far this can be extended to regular semigroups (especially $F(X))$, we first make the following

**Definition 3.4.** Let $S$ be a regular semigroup. A subset $N$ of $S$ is said to
be a normal subsemigroup of $S$ if $N$ is a regular subsemigroup of $S$ such that $x \in N$ implies $yxy' \in N$ for all $y \in S$ and for all inverses $y'$ of $y$.

It can be seen that any group congruence on a regular semigroup is induced by a normal subsemigroup as in 1.1.7(2).

**Proposition 3.5.** Let $\rho$ be a group congruence on a regular semigroup $S$ and $N$ the congruence class containing all the idempotents of $S$. Then $N$ is a normal subsemigroup of $S$ and the following are equivalent for $x, y \in S$:

1. $(x, y) \in \rho$;
2. $xy' \in N$ for all inverses $y'$ of $y$;
3. $y'x \in N$ for all inverses $y'$ of $y$.

**Proof.** Let $e$ be a fixed idempotent in $S$. Then $N = \rho(e)$. Now if $x, y \in N$, then $\rho(x) = \rho(e) = \rho(y)$, so that $\rho(xy) = \rho(e)$ and hence $xy \in N$. Also if $x \in N$ and $x'$ an inverse of $x$, then $xx' \in E(S)$, so that $\rho(xx') = \rho(e)$. Since $\rho(x) = \rho(e)$, we have $\rho(e)\rho(x') = \rho(xx') = \rho(e) = \rho(e)\rho(e)$, from which it follows that $\rho(x') = \rho(e)$, since $S/\rho$ is a group. So $x' \in N$. Thus $N$ is a regular subsemigroup of $S$. Moreover, if $x \in N$, $y \in S$ and $y'$ is an inverse of $y$, then $yy' \in E(S)$ so that

$$\rho(yxy') = \rho(y)\rho(x)\rho(y') = \rho(y)\rho(e)\rho(y') = \rho(y)\rho(y') = \rho(yy') = \rho(e)$$

since $\rho(e)$ is the identity of $S/\rho$. So $yxy' \in N$. Hence $N$ is a normal subsemigroup of $F(X)$.

Let $x, y \in S$. We will show that (1) and (2) are equivalent. Let $(x, y) \in \rho$ and $y'$ an inverse of $y$. Then

$$\rho(xy') = \rho(x)\rho(y') = \rho(y)\rho(y') = \rho(yy') = \rho(e)$$
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so that $xy' \in N$. Conversely suppose (2) holds and let $y'$ be an inverse of $y$. Then $y = yy'y$ so that

$$
\rho(y) = \rho(yy')\rho(y) = \rho(e)\rho(y) = \rho(xy')\rho(y) = \rho(x)\rho(y'y) = \rho(x)\rho(e) = \rho(x)
$$

since $yy', y'y \in E(S)$ and $xy' \in N$ by (2). So $(x, y) \in \rho$. The equivalence of (1) and (3) can be similarly proved.

Note that in the above result we did not prove that $\rho(x) = Nx$ for all $x \in S$. This may not be true in general. Also an arbitrary normal subsemigroup of a regular semigroup may not induce a congruence as in 1.1.7(1). We now show that a certain type of normal subsemigroup of $F(X)$ does induce a group congruence.

Theorem 3.6. Let $N$ be a normal subsemigroup of $F(X)$ with $N + K(X) = N$. Then the relation $\rho_N$ on $F(X)$ defined by

$$
\rho_N = \{ (s, t) \in F(X) \times F(X) : st' \in N \text{ for all inverses } t' \text{ of } t \}
$$

is a group congruence on $F(X)$ with $N = \rho_N(1)$. Also $\rho_N$ is equal to the relation

$$
\{ (s, t) \in F(X) \times F(X) : t's \in N \text{ for all inverses } t' \text{ of } t \}.
$$

Conversely, if $\rho$ is a group congruence on $F(X)$, then $N = \rho(1)$ is a normal subsemigroup of $F(X)$ with $N + K(X) = N$ and $\rho = \rho_N$.

Proof. Let $\pi : CL(X) \to CL(X)/K(X)$ be the canonical projection defined by $\pi(t) = t + K(X)$. Then by 1.7, $F(X) = \pi^{-1}(\mathcal{G})$, where $\mathcal{G}$ is the group of units of $CL(X)/K(X)$. 

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Let $N$ be a normal subsemigroup of $F(X)$ and $N = \pi(N)$. Then $N \subseteq \pi(F(X)) = \pi\pi^{-1}(G) = G$. Also since $\pi|N$ is a homomorphism of the regular semigroup $N$ into the group $G$, it follows that $N$ is a regular subsemigroup of $G$ and hence a subgroup of $G$. Now let $t \in N$, $s \in F(X)$ and $s'$ an inverse of $s$. Then by 1.6, $1 - ss', 1 - s's \in K(X)$ so that $\pi(s)\pi(s') = \pi(1) = \pi(s')\pi(s)$. Hence $\pi(s') = \pi(s)^{-1}$ and so $\pi(s)\pi(t)\pi(s)^{-1} = \pi(sts') \in N$, since $N$ is a normal subsemigroup of $F(X)$. It follows that $N$ is a normal subgroup of $G$.

Let $s, t \in F(X)$. If $(s, t) \in \rho_N$, and $t'$ an inverse of $t$, then $st' \in N$ so that $\pi(s)\pi(t)^{-1} \in N = \pi(N)$, so that $st' \in \pi^{-1}\pi(N)$. Now $\pi^{-1}\pi(N) = N$, for $N \subseteq \pi^{-1}\pi(N)$ and if $u \in \pi^{-1}\pi(N)$, then $\pi(u) = \pi(v)$ for some $v \in N$ and so $u \in \pi(v) = v + K(X) \subseteq N + K(X) = N$. Hence $st' \in \pi^{-1}\pi(N) = N$ so that $(s, t) \in \rho_N$. Thus

\[(*) \quad (s, t) \in \rho_N \iff \pi(s)\pi(t)^{-1} \in N.\]

Now $\{(A, B) \in G \times G : AB^{-1} \in N\}$ is a congruence on $G$ (I.1.7) and $\pi : F(X) \to G$ is a homomorphism, so that it easily follows from the above description of $\rho_N$ that it is a congruence on $F(X)$. Also if $e, f \in E(F(X))$, then $1 - e, 1 - f \in E(K(X))$ by 1.6, so that $\pi(e) = \pi(1) = \pi(f)$. Hence $\pi(e)\pi(f)^{-1} = \pi(1) = \pi(N)$, since $\pi(1)$ is the identity of $G$ and $N$ is a subgroup. So $(e, f) \in \rho_N$ by $(*)$. Thus $\rho_N$ is a group congruence. Again for $s, t \in F(X)$, $\pi(s)\pi(t)^{-1} \in N$ if and only if $\pi(t)^{-1}\pi(s) \in N$, since $N$ is a normal subgroup of $G$ (I.1.7). From this it is easily seen that $(s, t) \in \rho_N$ if and only if $t's \in N$ for every inverse $t'$ of $t$. Finally, since 1 is the only inverse of $1 \in F(X)$, it follows from the definition of $\rho_N$ that $t \in \rho_N(1)$ if and only if $t \in N$. 

Conversely, if $\rho$ is a congruence on $F(X)$ and $N = \rho(1)$, then by 3.5, $N$ is a normal subsemigroup of $F(X)$ and $\rho = \rho_N$. Also if $t \in N$ and $s \in K(X)$, then $(t, t + s) \in \sigma$ by 3.3, so that $(t, t + s) \in \rho$, since $\sigma$ is the minimum group congruence on $F(X)$. So $t + s \in \rho(t) = N$ and hence $N + K(X) \subseteq N$. Again, since $0 \in K(X)$, we have $N \subseteq N + K(X)$. Thus $N + K(X) = N$. 

To extend the above result to normal subsemigroups of $F(X)$ which are not necessarily invariant under $K(X)$, we first note the following

**Proposition 3.7.** If $N$ is a normal subsemigroup of $F(X)$, then so is $N + K(X)$. 

**Proof.** Let $N$ be a normal subsemigroup of $F(X)$ and $M = N + K(X)$. If $s, t \in N$ and $u, v \in K(X)$ then $(s + u)(t + v) = st + (sv + ut + uv)$, where $st \in N$, since $N$ is a semigroup; and $sv + ut + uv \in K(X)$, since $K(X)$ is an ideal in the algebra $CL(X)$. Hence $(s + u)(t + v) \in N + K(X) = M$. It follows that $M$ is a subsemigroup of $F(X)$.

Now let $t \in M$ and let $t'$ be an inverse of $t$ in $F(X)$. Then there exist $s \in N$ and $u \in K(X)$ with $t = s + u$, so that $\pi(t') = \pi(s)$, where $\pi$ is the canonical projection of $CL(X)$ onto $CL(X)/K(X)$. Also since $N$ is regular, $s$ has an inverse $s'$ in $N$. Then $\pi(s') = \pi(s)^{-1}$ and $\pi(t') = \pi(t)^{-1}$, as in the proof of 3.6, so that $\pi(s') = \pi(s)^{-1} = \pi(t)^{-1} = \pi(t')$. Hence $t' \in \pi(s') = s' + K(X) \subseteq N + K(X) = M$. Thus $M$ is regular.

If $t \in M$, $s \in F(X)$ and $s'$ an inverse of $s$, then $t = u + v$ for some $u \in N$ and $v \in K(X)$ so that $sts' = sus' + sus' \in N + K(X) = M$, since $N$ is a normal subsemigroup of $F(X)$ and $K(X)$ is an ideal of $CL(X)$. So $M$ is a normal subsemigroup of $F(X)$. 

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It follows from 3.6 and 3.7 that if $N$ is a normal subsemigroup of $F(X)$, then $\rho_{N+K(X)}$ is a group congruence on $F(X)$ and conversely, any group congruence on $F(X)$ arises in this way. We now give an alternate description of $\rho_{N+K(X)}$.

**Proposition 3.8.** Let $N$ be a normal subsemigroup of $F(X)$ and $M = N + K(X)$. Then for $s, t \in F(X)$, the following are equivalent:

1. $(s, t) \in \rho_M$;
2. there exists $u \in N$ such that $s - ut \in K(X)$;
3. there exists $u \in N$ such that $s - tu \in K(X)$.

**Proof.** We prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Let $(s, t) \in \rho_M$ and $t'$ an inverse of $t$. Then $st' \in M$. Let $\pi$ denote the canonical projection of $CL(X)$ onto $CL(X)/K(X)$. Then $\pi(t') = \pi(t)^{-1}$ as seen in the proof of 3.6, so that

$$\pi(s)\pi(t)^{-1} = \pi(st') \in \pi(M) = \pi(N + K(X)) = \pi(N).$$

Hence there exists $u \in N$ with $\pi(s)\pi(t)^{-1} = \pi(u)$. So $\pi(s) = \pi(ut)$, from which it follows that $s - ut \in K(X)$. Thus we have (1) $\Rightarrow$ (2).

Next assume (2). Then $\pi(s) = \pi(ut) = \pi(u)\pi(t)$ and $\pi(t) = t + K(X)$ is invertible in $CL(X)/K(X)$ by 1.7, so that $\pi(s)\pi(t)^{-1} = \pi(u) \in \pi(N)$. Now $\pi(N)$ is a normal subgroup of $G$ as seen in the proof of 3.6, so that $\pi(s)\pi(t)^{-1} \in \pi(N)$ implies $\pi(t)^{-1}\pi(s) \in \pi(N)$ (I.1.7). So there exists $v \in N$ with $\pi(t)^{-1}\pi(s) = \pi(v)$ and hence $\pi(s) = \pi(t)\pi(v) = \pi(tv)$. It follows that $s - tv \in K(X)$, giving (3).

Finally assume (3) and let $t'$ be an inverse of $t$. By (3) we have $\pi(s) = \pi(tu) = \pi(t)\pi(u)$ so that $\pi(t)^{-1}\pi(s) = \pi(u) \in \pi(N)$. Since $\pi(N)$ is a normal subgroup of $G$, it follows that $\pi(s)\pi(t)^{-1} \in \pi(N)$ so that $\pi(st') = \pi(st)^{-1} \in \pi(N)$.
\[ \pi(s)\pi(t') = \pi(s)\pi(t)^{-1} \in \pi(N) = \pi(N + K(X)) = \pi(M). \]

Also since \( M + K(X) = M \), we have \( \pi^{-1}\pi(M) = M \) as in the proof of 3.6. Thus \( st' \in M \) and so \( (s, t') \in \rho_M \).

Now if for each normal subsemigroup of \( F(X) \) we denote \( \rho_{N+K(X)} \) by \( \sigma_N \), then from 3.6 and 3.8 we have the following

**Theorem 3.9.** Let \( N \) be a normal subsemigroup of \( F(X) \). Then the relation \( \sigma_N \) on \( F(X) \) defined by

\[ \sigma_N = \{(s, t) \in F(X) \times F(X) : s - ut \in K(X) \text{ for some } u \in N\} \]

is a group congruence on \( F(X) \) with \( \sigma_N(t) = Nt + K(X) \) for each \( t \in F(X) \). Also \( \sigma_N \) is equal to the relation

\[ \{(s, t) \in F(X) \times F(X) : s - tu \in K(X) \text{ for some } u \in N\} \]

and \( \sigma_N(t) = tN + K(X) \) for each \( t \in F(X) \). Conversely, if \( \rho \) is a group congruence on \( F(X) \) and \( N = \rho(1) \), then \( \rho = \sigma_N \).

**Proof.** Let \( N \) be a normal subsemigroup of \( F(X) \). Then by 3.7, \( M = N + K(X) \) is also a normal subsemigroup of \( F(X) \) and since \( K(X) \) is an additive subgroup of \( CL(X) \) (II.5.4), we have \( M + K(X) = M \). So by 3.6, \( \rho_M \) is a group congruence on \( F(X) \). Now by 3.8, \( \rho_M \) is equal to \( \sigma_N \) as defined above. Let \( s, t \in F(X) \). If \( s \in \sigma_N(t) \), then \( s - ut \in K(X) \) for some \( u \in N \) so that \( s = ut + (s - ut) \in Nt + K(X) \). Conversely, if \( s \in Nt + K(X) \), then there exist \( u \in N \) and \( v \in K(X) \) such that \( s = ut + v \) and hence \( s \in F(X) \) by 1.8 and \( (s, t) \in \sigma_N \), by definition. Again by 3.8, \( \sigma_N \) is equal to the second relation defined in the theorem and from this, it follows as before that \( \sigma_N(t) = tN + K(X) \) for each \( t \in F(X) \).
Conversely, if $\rho$ is a group congruence on $F(X)$ then by 3.6, $N = \rho(1)$ is a normal subgroup of $F(X)$ with $N + K(X) = N$ and $\rho = \rho_N$. So, $\rho = \rho_N = \rho_{N+K(X)} = \sigma_N$, using 3.8.

If $m(X) = 0$, then the normal subsemigroups of $F(X)$ which are invariant under $K(X)$ can be described in terms of normal subgroups of $G(X)$.

**Proposition 3.10.** Let $m(X) = 0$. Then $N$ is a normal subsemigroup of $F(X)$ with $N + K(X) = N$ if and only if there exists a normal subgroup $H$ of $G(X)$ such that $N = H + K(X)$.

**Proof.** Let $H$ be a normal subgroup of $G(X)$ and $N = H + K(X)$. Since $H$ is in particular a regular subsemigroup of $F(X)$, it follows as in the proof of 3.7 that $N$ is a regular subsemigroup of $F(X)$. Let $t \in N$ and $s \in F(X)$. Since $m(X) = 0$, we have $i(s) = 0$ by 2.16 and so there exists $u \in G(X)$ with $s \leq u$, by 2.3. Hence $s - u \in K(X)$ by 1.17(3). Also, since $t \in N = H + K(X)$, there exists $v \in H$ with $t - v \in K(X)$. Now if $\pi$ is the canonical projection of $CL(X)$ onto $CL(X)/K(X)$, then $\pi(s) = \pi(u)$ and $\pi(t) = \pi(v)$. So for any inverse $s'$ of $s$,

$$ \pi(sts') = \pi(s)\pi(t)\pi(s)^{-1} = \pi(u)\pi(v)\pi(u)^{-1} = \pi(uvu^{-1}) \in \pi(H), $$

since $H$ is a normal subgroup of $G(X)$. Hence $sts^{-1} \in \pi^{-1}\pi(H)$. Also

$$ \pi^{-1}\pi(H) = \pi^{-1}\pi(H + K(X)) = \pi^{-1}\pi(N) = N, $$

as seen in the proof of 3.6. Thus $sts' \in N$ and it follows that $N$ is a normal subsemigroup of $F(X)$. Further $N + K(X) = H + K(X) = N$.

Conversely, let $N$ be a normal subsemigroup of $F(X)$ with $N + K(X) = N$ and let $H = N \cap G(X)$. Since $N$ and $G(X)$ are subsemigroups of $F(X)$,
so is $H$. Let $u \in H$. Then $u \in G(X)$ and so there exists $u^{-1} \in G(X)$. Since $u^{-1}$ is the unique inverse of $u$ in $CL(X)$ and $N$ is regular, we have $u^{-1} \in N$ and hence $u^{-1} \in H$. Thus $H$ is a subgroup of $G(X)$. Let $u \in H$ and $v \in G(X)$. Then $u \in N$ and $v^{-1}$ is an inverse of $v$ in $F(X)$ so that $vuv^{-1} \in N$, since $N$ is a normal subsemigroup of $F(X)$. So $vuv^{-1} \in N \cap G(X) = H$ and hence $H$ is a normal subgroup of $G(X)$. Also since $H \subseteq N$, we have $H + K(X) \subseteq N + K(X) = N$. On the other hand if $t \in N$, then $i(t) = 0$, since $m(X) = 0$; and so there exists $u \in G(X)$ with $u - t \in K(X)$, using 2.3 and 1.17(3). Hence $u = t + (u - t) \in N + K(X) = N$ so that $u \in N \cap G(X) = H$. Now $t = u + (t - u) \in H + K(X)$ and so $N \subseteq H + K(X)$. Thus $N = H + K(X)$.

The above result, together with 3.9 shows that if $m(X) = 0$, then all group congruences on $F(X)$ can be described in terms of normal subgroups of $G(X)$.

**Theorem 3.11.** Let $m(X) = 0$. If $H$ is a normal subgroup of $G(X)$, then the relation $\sigma_H$ defined on $F(X)$ by

$$\sigma_H = \{(s, t) \in F(X) \times F(X) : s - ut \in K(X) \text{ for some } u \in H\}$$

is a group congruence on $F(X)$ with $\sigma_H(t) = Ht + K(X)$ for all $t \in F(X)$. Also $\sigma_H$ is equal to the relation

$$\{(s, t) \in F(X) \times F(X) : s - tu \in K(X) \text{ for some } u \in H\}$$

and $\sigma_H(t) = tH + K(X)$ for each $t \in F(X)$. Conversely if $\rho$ is a group congruence on $F(X)$, then $H = G(X) \cap \rho(1)$ is a normal subgroup of $G(X)$ with $\rho = \sigma_H$. 

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Proof. Let \( H \) be a normal subgroup of \( G(X) \) and \( N = H + K(X) \). Then by 3.10, \( N \) is a normal subsemigroup of \( F(X) \) with \( N + K(X) = N \). Let \( \sigma_H \) be defined as in the theorem, and \( \sigma_N \) as in 3.8. Since \( H \subseteq N \) it easily follows from the definition that \( \sigma_H \subseteq \sigma_N \). Now suppose \((s,t) \in \sigma_N \). Then there exists \( u \in N \) such that \( s - ut \in K(X) \). Again since \( u \in N = H + K(X) \), there exists \( v \in H \) and \( w \in K(X) \) with \( u = v + w \). Hence \( s - vt = s - (u - w)t = (s - ut) + wt \in K(X) \), since \( K(X) \) is an ideal in the algebra \( CL(X) \). This shows that \((s,t) \in \sigma_H \) and so \( \sigma_N \subseteq \sigma_H \). Thus \( \sigma_H = \sigma_N \). By 3.8, \( \sigma_H = \sigma_N \) is a group congruence on \( F(X) \) and

\[
\sigma_H(t) = \sigma_N(t) = N(t) + K(X) = (H + K(X))t + K(X) = Ht + K(X) + K(X) = Ht + K(X)
\]

since \( K(X) \) is an ideal in the algebra \( CL(X) \). Similarly it can be shown that the second relation given in the theorem is equal to the second form of \( \sigma_N \) given in 3.8 and hence is equal to \( \sigma_H \). Also it follows from this form of \( \sigma_H \) that \( \sigma_H(t) = tH + K(X) \) for each \( t \in F(X) \).

Conversely, let \( \rho \) be a group congruence on \( F(X) \) and \( N = \rho(1) \). Then by 3.6, \( N \) is a normal subsemigroup of \( F(X) \) with \( N + K(X) = N \) and so \( H = N \cap G(X) \) is a normal subgroup of \( G(X) \) with \( H + K(X) = N \), as seen in the proof of the second part of 3.10. Also by 3.9, \( \rho = \sigma_N \). Again, since \( N = H + K(X) \), it follows as before that \( \sigma_N = \sigma_H \). Thus \( \rho = \sigma_H \). ◼

Finally, we consider a particular group congruence on \( F(X) \), namely the one induced by the index map.
Proposition 3.12. Let $\tau$ be the relation on $F(X)$ defined by
\[ \tau = \{(s,t) \in F(X) \times F(X) : i(s) = i(t)\}. \]
Then $\tau$ is a group congruence on $F(X)$ and $F(X)/\tau$ is isomorphic with $m\mathbb{Z}$, where $m = m(X)$.

Proof. We have seen in 2.4 that $i : F(X) \to \mathbb{Z}$ is a homomorphism; also $\tau = \ker i$ by definition so that $\tau$ is a congruence on $F(X)$ (1.1.9). Moreover, if $e, f \in E(F(X))$ then $i(e) = 0 = i(f)$ by 2.5. Hence it follows from 3.2 that $\tau$ is a group congruence on $F(X)$. Again $F(X)/\tau$ is isomorphic with $i(F(X))$ (1.1.9); and $i(F(X)) = m\mathbb{Z}$, where $m = m(X)$, by 2.10.

From the above result and 3.3, we can easily deduce the following result, which is well known in the case when $X$ is a Banach space (cf. [2], [19], [29], [40]).

Corollary 3.13. If $t \in F(X)$ and $s \in K(X)$, then $i(t + s) = i(t)$.

Proof. Let $t \in F(X)$ and $s \in K(X)$. Then $t + s \in F(X)$ by 1.8; and $(t, t + s) \in \sigma$, where $\sigma$ is the congruence defined in 3.3. Since $\sigma$ is the minimum group congruence and $\tau$ is a group congruence on $F(X)$, we have $\sigma \subseteq \tau$ and hence $(t, t + s) \in \tau$. Thus $i(t + s) = i(t)$.

Using 3.9, we have the following description of $\tau$.

Proposition 3.14. Let $s, t \in F(X)$. Then the following are equivalent:

1. $i(s) = i(t)$;
2. there exists $u \in G(X)$ such that $s - ut \in K(X)$;
3. there exists $u \in G(X)$ such that $s - tu \in K(X)$.
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Proof. Let $N = \tau(1)$ with $\tau$ defined as in 3.12. Since $\tau$ is a group congruence, we have $\tau = \sigma_N$ where $\sigma_N$ is defined as in 3.9. Now if $t \in N$, then $i(t) = i(1) = 0$ so that by 2.3, there exists $u \in G(X)$ with $t \leq u$. Hence $t - u$ is in $K(X)$ by 1.17(3) and so $t = u + (t - u) \in G(X) + K(X)$. Conversely if $t \in G(X) + K(X)$, then $t = u + s$ for some $u \in G(X)$ and $s \in K(X)$ so that $i(t) = i(u) = 0$ using 3.13; and hence $t \in N$. Thus $N = G(X) + K(X)$. Consequently, $\sigma_N = \sigma_{G(X)}$ as seen in the proof of 3.11. Thus $\tau = \sigma_{G(X)}$, and the result follows from the definition of $\sigma_{G(X)}$. 

III.4. Idempotent separating congruences on $F(X)$

In the last section, we considered congruences on $F(X)$ for which all idempotents belong to the same congruence class. At the other extreme are the congruences for which no two distinct idempotents belong to the same class. In this section we characterize the largest such congruence on $F(X)$. Also we describe all such congruences in terms of subgroups of the multiplicative group of non-zero scalars.

Definition 4.1. A congruence $\rho$ on a semigroup $S$ is said to be an idempotent separating congruence, if for $e, f \in E(S)$, $(e, f) \in \rho$ implies $e = f$.

It easily follows from I.1.4 that any congruence contained in the Green's equivalence $\mathcal{H}$ is idempotent separating. Conversely, it can be shown that if $\rho$ is an idempotent separating congruence on a regular semigroup $S$, then $\rho \subseteq \mathcal{H}$ ([20], II.4.8). Now if $\eta$ is an equivalence on an arbitrary semigroup $S$, then there exists a congruence $\eta^b$ on $S$ such that $\eta^b \subseteq \eta$ and any congruence contained in $\eta$ is contained in $\eta^b$ ([20], I.5.13). It follows that on any regular semigroup $S$, the relation $\mu = \mathcal{H}^b$ is the maximum idempotent separating
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congruence (in the sense that $\mu$ is idempotent separating, and contains all such congruences on $S$). We now characterize this congruence on $F(X)$.

**Theorem 4.2.** The maximum idempotent separating congruence $\mu$ on $F(X)$ is given by

$$\mu = \{(s,t) \in F(X) \times F(X) : s = \lambda t \text{ for some non-zero scalar } \lambda\}.$$ 

**Proof.** It is easily seen that $\mu$ is a congruence on $F(X)$. Also if $e$ and $f$ are in $E(F(X))$ with $(e,f) \in \mu$, then $f = \lambda e$ for a non-zero scalar $\lambda$, so that $\lambda^2 e = (\lambda e)^2 = f^2 = f = \lambda e$. This gives $\lambda = 1$, since $e \neq 0$ and $\lambda \neq 0$. Thus $e = f$ and so $\mu$ is idempotent separating.

Now let $\rho$ be an idempotent separating congruence on $F(X)$. Since $F(X)$ is regular, we have $\rho \subseteq \mathcal{H}$ ([20], II.4.8). Let $(s,t) \in \rho$. Then $s \mathcal{H} t$ and so in particular $s \mathcal{L} t$. Hence $N(s) = N(t)$ by 1.16. We now show that for every $x \in X$ with $sx \neq 0$ there exists a scalar $\lambda_x \neq 0$ with $sx = \lambda_x t x$.

Let $x \in X$ with $sx \neq 0$. Then $tx \neq 0$, since $N(s) = N(t)$. Let $A$ be the one dimensional subspace of $X$ generated by $x$ and let $B = t(A)$. Then there exists $e \in F(X)$ with $N(e) = B$; and $(es, et) \in \rho$, since $\rho$ is a congruence. So $N(es) = N(et)$ as before. Now $N(es) = s^{-1}(N(e)) = s^{-1}(B)$ and $N(et) = t^{-1}(N(e)) = t^{-1}(B)$. Hence $x \in A \subseteq t^{-1}t(A) = t^{-1}(B) = s^{-1}(B)$ and so $sx \in B = t(A)$. Since $A = \{\lambda x : \lambda \in K\}$, it follows that $sx = \lambda_x t x$ for some $\lambda_x \in K$. Also since $sx \neq 0$, we have $\lambda_x \neq 0$.

Next, we show that $\lambda_x$ is independent of $x$. Let $x, y \in X$ with $sx, sy \neq 0$. First suppose that $sx$ and $sy$ are linearly dependent. Then there exists $\alpha \in K$ with $sy = \alpha sx$ so that $y - \alpha x \in N(s) = N(t)$. Hence $ty = \alpha tx$
and so, $\lambda_y ty = sy = \alpha sx = \alpha \lambda_x tx = \lambda_x ty$. Since $ty \neq 0$, this gives $\lambda_x = \lambda_y$. Now let $sx$ and $sy$ be linearly independent. Then so are $tx$ and $ty$, for $\alpha(tx) + \beta(ty) = 0$ implies $\alpha x + \beta y \in N(t) = N(s)$ and hence $\alpha(sx) + \beta(sy) = 0$. Let $z = x - y$. Then $sz = sx - sy \neq 0$, by the independence of $sx$ and $sy$. So

$$\lambda_x tx - \lambda_y ty = sx - sy = sz = \lambda_x tz = \lambda_x (tx - ty).$$

This gives $\lambda_x = \lambda_y$, since $tx$ and $ty$ are linearly independent.

Thus there exists $\lambda \neq 0$ such that $sx = \lambda tx$ for all $x \in X$ with $sx \neq 0$. Since $N(s) = N(t)$, this holds also for those $x \in X$ with $sx = 0$. Hence $s = \lambda t$ and so $(s, t) \in \mu$. This shows that $\rho \subseteq \mu$. 

The following result gives alternate characterizations of $\mu$.

**Proposition 4.3.** For $s, t \in F(X)$, the following are equivalent:

1. $(s, t) \in \mu$;
2. $s(Y) = t(Y)$ for all subspaces of $Y$ of $X$;
3. $s(Y) = t(Y)$ for all closed subspaces $Y$ of $X$.

**Proof.** If (1) holds, then by 4.2, $s = \lambda t$ for some non-zero scalar $\lambda$, and so for each subspace $Y$ of $X$, $s(Y) = \lambda t(Y) = t(Y)$, since $t(Y)$ is a subspace of $X$. This gives (2). Evidently, (2) implies (3).

Finally assume (3). For each $x \in X$, let $A_x = \{\lambda x : \lambda \in K\}$. Then $A_x$ is a closed subspace of $X$ (I.3.15) and so $s(A_x) = t(A_x)$ for each $x \in X$. Let $x \in N(s)$. Then $t(A_x) = s(A_x) = \{0\}$ so that $tx = 0$, and hence $x \in N(t)$. Thus $N(s) \subseteq N(t)$ and similarly, $N(t) \subseteq N(s)$. Now for each $x \in X$ with $sx \neq 0$, we have $sx \in s(A_x) = t(A_x)$, so that there exists $\lambda_x \neq 0$ with
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$sx = \lambda_xtx$. It now follows as in proof of 4.2 that $(s, t) \in \mu$. Thus (3) implies (1).

To prove that $\rho = \rho_G$, first suppose that $s, t \in F(X)$ with $(s, t) \in \rho$.

Then there exists $\lambda \in G$ with $s = \lambda t$. Hence by definition of $G$, $\rho(\lambda1) = \rho(1) = \rho(\lambda t) = \rho(s)$.

If $m(X) \neq 0$ then each idempotent separating congruence on $F(X)$ is determined by a subgroup of the multiplicative group of non-zero scalars.

**Theorem 4.4.** Let $m(X) \neq 0$. If $G$ is a subgroup of the multiplicative group $K'$ of non-zero scalars and $\rho_G$ is defined on $F(X)$ by

$$\rho_G = \{(s, t) \in F(X) \times F(X) : s = \lambda t \text{ for some } \lambda \in G\}$$

then $\rho_G$ is an idempotent separating congruence on $F(X)$. Conversely if $\rho$ is an idempotent separating congruence on $F(X)$, then $G = \{\lambda \in K' : (\lambda 1, 1) \in \rho\}$

is a subgroup of $K'$ and $\rho = \rho_G$.

**Proof.** Let $G$ be a subgroup of $K'$ and $\rho_G$ be defined as above. Then it is easily seen that $\rho_G$ is a congruence on $F(X)$ and that $\rho_G \subseteq J_1$, where $\mu$ is defined as in 4.2. Hence $\rho_G$ is an idempotent separating congruence.

Conversely, let $\rho$ be an idempotent separating congruence on $F(X)$ and $G$ be defined as in the theorem. Since $\rho$ is reflexive, $1 \in G$ and hence $G$ is non-empty. If $\lambda, \eta \in G$, then

$$\rho((\lambda \eta)1) = \rho(\lambda 1)\rho(\eta 1) = \rho(1)\rho(1) = \rho(1)$$

so that $\lambda \eta \in G$. Again, for $\lambda \in G$

$$\rho(\lambda^{-1}1) = \rho(\lambda^{-1}1)\rho(1) = \rho(\lambda^{-1}1)\rho(\lambda 1) = \rho(1),$$

so $\lambda^{-1}1 \in G$. Hence $G$ is a subgroup of $K'$. Hence $\rho = \rho_G$. Q.E.D.
so that $\lambda^{-1} \in G$. Thus $G$ is a subgroup of $K'$.

To prove that $\rho = \rho_G$, first suppose that $s, t \in F(X)$ with $(s, t) \in \rho_G$. Then there exists $\lambda \in G$ with $s = \lambda t$. Hence by definition of $G$, $\rho(\lambda 1) = \rho(1)$ so that

$$\rho(s) = \rho(\lambda t) = \rho(\lambda 1)\rho(t) = \rho(1)\rho(t) = \rho(t).$$

It follows that $\rho_G \subseteq \rho$. To prove the reverse inclusion, let $(s, t) \in \rho$. Since $\rho$ is an idempotent separating congruence, we have $\rho \subseteq \mu$ and so there exists $\lambda \in K'$ with $s = \lambda t$, by 4.2. Moreover, since $m(X) \neq 0$, $F(X)$ is simple by 2.15 and so $t \not\in 1$. Consequently, there exist $u, v \in F(X)$ with $utv = 1$. Now

$$\lambda 1 = u(\lambda t)v = usv$$

and $\rho(s) = \rho(t)$ so that

$$\rho(\lambda 1) = \rho(u)\rho(s)\rho(v) = \rho(u)\rho(t)\rho(v) = \rho(utv) = \rho(1).$$

It follows that $\lambda \in G$ and hence $(s, t) \in \rho_G$. Thus $\rho \subseteq \rho_G$ also. 

Note that in the first part of the above proof, we did not make use of the hypothesis that $m(X) \neq 0$. So $\rho_G$ is an idempotent separating congruence on $F(X)$, even if $m(X) = 0$. But in this case, not all idempotent separating congruences on $F(X)$ arise in this way. We consider this case below. Here $N$ denotes the set $\{0, 1, 2, \ldots \}$.

**Theorem 4.5.** Let $m(X) = 0$. If $\Gamma = \{G_n : n \in N, n \leq \dim X\}$ is a set of subgroups of $K'$ such that $G_m \subseteq G_n$ whenever $m \leq n$ and if $\rho_\Gamma$ is defined on $F(X)$ by

$$\rho_\Gamma = \{(s, t) \in F(X) \times F(X) : s = \lambda t \text{ for some } \lambda \in G_{\alpha(t)}\}$$

then $\rho_\Gamma$ is an idempotent separating congruence on $F(X)$. Conversely, if $\rho$ is an idempotent separating congruence on $F(X)$ and $G_n$ is defined for
III.4. Idempotent separating congruences on $F(X)$

each $n \in \mathbb{N}$ with $n \leq \dim X$ by

$$G_n = \{ \lambda \in K' : (\lambda e, e) \in \rho \text{ for all } e \in E(F(X)) \text{ with } \alpha(e) = n \}$$

then $G_m \subseteq G_n$ whenever $m \leq n$ and $\rho = \rho_\Gamma$, where $\Gamma$ is the set of all $G_n$.

Proof. Let $\Gamma = \{ G_n : n \in \mathbb{N}, n \leq \dim X \}$ and let $\rho_\Gamma$ be defined as in the theorem. Let $t \in F(X)$. Then $\alpha(t) \in \mathbb{N}$ and $\alpha(t) \leq \dim X$, so that there exists $G_{\alpha(t)} \in \Gamma$; and since $G_{\alpha(t)}$ is a subgroup of $K'$, we have $1 \in G_{\alpha(t)}$. Hence $(t, t) \in \rho_\Gamma$ and it follows that $\rho_\Gamma$ is reflexive. Let $(s, t) \in \rho_\Gamma$. Then there exists $\lambda \in G_{\alpha(t)}$ with $s = \lambda t$, so that $\alpha(s) = \alpha(\lambda t) = \alpha(t)$. So $\lambda^{-1}$ is in $G_{\alpha(t)} = G_{\alpha(s)}$ and $t = \lambda^{-1}s$, from which we have $(t, s) \in \rho_\Gamma$, proving the symmetry of $\rho_\Gamma$. If $(s, t), (t, u) \in \rho_\Gamma$, then there exist $\lambda \in G_{\alpha(t)}, \eta \in G_{\alpha(u)}$ with $s = \lambda t, t = \eta u$, so that $s = \lambda \eta u$; also since $t = \eta u$, we have $\alpha(t) = \alpha(u)$ so that $\lambda \in G_{\alpha(t)} = G_{\alpha(u)}$ and hence $\lambda \eta \in G_{\alpha(u)}$. It follows that $(s, u) \in \rho_\Gamma$ and hence $\rho_\Gamma$ is transitive. Now let $(s, t) \in \rho_\Gamma$ and $u \in F(X)$. Then $s = \lambda t$ with $\lambda \in G_{\alpha(t)}$ so that $us = \lambda(ut)$. Also $N(t) \subseteq N(ut)$ which gives $\alpha(t) \leq \alpha(ut)$ and hence $G_{\alpha(t)} \subseteq G_{\alpha(ut)}$. Consequently, $\lambda \in G_{\alpha(ut)}$ and so $(us, ut) \in \rho_\Gamma$. Again $su = \lambda(tu)$; and $R(tu) \subseteq R(t)$, so that $\beta(t) \leq \beta(tu)$. Now $m(X) = 0$ so that by 2.10 all operators in $F(X)$ have index zero and hence $\alpha(t) = \beta(t) \leq \beta(tu) = \alpha(tu)$. So $\lambda \in G_{\alpha(t)} \subseteq G_{\alpha(tu)}$ from which we have $(su, tu) \in \rho_\Gamma$. Thus $\rho$ is a congruence on $F(X)$. Also it is easily seen that $\rho_\Gamma \subseteq \mu$ and hence $\rho_\Gamma$ is idempotent separating.

Conversely, let $\rho$ be an idempotent separating congruence on $F(X)$ and let $G_n$ be defined as in the theorem for each $n \in \mathbb{N}$ with $n \leq \dim X$. Consider a fixed $n \in \mathbb{N}$ with $n \leq \dim X$. Then there exists a subspace $A$ of $X$ with $\dim A = n$ and hence there exists $e \in E(F(X))$ with $\alpha(e) = n$. Since $\rho$ is reflexive, we have $(e, e) \in \rho$ and so $1 \in G_n$. Thus $G_n$ is non-empty.
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Let $\lambda, \eta \in G_n$. Then

$$\rho(\lambda \eta e) = \rho(\lambda e) \rho(\eta e) = \rho(e) \rho(e) = \rho(e)$$

so that $\lambda \eta \in G_n$. Again, for $\lambda \in G_n$,

$$\rho(\lambda^{-1} e) = \rho(\lambda^{-1} e) \rho(e) = \rho(\lambda^{-1} e) \rho(e) = \rho(e),$$

so that $\lambda^{-1} \in G_n$. Thus $G_n$ is a subgroup of $K'$ for every $n \in N$ with $n \leq \dim X$. Let $m, n \in N$ with $m \leq n \leq \dim X$ and $\lambda \in G_m$. Let $e \in E(F(X))$ with $\alpha(e) = n$. Since $m \leq n = \dim N(e)$, there exists a subspace $B$ of $N(e)$ with $\dim B = m$ and hence there exists $f \in E(F(X))$ with $N(f) = B$. Now $N(f) \subseteq N(e)$, so that by II.2.1, we have $ef = e$.

Again $\alpha(f) = m$ and $\lambda \in G_m$, so that $(\lambda f, f) \in \rho r$. So,

$$\rho(\lambda e) = \rho(\lambda f) = \rho(e) \rho(\lambda f) = \rho(e) \rho(f) = \rho(e).$$

From this, it follows that $\lambda \in G_n$ and hence $G_m \subseteq G_n$.

Let $\Gamma = \{G_n : n \in N, n \leq \dim X\}$ and $\rho_r$ defined as in the theorem. Let $(s, t) \in \rho r$ so that there exists $\lambda \in G_{\alpha(t)}$ with $s = \lambda t$. Since $F(X)$ is regular, there exists $e \in E(F(X))$ with $t \in E$ (I.1.17). Then $te = t$ by I.1.14 and $\alpha(e) = \alpha(t)$, using 1.16. So $\lambda \in G_{\alpha(e)}$ and hence $(\lambda e, e) \in \rho$. Thus we have

$$\rho(s) = \rho(\lambda t) = \rho(\lambda te) = \rho(t) \rho(\lambda e) = \rho(t) \rho(e) = \rho(t)$$

and so $(s, t) \in \rho$. Conversely, let $(s, t) \in \rho$. Then $(s, t) \in \mu$ and so there exists $\lambda \in K'$ with $s = \lambda t$, by 4.2. To prove that $\lambda \in G_{\alpha(t)}$, let $e \in E(F(X))$ with $\alpha(e) = \alpha(t)$. Then by 2.13, $t \in J$ and hence $t \in J$, since $\mathcal{D} \subseteq J$ in any semigroup (I.1.11). So there exist $u, v \in F(X)$ with $e = utv$. Now $\lambda e = u(\lambda t)v = usv$ and $\rho(s) = \rho(t)$ so that

$$\rho(\lambda e) = \rho(u) \rho(s) \rho(v) = \rho(u) \rho(t) \rho(v) = \rho(utv) = \rho(e).$$

It follows that $\lambda \in G_{\alpha(t)}$ and hence $(s, t) \in \rho r$. Thus $\rho = \rho r$. 

$\blacksquare$
III.5. The semigroup of Weyl operators

One consequence of the additivity of the index is that the set of all Fredholm operators of index zero forms a subsemigroup of $F(X)$. Some properties of this semigroup are considered here. We show that it is a full, regular subsemigroup of $F(X)$, which is completely semisimple and satisfies a stronger form of unit regularity. Also, using the theorem that the sum of an invertible operator and a compact operator is a Fredholm operator of index zero, we extend the relations between $F(X)$ and $H(X)$ to analogous relations between $F(X)$ and $\hat{K}(X)$.

The following definition is from [19].

Definition 5.1. An operator $t$ on $X$ is said to be a Weyl operator if $t$ is in $F(X)$ and $i(t) = 0$. The set of all Weyl operators on $X$ is denoted by $F_0(X)$.

The result below describes some properties of $F_0(X)$. Note that a subsemigroup $T$ of a semigroup $S$ is said to be a full subsemigroup of $S$ if $T$ contains all the idempotents of $S$.

Theorem 5.2. $F_0(X)$ is a full, regular subsemigroup of $F(X)$. Also $F_0(X)$ is completely semisimple and unit regular with $G(X)$ as its group of units.

Proof. Let $s, t \in F_0(X)$. Then by 2.4, $i(st) = i(s) + i(t) = 0$, so that $st \in F_0(X)$. Also if $t \in F_0(X)$ and $t'$ an inverse of $t$ in $F(X)$, then by 2.5, $i(t') = -i(t) = 0$ and hence $t' \in F_0(X)$. Again by 2.5, $i(e) = 0$ and so $e \in F_0(X)$, for all $e \in E(F(X))$. Since for each $u \in G(X)$, $\alpha(u) = 0 = \beta(u)$ and hence $i(u) = 0$, we have $G(X) \subseteq F_0(X)$.

Now if $t \in F_0(X)$, then $i(t) = 0$ so that by 2.3(3), there exists $u \in G(X)$ with $t \leq u$. Hence $F_0(X)$ is unit regular ([41], 2.1). Let $e, f \in E(F_0(X))$
with \( eDf \) in \( F_0(X) \) and \( eWf \). Then there exists \( t \in F_0(X) \) with \( eLt \leq f \), so that by 1.16 we have \( \alpha(e) = \alpha(t) \) and \( \beta(t) = \beta(f) \). But \( \alpha(t) = \beta(t) \), since \( t \in F_0(X) \) and hence \( \alpha(e) = \alpha(f) \). Again, since \( eWf \), we have \( N(f) \subseteq N(e) \) by II.2.1. From these it follows that \( N(e) = N(f) \), so that \( eLf \) by 1.16(2).

Now from \( eLf \) and \( eWf \) we have \( e = f \) (cf. [32], 1.3(b)). Thus \( F_0(X) \) is completely semisimple.

By the above result all operators in \( F_0(X) \) are unit regular. In fact \( F_0(X) \) is precisely the set of all unit regular operators in \( F(X) \) as the next result shows.

**Proposition 5.3.** For \( t \in F(X) \), the following are equivalent:

1. \( t \in F_0(X) \);
2. \( t \in F(X) \) and \( t \) is unit regular;
3. \( t = u + s \) for some \( u \in G(X) \) and \( s \in K(X) \).

**Proof.** From 5.2, we have \( 1 \Rightarrow 2 \). Now assume \( 2 \). Since \( t \) is unit regular, there exists \( u \in G(X) \) with \( tu = t \). Then \( e = tu \in E(R_t) \) with \( t = eu^{-1} \);

also \( R_t = R_e \leq R_1 = R_{u^{-1}} \), since \( e \leq 1 \) and \( u^{-1} \not\leq 1 \). Thus \( t \leq u^{-1} \), by I.1.26 and so by 1.17(3), there exists \( s \in K(X) \) with \( t = u^{-1} + s \). This gives \( 3 \). Finally if \( 3 \) holds, then by 3.13 and 5.2, we have \( i(t) = i(u) = 0 \) and hence \( t \in F_0(X) \).

Since \( F_0(X) \) is regular, the Green's relations \( \mathcal{R} \) and \( \mathcal{L} \) on \( F_0(X) \) are given by (1) and (2) of 1.16. But two elements of \( F_0(X) \) which are \( D \)-related in \( F(X) \) may not be \( D \)-related in \( F_0(X) \).

**Proposition 5.4.** The following are equivalent for \( s, t \in F_0(X) \):
(1) \( sDt \) in \( F_0(X) \);
(2) \( \alpha(s) = \alpha(t) \);
(3) \( \beta(s) = \beta(t) \).

Proof. Let \( sDt \) in \( F_0(X) \). Then there exists \( u \in F_0(X) \) with \( sLu \mathcal{R} t \) so that by 1.16, \( \alpha(s) = \alpha(u) \) and \( \beta(u) = \beta(t) \). Now since \( u \) and \( t \) are in \( F_0(X) \), we also have \( \alpha(u) = \beta(u) \) and \( \beta(t) = \alpha(t) \). Hence \( \alpha(s) = \alpha(t) \), giving (2). If (2) holds, then \( \beta(s) = \alpha(s) = \alpha(t) = \beta(t) \), so that we have (3). Finally suppose (3) holds. Then by 2.13, \( sDt \) in \( F(X) \) and so there exists \( u \in F(X) \) with \( sLu \mathcal{R} t \). Hence \( \alpha(u) = \alpha(s) \) and \( \beta(t) = \beta(u) \). Also since \( s \in F_0(X) \), we have \( \alpha(s) = \beta(s) \); and by (3), \( \beta(s) = \beta(t) \). So \( \alpha(u) = \beta(u) \). Thus \( u \in F_0(X) \) and hence \( sDt \) in \( F_0(X) \).

We now consider some properties of pairs of \( \mathcal{D} \) related idempotents in \( F_0(X) \). First, we note the following notions from the general theory of semigroups. Let \( S \) be a semigroup and \( e, f \in E(S) \). By an \( E \)-chain linking \( e \) to \( f \), we mean a sequence \( e = e_0, e_1, \ldots, e_n = f \) of idempotents in \( S \) with \( e_k(R \cup L)e_{k+1} \), for \( k = 0, 1, \ldots, n - 1 \). The length of the \( E \)-chain \( e_0, e_1, \ldots, e_n \) is defined to be \( n \). The distance \( d(e, f) \) between \( e \) and \( f \) is the length of the shortest \( E \)-chain linking \( e \) to \( f \) ([5], [31]). Note that if there exists an \( E \)-chain linking \( e \) to \( f \), then \( e \mathcal{D} f \); but the converse may not be true.

**Proposition 5.5.** Let \( e, f \) be \( \mathcal{D} \) related idempotents in \( F_0(X) \). Then there exists an \( E \)-chain \( eLgRhf \) linking \( e \) to \( f \). In particular, \( d(e, f) \leq 3 \).

Proof. Let \( N(e) = A \) and \( R(e) = B \). Since \( e \mathcal{D} f \) in \( F_0(X) \), we have \( \alpha(e) = \alpha(f) \) by 5.4 and so \( N(e) \) and \( N(f) \) are subspaces of \( X \) with equal finite
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dimension. Hence by II.5.8, there exists a subspace $C$ of $X$ with $X = A \oplus C = B \oplus C$ and so there exist $g, h \in E(F(X))$ with $N(g) = A$, $R(g) = C$ and $N(h) = B$, $R(h) = C$. Now $g, h \in E(F_0(X))$ by 5.2 and $eLgRh \subseteq Lf$, by 1.16. This proves the result.

Now if $e$ is an idempotent in a monoid $S$ and $f = ueu^{-1}$, then it is easily seen that $f$ is an idempotent and $eLueRf$ so that $eDf$. We next show that the converse holds in $F_0(X)$. The following definition is from [41].

**Definition 5.6.** A unit regular semigroup $S$ is said to be strongly unit regular if for $e, f \in E(S)$, $eDf$ implies the existence of a unit $u$ in $S$ with $f = ueu^{-1}$.

**Proposition 5.7.** $F_0(X)$ is a strongly unit regular semigroup.

**Proof.** By 5.2, $F_0(X)$ is unit regular. Let $e, f \in E(F_0(X))$ with $eDf$ in $F_0(X)$. Then by 5.4, $\alpha(e) = \alpha(f)$ so that $\dim N(e) = \dim N(f)$. From this it follows that there exists a topological isomorphism $\phi_0$ of $N(e)$ onto $N(f)$. ([19], 6.2.5; [3], 23.2). Again since $eDf$ in $F(X)$, there exists a topological isomorphism $t_0$ of $R(e)$ onto $R(f)$. Define

$$u = t_0e + s_0(1 - e) \quad \text{and} \quad u' = t_0^{-1}f + s_0^{-1}(1 - f).$$

Then $u, u' \in CL(X)$. Also $uu' = f + (1 - f) = 1$, since $es_0^{-1} = 0 = (1 - e)t_0^{-1}$ and $u' = e + (1 - e) = 1$, since $fs_0 = 0 = (1 - f)t_0$. Hence $u, u' \in G(X)$ with $u' = u^{-1}$. Moreover, $ueu^{-1} = u' = t_0eu' = f$. ■

It is proved in [41], Proposition 2.2 that if $S$ is a unit regular semigroup with group of units $G$, then $S \setminus G = \{x \in S : x \notin G\}$ is an ideal in $S$. Now
in $F_0(X)$, we have $F_0(X) \setminus G(X) = \{ t \in F_0(X) : \alpha(t) \geq 1 \}$. The following result generalizes this.

**Proposition 5.8.** For each non-negative integer $n$, the set

$$I_n(X) = \{ t \in F_0(X) : \alpha(t) \geq n \}$$

is an ideal in $F_0(X)$. In fact $I_n(X)$ is the principal ideal generated by any $t \in F_0(X)$ with $\alpha(t) = n$. Moreover, any ideal in $F_0(X)$ is equal to $I_n(X)$ for some $n$.

**Proof.** Let $t \in I_n(X)$ and $s \in F_0(X)$. Then $N(t) \subseteq N(st)$, so that $n \leq \alpha(t) \leq \alpha(st)$ and hence $st \in I_n(X)$. Again $R(ts) \subseteq R(t)$, so that $\beta(t) \leq \beta(ts)$ (cf. equation (2) in the proof of 2.2). Now since $t, st \in F_0(X)$, we have $\alpha(t) = \beta(t)$ and $\alpha(ts) = \beta(ts)$ and so $\alpha(ts) \geq \alpha(t) \geq n$. Thus $ts \in I_n(X)$ and hence $I_n(X)$ is an ideal in $F_0(X)$.

Now let $t \in F_0(X)$ with $\alpha(t) = n$ and let $J(t)$ denote the principal ideal in $F_0(X)$ generated by $t$. Then $J(t) \subseteq I_n(X)$, since $I_n(X)$ is an ideal containing $t$ in $F_0(X)$. Conversely, let $s \in I_n(X)$. Then $n \leq \alpha(s) = \dim N(s)$ and so $N(s)$ has a subspace $A$ with $\dim A = n$. Hence there exists $e \in E(F(X)) = E(F_0(X))$ with $N(e) = A$. Since $\alpha(e) = n = \alpha(t)$, we have $eDt$ by 2.13. It follows that $eJt$ (I.1.11); and so $J(e) = J(t)$ by definition of $J$. Again from $N(e) = A \subseteq N(s)$ we have $s(1-e) = 0$, which gives $s = se = see \in J(e) = J(t)$. Thus $I_n(X) \subseteq J(t)$ also.

Finally, let $I$ be an ideal in $F_0(X)$. Let $n = \min\{ \alpha(t) : t \in I \}$. Then there exists $s \in I$ with $\alpha(s) = n$, so that $I_n(X) = J(s) \subseteq I$. Also if $t \in I$, then $\alpha(t) \geq n$, by definition of $n$ and so $t \in I_n(X)$. Thus $I = I_n(X)$. □
From 5.3, we see that if \( s \in K(X) \) then \( 1 + s \in F_0(X) \). This can be extended to compact operators (II.5.11) as in the following result, which is proved in [16] (Chapter 5, Part 2, Section 4, Theorem 5).

**Proposition 5.9.** If \( s \in \hat{K}(X) \), then \( 1 + s \in F_0(X) \). \( \blacksquare \)

From this, we get another characterization of Weyl operators.

**Proposition 5.10.** \( t \in F_0(X) \) if and only if \( t = u + s \) for some \( u \in G(X) \) and \( s \in \hat{K}(X) \).

**Proof.** If \( t \in F_0(X) \), then by 5.3, there exist \( u \in G(X) \) and \( s \in K(X) \) with \( t = u + s \). Also by II.5.12, \( s \in \hat{K}(X) \). Conversely, let \( u \in G(X) \) and \( s \in \hat{K}(X) \). Then \( u^{-1}s \in \hat{K}(X) \), since \( \hat{K}(X) \) is an ideal (II.5.12). So \( 1 + u^{-1}s \in F_0(X) \), by 5.9. Hence \( u + s = u(1 + u^{-1}s) \in F_0(X) \), by 5.2. \( \blacksquare \)

Using 5.9, we can give a characterization of Fredholm operators in terms of compact operators, analogous to 1.7 (cf. [2], [19], [29], [40]).

**Theorem 5.11.** Let \( t \in CL(X) \). Then \( t \in F(X) \) if and only if \( t + \hat{K}(X) \) is invertible in the algebra \( CL(X)/\hat{K}(X) \).

**Proof.** Let \( t \in F(X) \). Then by 1.11, \( t \) has a \( K \)-regularizer \( s \) in \( F(X) \). So \( 1 - st, 1 - ts \in \hat{K}(X) \), since \( K(X) \subseteq \hat{K}(X) \) by II.5.12. Hence

\[
(t + \hat{K}(X))(s + \hat{K}(X)) = ts + \hat{K}(X) = 1 + \hat{K}(X)
\]

and

\[
(s + \hat{K}(X))(t + \hat{K}(X)) = st + \hat{K}(X) = 1 + \hat{K}(X)
\]

so that \( t + \hat{K}(X) \) is invertible in \( CL(X)/\hat{K}(X) \).
Conversely suppose that \( t + \hat{K}(X) \) is invertible in \( CL(X)/\hat{K}(X) \) and let \((t + \hat{K}(X))^{-1} = s + \hat{K}(X)\). Then \( st - 1 \) and \( ts - 1 \) are in \( \hat{K}(X) \) and so \( st, ts \) are in \( F_0(X) \) by 5.9. Hence by 5.3, there exist \( u, v \in G(X) \) with \( u-st, v-ts \in K(X) \). So \( 1-u^{-1}st = u^{-1}(u-st) \) and \( 1-ts v^{-1} = (v-ts)v^{-1} \) are in \( K(X) \), since \( K(X) \) is an ideal. Let \( s_1 = u^{-1}s \) and \( s_2 = sv^{-1} \). If \( \pi \) denotes the canonical projection from \( CL(X) \) to \( CL(X)/K(X) \) defined by \( \pi(w) = w + K(X) \), then \( \pi(s_1)\pi(t) = \pi(u^{-1}st) = \pi(1) \) and \( \pi(t)\pi(s_2) = \pi(tsv^{-1}) = \pi(1) \). Hence

\[
\pi(s_1) = \pi(s_1)(\pi(t)\pi(s_2)) = (\pi(s_1)\pi(t))\pi(s_2) = \pi(s_2).
\]

Thus \( s_1 + K(X) = s_2 + K(X) \) is the inverse of \( t + K(X) \) in \( CL(X)/K(X) \).

From 1.7, it follows that \( t \in F(X) \).

We can also extend 1.8 and 3.13 to compact operators.

**Proposition 5.12.** If \( t \in F(X) \) and \( s \in \hat{K}(X) \), then \( t + s \in F(X) \) and

\[
i(t + s) = i(t).
\]

**Proof.** Let \( t \in F(X) \) and \( s \in \hat{K}(X) \). Then \( t + \hat{K}(X) \) is invertible in \( CL(X)/\hat{K}(X) \) by 5.11. Also \( t+s+\hat{K}(X) = t+\hat{K}(X) \). Hence \( t+s \in F(X) \), again by 5.11.

Now let \( t' \) be an inverse of \( t \). Then \( tt' \in E(F(X)) \) and so \( 1 - tt' \in K(X) \), by 1.6. Hence \( 1 - tt' \in \hat{K}(X) \) and so \( (t+s)t' - 1 = (tt' - 1) + st' \) is in \( \hat{K}(X) \), since \( \hat{K}(X) \) is an ideal in the algebra \( CL(X) \). By 5.9, we now have \( (t+s)t' \in F_0(X) \) so that \( i(t+s) + i(t') = i((t+s)t') = 0 \), using 2.4. Thus \( i(t+s) = -i(t') = i(t) \), by 2.5.