Chapter 1

Introduction and Basic Concepts

1.1 Non-Gaussian time series modelling: a prelude

In our endeavours to understand the changing world around us observations of one kind or another are frequently made sequentially over time. Such a record of values of any fluctuating quantity measured at different points of time is called a time series. An observed time series can be assumed to be a realization of a stochastic process, the population of all such realizations that could have occurred.

The basic objective of time series analysis is to uncover the dynamical law governing its generation. This enables us to understand the important features of the observed data and to develop an empirical time series model which may be used for forecasting and other purposes.

Until recently the models considered for time series analysis were linear with constant coefficients and Gaussian marginal distributions. Gaussianity, linearity and stationarity were regarded as the basic rules for many decades. The much celebrated Box-Jenkins autoregressive
moving average (ARMA) models were based on these assumptions. A
detailed account of the statistical theory relating to Gaussian ARMA
models can be found in Box and Jenkins (1976), Abraham and Ledolter
(1983) and Brockwell and Davies (1991). Such models in general have
an approximately normal marginal distribution, no matter how the in­
novations are qistributed. Mallows (1967) proved that all linear pro­
cesses are approximately normal. Lusk and Wright (1982) is also con­
cerned with similar problems.

The need for developing non-Gaussian time series models have been
long felt from the fact that most of the observed time series data have
non-Gaussian marginal distributions. For example, in economics, Nel­
sen and Granger (1979) considered a set of 21 time series data of which
only six were found to be Gaussian. Lawrance and Kottegoda (1977)
explain the need for using time series models having non-Gaussian
marginal distributions for modelling river flow and other hydrological
time series data. Brown et al. (1984) stress the need for constructing
time series models with Weibull marginal distribution for the modelling
of wind velocity data. Gibson (1986) described the use of autoregres­
sive processes with Laplace marginal distribution for image source mod­
elling. Anderson and Arnold (1993) describe the use of Linnik marginal
distribution in modelling stock price returns and other financial data.
Davies et al. (1980) discuss the applications of ARMA processes with
non-normal residuals in surface metrology.

The ARMA models are not ideally suited for data exhibiting time
irreversibility. Raftery (1982) points out that in cases such as river flow
series and windspeed series, which are often skewed, the marginal dis-
tribution and time irreversibility are themselves of vital interest and it is preferable to develop models which incorporate these features explicitly, particularly if one is interested in more than just point forecasts. Such data are time irreversible and simple monotonic transformations will not rectify the asymmetry.

Weiss (1975) established that ARMA processes with an autoregressive component are time reversible if and only if they are Gaussian. The implications of this result are far reaching. Stationary series which show evidence of directionality cannot be modelled by Gaussian ARMA models. They need be modelled by non-Gaussian ARMA models or some type of non-linear models. Lawrance (1991) explains the directional nature of most time series data and cites examples like Wolfer sunspot series and Canadian lynx data.

Gaussian ARMA models are not suited for modelling data exhibiting sudden bursts of large amplitude at irregular time epochs. Having symmetric joint distributions, stationary Gaussian ARMA models are not ideally suited for data exhibiting strong asymmetry. The conditional expectations of lag $j$ namely, $E(X_t/X_{t-j})$ for $j = 1, 2, \ldots$ are all linear for Gaussian ARMA models because of the joint normality property. This characteristic may sometimes weaken the usefulness of ARMA models for data exhibiting strong cyclicity.

Gaussian processes are also characterized in terms of their reversed conditional expectations. Lawrance (1978) showed that a first order linear autoregressive model for which $E(X_t/X_{t+1})$ is linear in $X_{t+1}$ is necessarily Gaussian. In order to remove these limitations of Gaussian
ARMA models we may either retain the ARMA framework and allow the innovations to be non-Gaussian or we may completely abandon the linearity assumption.

Further motivation for the development of non-Gaussian time series models is provided by the existence of many point processes which exhibit short term autocorrelation between the interevent intervals. Despite the abundance of physically motivated point process models, there are few simple models, analogous to the ARMA class, suitable for general use in this case (see Lawrance (1980), Gaver and Lewis (1980)). The increased computational power and speed of modern computers also influenced time series analysts to turn towards the more natural and realistic non-linear, non-Gaussian and non-stationary time series models.

A direct approach towards the construction of non-Gaussian time series models is to develop time series driven by innovations with a prespecified common distribution. Li and McLeod (1988) fitted autoregressive models with lognormal and gamma innovations to the sunspot and Canadian lynx data respectively and obtained fairly good results. Damsleth and El-Shaarawi (1989) constructed a first order autoregressive model with double exponential innovations and applied it to a series of weakly measurements of sulphate concentration. They obtained a significantly better fit when compared with the Gaussian model.

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Subba Rao (1981, 1983, 1988), Subba Rao and Gabr (1984), Ozaki (1981, 1982), Haggan and Ozaki (1981) etc., are works in this field. Tong (1983) constructs threshold models in an attempt to model non-linear time series. This is motivated by the theory of limit cycles and cyclical vibrations. A threshold model may be regarded as a 'piece-wise linear' approximation to the general non-linear model. Subba Rao (1981) and Granger and Anderson (1978) consider a bilinear model, which is obtained as an extension of the linear ARMA model, by adding a bilinear form in \( \{X_t\} \) and the innovations \( \{e_t\} \). Ozaki (1981) considers some type of exponential autoregressive models, which are entirely different from the exponential autoregressive (EAR) models of Gaver and Lewis (1980), in an attempt to reproduce the effects of 'amplitude dependent frequency' which are characteristic of non-linear random vibrations. These models can also give rise to 'jump phenomena' and limit cycle behaviour.

Priestley (1980) describes a general class of non-linear models called state dependent models, which includes the bilinear, threshold and Ozaki's exponential models as special cases and allows much greater flexibility in the character of the non-linear structure. Nicholls and Quinn (1982) consider random coefficient autoregressive models and describe various conditions for stationarity and stability of the models.

Yet another approach, which is more realistic, is the construction of models which have a predesignated marginal distribution. This thesis is concerned with this approach. The pioneering work in this field of non-Gaussian autoregressive time series modelling with a specified stationary marginal distribution was by Gaver and Lewis (1980). A
number of models for stationary time series of continuous variables with negative exponential and gamma marginal distributions were developed in a series of papers by Gaver and Lewis (1980) and Lawrance and Lewis (1977, 1980, 1981, 1982, 1985). The original formulations of these models were as first order autoregressive processes. Then they were extended to moving average and mixed autoregressive moving average structures. The EAR(1) model of Gaver and Lewis (1980) and the new exponential autoregressive (NEAR(1)) model of Lawrance and Lewis (1981) have been studied by many authors. Raftery (1982) constructed time series models with Weibull, gamma, Erlang and Kritskii-Menkel marginal distributions. Jayakumar and Pillai (1993) extended the Gaver-Lewis (1980) process to a larger class called semi-Mittag-Leffler autoregressive process.

Anderson and Arnold (1993) developed time series models having Linnik marginal distribution which could be used for modelling financial time series. Dewald and Lewis (1985) developed an autoregressive model with Laplace marginals. McKenzie (1982) developed a product autoregression model in gamma variables. Sim (1987) fitted a mixed gamma ARMA(1,1) model for stream flow data of Perak river in Malaysia. It was established that the simulated data bears close resemblance to the historical sequence in terms of mean, variance, skewness and autocorrelation coefficients. Lewis et al. (1989) developed a first order beta–gamma process employing the beta–gamma transformation, which can be used for modelling sequence of times between failures of a computer system. Sim (1993) develops an autoregressive model with logistic marginal distribution. Andel (1988) develops an autoregressive
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minification processes using a ‘thickening’ operation which results in self-recomposition instead of self-decomposition.

However, it may be noted that no rigorous and systematic study on the stationary solutions with respect to which various types of non-Gaussian time series models are defined, has not been made. Cox (1981) and Bondesson (1981) established that the linear autoregressive model $X_n = aX_{n-1} + \epsilon_n$ is properly defined for $a \in (0,1)$ if and only if the marginal distribution of $X_n$ belongs to the class $L$. Hart (1984), McKenzie (1982, 1986, 1988a), Jayakumar and Pillai (1992, 1993) also discuss such problems. Chan (1988) discusses the existence of the stationary ergodic NEAR(p) model. Jin Guan and Yuan (1991) discuss the existence of the $p$-th order integer valued autoregressive INAR(p) process. Pillai and Jayakumar (1994a) describe the role of specialized class $L$ property in defining a $p$-th order autoregressive process having the EAR(p) structure described in Lawrance and Lewis (1980).

The present study aims at exploring the classes of distributions for which stationary solutions to different types of autoregressive models exist.

1.2 Summary of the present work

This thesis is primarily concerned with some theoretical aspects of non-Gaussian autoregressive time series modelling. Here we consider autoregressive models with linear additive structure and minification structure in detail. In our study we obtain the class of possible marginal distributions under which the models of consideration are
strictly Markovian and stationary. This study blends together the modern concepts in probability theory such as geometric infinite divisibility and autoregressive time series modelling.

In chapter 1 we introduce the basic concepts and definitions which are of frequent use in the sequel. Chapter 2 deals with the role of geometrically infinitely divisible distributions introduced by Klebanov et al. (1984) in autoregressive time series modelling. We establish that autoregressive models of the form

\[ X_n = \epsilon_n + \begin{cases} 0 & \text{with probability } p \\ X_{n-1} & \text{with probability } 1 - p \end{cases} \]

for \( 0 < p < 1 \), is stationary Markovian if and only if the marginal distribution of \( X_n \) is geometrically infinitely divisible. It is also established that the innovations \( \epsilon_n \) are geometrically infinitely divisible. In section 2.3 we obtain the stationary solution of a \( k \)-th order autoregressive model which is an extension of the above model. Section 2.4 deals with the importance of the Mittag–Leffler distribution introduced by Pillai (1990a) in this context. It is proved that the innovations \( \{\epsilon_n\} \) and the process \( \{X_n\} \) are identically distributed if and only if \( X_n \)'s are marginally distributed as semi-Mittag–Leffler. In section 2.5 we obtain the structural relationship between the stationary marginal distributions of \( \{\epsilon_n\} \) and \( \{X_n\} \) in terms of a unique positive measure \( W \) introduced by Fujita (1993) while generalizing the results of Pillai (1990a). Here we introduce a first order autoregressive Mittag–Leffler process which is analytically very tractable and can be regarded as a generalization of the TEAR(1) model. In section 2.6 we bring out the connections to \( p \)-thinning and Cox and renewal processes also.
In Chapter 3 we introduce some generalized autoregressive models in Mittag–Leffler variables. This generalizes the model of Jayakumar and Pillai (1993). In section 3.5 the semi–Mittag–Leffler tailed autoregressive process is introduced and studied in detail. This can be used for modelling stream flow data of rivers that are dry for part of the year. In section 3.6 we extend the model to the entire real line to construct the semi–\(\alpha\)–Laplace tailed autoregressive process. Section 3.7 deals with a first order \(\alpha\)–Laplace tailed autoregressive process and its special cases. Various properties of these models namely joint distribution, distribution of the sums, etc. are studied here.

Chapter 4 deals with a new class of distributions called semi–Weibull distribution, which includes the Weibull and exponential distributions as special cases. Its properties are explored. A first order semi–Weibull autoregressive minification process is developed and studied in section 4.3. As a special case we study a first order Weibull autoregressive minification process in detail. Its sample path behaviour and other properties are studied in section 4.5. The bivariate distribution, time reversibility, conditional expectations etc. are studied here. In section 4.6 the autocorrelation function is derived and a simulation study is made to bring out the dependence structure. In section 4.7 the model is extended to construct a \(k\)-th order autoregressive model as well as a moving average model. This model can be used for modelling wind velocity data.

In chapter 5, a new class of distributions called distributions of universal geometric minimum is introduced and its properties studied. It is shown that this class forms the stationary solution of an autoregressive
minification structure of the form

\[ X_n = \begin{cases} 
\varepsilon_n & \text{with probability } p \\
\min(X_{n-1}, \varepsilon_n) & \text{with probability } 1 - p
\end{cases} \]

where \( 0 < p < 1 \). We obtain some characterizations and generalize the results to the \( k \)-th order case also. This class is a subclass of distributions with complete monotone derivative which is a subclass of geometrically infinitely divisible distributions. We establish the role of the semi-Pareto distribution of Pillai (1991) in this context. It is shown that \( \{X_n\} \) and \( \{\varepsilon_n\} \) are identically distributed if and only if \( X_n \) are distributed as semi-Pareto. A new semi-Pareto process is developed in section 5.5. It is shown that the new minification process introduced here is a subclass of mixed Poisson and renewal process, where the mixing distribution is geometrically infinitely divisible. In section 5.6 we introduce an analogous concept called universal geometric maximum and a corresponding maximal process is developed.

Chapter 6 is concerned with another new class namely, distributions having self-decomposable survival functions. In section 6.2 the concept of survival self-decomposability is introduced and studied. In section 6.3 various characterizations are obtained and it is shown that this class is unimodal. This class forms the stationary solution of the autoregressive minification structure

\[ X_n = \min(c^{-1}X_{n-1}, \varepsilon_n) \]

for \( 0 < c < 1 \). In section 6.4 the importance of semi-Pareto distribution is established. We obtain a characterization of the semi-Pareto
distribution also. In section 6.5 the model is extended to the $k$-th order and the stationary solution is obtained using a property called specialized survival self-decomposability. It is established that this property characterizes the Pareto type III distribution.

1.3 Preliminary concepts

In this section we discuss some preliminary concepts which will be of frequent use in the sequel.

1.3.1 Infinite divisibility

A random variable $X$ is said to be infinitely divisible if for every $n \in \mathcal{N}$, there exists independently and identically distributed random variables $Y_{1n}, Y_{2n}, \ldots, Y_{nn}$ such that $X \stackrel{d}{=} Y_{1n} + Y_{2n} + \cdots + Y_{nn}$, where $\stackrel{d}{=}$ denotes equality in distributions. In terms of distribution functions, a distribution function $F$ is said to be infinitely divisible if for every positive integer $n$, there exists a distribution function $F_n$ such that $F = \underbrace{F_n \ast F_n \ast \cdots \ast F_n}_{n \text{ times}}$, where $\ast$ denotes convolution.

This is equivalent to the existence of a characteristic function $\varphi_n(t)$ for every $n \in \mathcal{N}$ such that $\varphi(t) = [\varphi_n(t)]^n$ where $\varphi(t)$ is the characteristic function of $X$.

Infinitely divisible distributions occur in various contexts in the modelling of many real phenomena. For instance when modelling the amount of rain $X$ that falls in a period of length $T$, one can divide $X$
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into more general independent parts from the same family. i.e.,

\[ X \overset{d}{=} X_{t_1} + X_{t_2-t_1} + \cdots + X_{T-t_{n-1}}. \]

Similarly, the amount of money \( X \) paid by an insurance company during a year must be expressible as the sum of the corresponding amounts \( X_1, X_2, \ldots, X_{52} \) in each week, i.e.,

\[ X \overset{d}{=} X_1 + X_2 + \cdots + X_{52}. \]

A large number of distributions such as normal, exponential, Weibull, gamma, Cauchy, Laplace, logistic, lognormal, Pareto, geometric, Poisson, etc., are infinitely divisible. Various properties and applications of infinitely divisible distributions can be found in Laha and Rohatgi (1979) and Steutel (1979).

1.3.2 Geometric infinite divisibility

The concept of geometric infinite divisibility (g.i.d.) was introduced by Klebanov et al. (1984). A random variable \( Y \) is said to be g.i.d. if for every \( p \in (0, 1) \), there exists a sequence of independently and identically distributed random variables \( X_1^{(p)}, X_2^{(p)}, \ldots \) such that

\[
Y \overset{d}{=} \sum_{j=1}^{N(p)} X_j^{(p)}
\]
\[ P\{N(p) = k\} = p(1 - p)^{k-1}, \quad k = 1, 2, \ldots \]

where \( Y, N(p) \) and \( X_j^{(p)}, j = 1, 2, \ldots \) are independent. The relation (1.3.1) is equivalent to

\[
\varphi(t) = \sum_{j=1}^{\infty} [g(t)]^j \cdot p(1 - p)^{j-1} = \frac{pg(t)}{1 - (1 - p)g(t)}
\]

where \( \varphi(t) \) and \( g(t) \) are the characteristic functions of \( Y \) and \( X_j^{(p)} \) respectively.

The class of g.i.d. distributions is a proper subclass of infinitely divisible distributions. The g.i.d. distributions play the same role in 'geometric summation' as infinitely divisible distributions play in the usual summation of independent random variables. Klebanov et al. (1984) established that a distribution \( F \) with characteristic function \( \varphi(t) \) is g.i.d. if and only if \( \exp \left\{ 1 - \frac{1}{\varphi(t)} \right\} \) is infinitely divisible. Exponential and Laplace distributions are obvious examples of g.i.d. distributions. Pillai (1990b), Mohan et al. (1993) discuss properties of g.i.d. distributions. It may be noted that normal distribution is not geometrically infinitely divisible.
1.3.3 Bernstein functions

A $C^\infty$-function $f$ from $(0, \infty)$ to $\mathbb{R}$ is said to be completely monotone if $(-1)^n \frac{d^n f}{dx^n} \geq 0$ for all integers $n \geq 0$.

A $C^\infty$-function $f$ from $(0, \infty)$ to $\mathbb{R}$ is said to be a Bernstein function, if $f(x) \geq 0$, $x > 0$ and $(-1)^n \frac{d^n f}{dx^n} \leq 0$ for all integers $n \geq 1$. Then $f$ is also referred to as a function with complete monotone derivative (c.m.d).

A completely monotone function is positive, decreasing and convex while a Bernstein function is positive, increasing and concave (see Berg and Forst (1975)).

Fujita (1993) established that a cumulative distribution function $G$ with $G(0) = 0$ is geometrically infinitely divisible, if and only if $G$ can be expressed as

$$G(x) = \sum_{n=1}^{\infty} (-1)^{n+1} W_n^*([0, x]), \quad x > 0$$

where $W_n^*(dx)$ is the $n$-fold convolution measure of a unique positive measure $W(dx)$ such that

$$\frac{1}{f(x)} = \int_0^\infty e^{-sx} W(ds), \quad x > 0$$
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where \( f(x) \) is a Bernstein function, satisfying the conditions

\[
\lim_{x \to 0} f(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.
\]

A distribution is said to have complete monotone derivative if its distribution function \( F(x) \) is Bernstein. Pillai and Sandhya (1990) proved that the class of distributions having complete monotone derivative is a proper subclass of g.i.d. distributions. This implies that all distributions with complete monotone densities are geometrically infinitely divisible. It is easier to verify the complete monotone criterion and using this approach we can establish the geometric infinite divisibility of many distributions such as Pareto, gamma and Weibull.

The class of non–degenerate generalized gamma convolutions with densities of the form given by

\[
f(x) = c x^{\beta-1} \prod_{j=1}^{M} (1 + c_j x)^{-r_j}, \quad x > 0
\]

is geometrically infinitely divisible for \( 0 < \beta \leq 1 \). Similarly distributions having densities of the form

\[
f(x) = cx^{\beta-1} \exp(-cx^\alpha); \quad 0 < \alpha \leq 1
\]

is g.i.d. for \( 0 < \beta \leq 1 \). Also the Bondesson family of distributions with
densities of the form

\[ f(x) = cx^{\beta-1} \prod_{j=1}^{M} \left[ 1 + \sum_{k=1}^{N_j} c_{jk} x^{\alpha_{jk}} \right]^{-r_j} \]

is g.i.d. for \( 0 \leq \beta \leq 1, \alpha_{jk} \leq 1 \) provided all parameters are strictly positive (see Bondesson(1992)).

1.3.4 Self-decomposability

Let \( \{X_n; \ n \geq 1\} \) be a sequence of independent random variables, and let \( \{b_n\} \) be a sequence of positive real numbers such that

\[ \lim_{n \to \infty} \max_{1 \leq k \leq n} P\{|X_k| \geq b_n \epsilon\} = 0 \text{ for every } \epsilon > 0. \]

Set \( S_n = \sum_{k=1}^{n} X_k \) for \( n \geq 1 \). Then the class of distributions which are the weak limits of the distributions of the sums \( b_n^{-1} S_n - a_n; \ n \geq 1 \) where \( a_n \) and \( b_n > 0 \) are suitably chosen constants, is said to constitute class \( L \). Such distributions are called self-decomposable.

A distribution \( F \) with characteristic function \( \varphi(t) \) is called self-decomposable, if and only if, for every \( \alpha \in (0, 1) \), there exists a characteristic function \( \varphi_{\alpha}(t) \) such that \( \varphi(t) = \varphi(\alpha t) \cdot \varphi_{\alpha}(t) \) for \( t \in \mathbb{R} \).

Clearly, apart from \( X \equiv 0 \), no lattice random variable can be self-decomposable. All non-degenerate self-decomposable distributions are absolutely continuous.
A discrete analogue of self-decomposability was introduced by Steutel and Van Harn (1979). A distribution on $\mathbb{N}_0 \equiv \{0, 1, 2, \ldots \}$ with probability generating function (p.g.f.) $P(z)$ is called discrete self-decomposable if $P(z) = P(1 - \alpha + \alpha z)P_\alpha(z); |z| \leq 1, \alpha \in (0, 1)$ where $P_\alpha(z)$ is a p.g.f.

If we define $G(z) = P(1 - z)$, then $G(z)$ is called the alternate probability generating function (a.p.g.f.). Then it follows that a distribution is discrete self-decomposable if and only if $G(z) = G(\alpha z)G_\alpha(z); |z| \leq 1, \alpha \in (0, 1)$ where $G_\alpha(z)$ is some a.p.g.f.

1.3.5 Stable distributions

A distribution function $F$ with characteristic function $\varphi(t)$ is stable if for every pair of positive real numbers $b_1$ and $b_2$, there exist finite constants $a$ and $b > 0$ such that $\varphi(b_1 t) \cdot \varphi(b_2 t) = \varphi(bt)e^{iat}$ where $i = \sqrt{-1}$.

Clearly, stable distributions are in class $L$ with the additional condition that the random variables $X_n; n \geq 1$ in section 1.3.4 are identically distributed also. $F$ is stable if and only if its characteristic function can be expressed as

$$\ln \varphi(t) = i\alpha t - c|t|^\beta[1 + i\gamma \omega(t, \beta)\text{sgn } t]$$
where \( \alpha, \beta, \gamma \) are constants with \( c \geq 0, 0 < \beta \leq 2, |\gamma| \leq 1 \) and

\[
\omega(t, \beta) = \begin{cases} 
\tan \frac{\pi \beta}{2}; & \beta \neq 1 \\
\frac{2}{\pi} \ln |t|; & \beta = 1.
\end{cases}
\]

The value \( c = 0 \) corresponds to the degenerate distribution, and \( \beta = 2 \) to the normal distribution. The case \( \gamma = 0, \beta = 1 \) corresponds to the Cauchy law (see Laha and Rohatgi (1979)).

### 1.3.6 Geometrically strictly stable distributions

A random variable \( Y \) is said to be geometrically strictly stable (g.s.s.) if for any \( p \in (0, 1) \) there exists a constant \( c = c(p) > 0 \) and a sequence of independent and identically distributed random variables \( Y_1, Y_2, \ldots \) such that

\[
Y \overset{d}{=} c(p) \sum_{j=1}^{N(p)} Y_j
\]

where \( P\{N(p) = k\} = p(1 - p)^{k-1}; \ k = 1, 2, \ldots \) and \( Y, N(p) \) and \( Y_j; j = 1, 2, \ldots \) are independent.

If \( \varphi(t) \) is the characteristic function of \( Y \), then it implies that

\[
\varphi(t) = \frac{p \varphi(ct)}{1 - (1-p)\varphi(ct)}; \quad p \in (0, 1).
\]

Among the geometrically strictly stable distributions, the Laplace distribution and exponential distribution possess all moments. A geometrically strictly stable random variable is clearly geometrically infinitely divisible.
A non-degenerate random variable \( Y \) is geometrically strictly stable if and only if its characteristic function is of the form

\[
\varphi(t) = \frac{1}{1 + \lambda |t|^\alpha \exp \left(-\frac{\pi}{2} \theta \alpha \text{sgn} t \right)}
\]

where \( 0 < \alpha \leq 2, \lambda > 0, |\theta| \leq \min(1, 2/\alpha - 1) \). When \( \alpha = 2 \), it corresponds to the Laplace distribution. Thus it is apparent that when ordinary summation of random variables is replaced by geometric summation, the Laplace distribution plays the role of the normal distribution, and exponential distribution replaces the degenerate distribution (see Klebanov et al. (1984)).

### 1.3.7 Mittag-Leffler distribution

The Mittag-Leffler distribution was introduced by Pillai (1990a) and has cumulative distribution function given by

\[
F_\alpha(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{k\alpha}}{\Gamma(1 + k\alpha)}; \quad 0 < \alpha \leq 1; \ x > 0.
\]

Its Laplace transform is given by \( \phi(t) = \frac{1}{1 + t^\alpha}; \ 0 < \alpha \leq 1; \ t \geq 0; \) and the distribution may be denoted by ML(\( \alpha \)). Here \( \alpha \) is called the exponent. It can be regarded as a generalization of the exponential distribution in the sense that \( \alpha = 1 \), corresponds to the exponential distribution. The Mittag-Leffler distribution is geometrically infinitely divisible and belongs to class \( L \). It is normally attracted to the stable
law with exponent $\alpha$. Kalyanaraman et al. (1990) conducted Monte-Carlo studies on Mittag-Leffler distributions and computed the numerical cumulative distribution function (c.d.f.) table. The heavy tailed nature of this distribution is evident from the empirical c.d.f. table and figure (see pages 31, 32).

If $U$ is exponential with unit mean and $Y$ is positive stable with exponent $\alpha$, then $X = U^{1/\alpha}Y$ is distributed as Mittag-Leffler ($\alpha$). Kalyanaraman (1989) describes how random variables following Mittag-Leffler distribution can be generated using this result. If $U$ is Mittag-Leffler ($\alpha$) and $V$ is exponential and $U$ and $V$ are independent, then $X = \frac{U}{V}$ is distributed as Pareto type III with survival function $\bar{F}_X(x) = P(X > x) = \frac{1}{1 + x^\alpha}$; $0 < \alpha \leq 1$.

For the Mittag-Leffler distribution, $E(X^\delta)$ exists for $0 \leq \delta < \alpha$ and is given by

$$E(X^\delta) = \frac{\Gamma(1-\delta/\alpha)\Gamma(1+\delta/\alpha)}{\Gamma(1-\delta)}.$$

A two parameter Mittag-Leffler distribution can also be defined with the corresponding Laplace transform $\phi(t) = \frac{\lambda^\alpha}{\lambda^\alpha + t^\alpha}$; $0 < \alpha \leq 1$. It may be denoted by $ML(\alpha, \lambda)$.

Jayakumar and Pillai (1993) considered a more general class called semi-Mittag-Leffler distribution which included the Mittag-Leffler distribution as a special case. A random variable $X$ with positive support is said to have a semi-Mittag-Leffler distribution if its Laplace trans-
form is given by
\[ \phi(t) = \frac{1}{1 + \eta(t)} \]
where \( \eta(t) \) satisfies the functional equation \( \eta(t) = a\eta(bt) \) where \( 0 < b < 1 \) and \( \alpha \) is the unique solution of \( ab^\alpha = 1 \). It may be denoted by \( \text{SML}(\alpha) \). Then it follows that \( \eta(bt) = b^\alpha h(t) \) where \( h(t) \) is a periodic function in \( t \) with period \( \frac{-\ln b}{2\pi\alpha} \). When \( h(t) \) is a constant, the distribution reduces to the Mittag-Leffler distribution. The semi-Mittag-Leffler distribution is also geometrically infinitely divisible and belongs to class \( L \).

### 1.3.8 \( \alpha \)-Laplace distribution

The \( \alpha \)-Laplace distribution has characteristic function given by
\[ \varphi(t) = \frac{1}{1 + |t|^\alpha}; \quad 0 < \alpha \leq 2, \quad -\infty < t < \infty. \]
This is also called Linnik's distribution. Pillai (1985) refers to it as the \( \alpha \)-Laplace distribution since \( \alpha = 2 \) corresponds to the Laplace distribution. It is unimodal, geometrically strictly stable and belongs to class \( L \). It is normally attracted to the symmetric stable law with exponent \( \alpha \). Also
\[
E(|X|^\delta) = \frac{2\delta}{\Gamma(1 + \delta)} \frac{\Gamma\left(1 + \frac{\delta}{\alpha}\right) \Gamma\left(1 - \frac{\delta}{\alpha}\right) \Gamma((1 + \delta)/2)}{\sqrt{\pi} \Gamma\left(1 - \frac{\delta}{2}\right)}
\]
where \( 0 < \delta < \alpha; \quad 0 < \alpha \leq 2. \)

If \( U \) and \( V \) are independent random variables where \( U \) is exponential with unit mean and \( V \) is symmetric stable with exponent \( \alpha \), then \( X = U^{1/\alpha}V \) is distributed as \( \alpha \)-Laplace. Using this result Devroye
(1990) develops an algorithm for generating random variables having \( \alpha \)-Laplace distribution.

Pillai (1985) introduced a larger class of distributions called semi-\( \alpha \)-Laplace distribution, with characteristic function given by

\[
\varphi(t) = \frac{1}{1 + \eta(t)}
\]

where \( \eta(t) \) satisfies the functional equation \( \eta(t) = a\eta(bt) \) for \( 0 < b < 1 \) and \( a \) is the unique solution of \( ab^\alpha = 1, \ 0 < \alpha \leq 2 \). Here \( b \) is called the order and \( \alpha \) is called the exponent of the distribution. If \( b_1 \) and \( b_2 \) are the orders of the distribution such that \( \frac{\log b_1}{\log b_2} \) is irrational, then \( \eta(t) = c|t|^\alpha \), where \( c \) is some constant. Pillai (1985) established that, for a semi-\( \alpha \)-Laplace distribution with exponent \( \alpha \), \( E|X|^{\delta} \) exists for \( 0 \leq \delta < \alpha \).

It can be shown that \( \varphi(t) = \frac{1}{1 + |t|^{\alpha}[1 - A \cos (k \log |t|)]} \) where

\[
k = \frac{2\pi}{\log b}, \quad 0 < b < 1
\]

is the characteristic function of a semi-\( \alpha \)-Laplace distribution for suitable choice of \( A \) and \( \alpha < 1 \).

The semi-\( \alpha \)-Laplace distribution is also geometrically infinitely divisible and belongs to class \( L \). It is useful in modelling household income data. Mohan et al. (1993) refer to it as a geometrically right semi-stable law.
1.3.9 Semi–Pareto distribution

The semi–Pareto distribution was introduced by Pillai (1991). A random variable $X$ with positive support has semi–Pareto distribution $SP(\alpha, p)$ if its survival function is given by $\bar{F}_X(x) = P(X > x) = \frac{1}{1+\psi(x)}$ where $\psi(x)$ satisfies the functional equation $p\psi(x) = \psi(p^{1/\alpha} x)$; $0 < p < 1$, $\alpha > 0$.

The above definition is analogous to that of the semi–stable law defined by Levy (see Pillai (1971)). It can be shown that $\psi(x) = x^\alpha h(x)$ where $h(x)$ is periodic in $\ln x$ with period $\frac{-2\pi\alpha}{\ln p}$. For example if $h(x) = \exp[\beta \cos(\alpha \ln x)]$, then it satisfies the above functional equation with $p = \exp(-2\pi)$ and $\psi(x)$ monotone increasing with $0 < \beta < 1$. The semi–Pareto distribution can be viewed as a more general class which includes the Pareto type III distribution when $\psi(x) = cx^\alpha$.

1.3.10 Stationary time series

A time series, $\{X_t\}$, is a family of real–valued random variables indexed by $t \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. More specifically, it is referred to as a discrete parameter time series.

The time series $\{X_t\}$ is said to be stationary if, for any $t_1, t_2, \ldots, t_n \in \mathbb{Z}$, any $k \in \mathbb{Z}$, and $n = 1, 2, \ldots$,

$$F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}(x_1, x_2, \ldots, x_n) = F_{X_{t_1+k}, X_{t_2+k}, \ldots, X_{t_n+k}}(x_1, x_2, \ldots, x_n)$$

where $F$ denotes the distribution function of the set of random variables.
which appear as suffices. This is called stationarity in the strict sense.

Less stringently, we say a process \( \{X_n\} \) is weakly stationary if the mean and variance of \( X_t \) remain constant over time and the covariance between any two values \( X_t \) and \( X_s \) depends only on the time difference and not on their individual time points.

\( \{X_t\} \) is called a Gaussian process if, for all \( t_1, t_2, \ldots, t_n \) the set of random variables \( \{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\} \) has a multivariate normal distribution.

Since a multivariate normal distribution is completely specified by its mean and variance, it follows that for a Gaussian process weak stationarity implies complete stationarity. But for non-Gaussian processes, this may not hold.

### 1.3.1 Autoregressive models

The era of linear time series models began with autoregressive models first introduced by Yule in 1927.

The standard form of an autoregressive model of order \( p \), denoted by AR\((p)\), is given by

\[
X_t = \sum_{j=1}^{p} a_j X_{t-j} + \epsilon_t; \quad t = 0, \pm 1, \pm 2, \ldots
\]

where \( \{\epsilon_t\} \) are independent and identically distributed random variables called innovations and \( a_1, a_2, \ldots, a_p \) are fixed parameters, with \( a_p \neq 0 \).
Another kind of model of great practical importance in the representation of observed time series is the moving average model. The standard form of a moving average model of order \( q \), denoted by \( \text{MA}(q) \), is given by

\[
X_t = \sum_{j=1}^{q} b_j \varepsilon_{t-j} + \varepsilon_t; \quad t \in \mathbb{Z}
\]

where \( b_1, b_2, \ldots, b_q \) are fixed parameters, with \( b_q \neq 0 \).

To achieve greater flexibility in the fitting of actually observed time series, it is more advantageous to include both autoregressive and moving average terms in the model. Such models called autoregressive-moving average models, denoted by \( \text{ARMA}(p,q) \), have the form

\[
X_t = \sum_{j=1}^{p} a_j X_{t-j} + \sum_{k=1}^{q} b_k \varepsilon_{t-k} + \varepsilon_t; \quad t \in \mathbb{Z}
\]

where \( \{a_j\}_{j=1}^{p} \) and \( \{b_k\}_{k=1}^{q} \) are real constants called parameters of the model. It can be seen that an AR\((p)\) model is the same as an ARMA\((p,0)\) model and a MA\((q)\) model is the same as an ARMA\((0,q)\) model.

With the introduction of various non-Gaussian and non-linear models, the standard form of autoregression was widened in several respects.

A more general definition of autoregression of order \( p \) is given in terms of the linear conditional expectation requirement that

\[
E(X_t/X_{t-1},X_{t-2},\ldots) = \sum_{j=1}^{p} a_j X_{t-j}
\]
This definition could apply to models which are not of the linear form (see Lawrance (1991)).

1.3.12 Threshold autoregressive models

A class of non-linear models called threshold autoregressive (TAR) models was introduced by Tong and Lim (1980), for analysing physically motivated time series data. Threshold models are in effect piecewise linear models. The threshold principle allows the analysis of a complex stochastic system by decomposing it into simpler subsystems.

Let \( \{X_t\} \) be a \( k \)-dimensional time series and, for each \( t \), let \( J_t \) be an indicator random variable, taking integer values \( \{1, 2, \ldots, l\} \). Then a threshold autoregressive model of order one denoted by TAR(1) can be given in the canonical form as

\[
X_t = B^{(J_t)} X_t + A^{(J_t)} X_{t-1} + H^{(J_t)} \epsilon_t + C^{(J_t)}
\]

where, for \( J_t = j \), \( A^{(j)} \) and \( H^{(j)} \) are \( k \times k \) non-random matrix coefficients, \( C^{(j)} \) is a \( k \times 1 \) vector of constants, and \( \{\epsilon_t\} \) is a sequence of independent and identically distributed \( k \)-dimensional random vectors with zero mean and a covariance matrix.

The class of exponential autoregressive (EAR) models introduced by Gaver and Lewis (1980) and extended by Lawrance and Lewis (1980) may be regarded as a subclass of threshold models with \( \{J_t\} \) indepen-
dent of \( \{X_t\} \). For example the EAR(2) model is given by

\[ X_t = \epsilon_t + \begin{cases} 
\alpha_1 X_{t-1} & \text{with probability } 1 - \alpha_2 \\
\alpha_2 X_{t-2} & \text{with probability } \alpha_2 
\end{cases} \]

where \( 0 < \alpha_1, \alpha_2 < 1 \) and \( t = 0, \pm 1, \pm 2, \ldots \). This can be represented in threshold form as

\[ X_t = a^{(J_t)} X_{t-1} + b^{(J_t)} X_{t-2} + \epsilon_t \]

where \( \{J_t\} \) is a sequence of independently and identically distributed random variables with probability distribution given as

\[ J_t = \begin{cases} 1 & \text{with probability } 1 - \alpha_2 \\
2 & \text{with probability } \alpha_2. \end{cases} \]

Here \( \{J_t\} \) is independent of \( \{X_t\} \) and \( \{\epsilon_t\} \) and

\[ a^{(1)} = \alpha_1, \quad a^{(2)} = 0, \quad b^{(1)} = 0, \quad b^{(2)} = \alpha_2 \]

where \( \alpha_1 \) and \( \alpha_2 \) are constants.

### 1.3.13 Random coefficient autoregressive models

Nicholls and Quinn (1982) introduced the random coefficient autoregressive (RCA) models, which have a standard form given by

\[ X_t = \sum_{i=1}^{k} [\beta_i + B_i(t)] X_{t-i} + \epsilon_t, \]
where \( \{\epsilon_t\} \) is a sequence of independent and identically distributed random variables; \( \beta_i, i = 1, 2, \ldots, k \) are constants; and \( \{B_i(t)\} \) is independent of \( \{\epsilon_t\} \).

The new exponential autoregressive (NEAR) models of Lawrance and Lewis (1981) form a subclass of RCA models. Let \( \{J(t); t \in \mathbb{Z}\} \) and \( \{\epsilon_t; t \in \mathbb{Z}\} \) be two independent sequences of independent and identically distributed random variables. Suppose the distribution of \( J(t) \) is given by

\[
J(t) = \begin{cases} 
0 & \text{with probability } \alpha_0 \\
1 & \text{with probability } \alpha_1 \\
& \vdots \\
p & \text{with probability } \alpha_p
\end{cases}
\]

where \( \alpha_0, \alpha_1, \ldots, \alpha_p \) are non-negative and \( \sum_{j=0}^{p} \alpha_j = 1 \). Let \( \beta_0 = 0, \beta_1, \ldots, \beta_p \) be \( p + 1 \) constants such that \( 0 \leq \beta_j \leq 1, 1 \leq j \leq p \). Writing \( j \) for \( J(t) \), a formulation of NEAR model of order \( p \) can be given by

\[
X_t = \beta_j X_{t-j} + \epsilon_t
\]

where \( J(t) \) is independent of \( X_{t-1}, X_{t-2}, \ldots \), (see Chan (1988)).

### 1.3.14 Time reversibility

A stationary time series \( \{X_t\} \) is said to be time reversible if \( \{X_t\} \overset{d}{=} \{X_{-t}\} \). This means that for every positive integer \( n \) and every \( t_1, t_2, \ldots, t_n \in \mathbb{Z} \), the vectors \( (X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \) and \( (X_{-t_1}, X_{-t_2}, \ldots, X_{-t_n}) \) have the same joint distributions. It implies that the probabilistic properties of the time series do not depend on the direction of time. Oth-
erwise, we say the series is time irreversible or directional. Lawrance (1991) remarks that 'in time series analysis, reversibility seems to be more unrealistic, than has hitherto been realised'.

Weiss (1975) established that ARMA processes with an autoregressive component are time reversible if and only if they are Gaussian. Hence time irreversibility is a characteristic feature of non-Gaussian time series models.

Two stationary processes \( \{X_t\} \) and \( \{Y_t\} \) are said to be mutually time reversible if \( \{X_t\} \overset{d}{=} \{Y_{-t}\} \).

Chernick et al. (1988) established that the EAR(1) model of Gaver and Lewis (1980) and the exponential minification process of Tavares (1980) are mutually time reversible. Littlejohn (1992) proved that a discrete minification process is time reversible if and only if it has Bernoulli marginal distribution. It is also established that the geometric autoregressive process of McKenzie (1986) and the geometric minification process of Littlejohn (1992) are mutually time reversible.
Table 1.1

Numerical c.d.f.'s of the Mittag-Leffler distributions

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Figure 1.1
Cumulative probability plots for the Mittag-Leffler distribution