

Introduction

The theory of group algebra is a nice blend of two different disciplines namely group theory and ring theory which is now a highly developed independent branch of study in Modern Abstract Algebra. The techniques and methods involved in group algebra have a strong group theoretic flavour and there is a deep ring theoretic under current. This itself shows that any question about group algebra will lead to an equivalent question in ring theory and group theory. The fundamental work in group algebra was done by Graham Higman in 1940. The other major contributions had been given by Perlis, Walker, Deskins, Jennings, Connell, Coleman, Auslander, McLaughlin, Passman, Cohn, Livingston, Passi, Seghal, May, Karpilovsky etc.

The research work titled "A study of the group algebra of a cyclic group over a special class of rings" is based on a structure theorem of Graham Higman. We state the theorem in the modified form of the original theorem (Karpilovsky[17]). "If G is an abelian group with exponent n and R is an integral domain containing a primitive n \text{th} root of unity, then RG \cong R^n". This lead to the question, is there a ring other than an integral domain or a field which satisfies the above theorem? The answer is in the
affirmative. The ring, which we are in search of is called a Halidon ring. We generalise the concept of primitive $n^{th}$ root of unity in such a way that the definition is suitable for a ring and using this concept we define a Halidon ring. Then we study the necessary and sufficient condition that a given ring may be Halidon ring and under what conditions the given Halidon ring may be a semisimple ring. To prove the above theorem of Higman for Halidon rings, we use a tool from the theory of matrices called the circulants. Thus we were able to bridge a link between the theory of circulant matrices and group algebras.

In chapter - I, we discuss the developments in group algebra in the direction of units and idempotents. The study of units and idempotents is an important area of study in group algebra. In this short survey, we discuss the theorems from important research papers on units and idempotents. This survey is intended for getting an idea about the developments in the area of units and idempotents in group algebras. The famous papers include papers of Higman, Karpilovsky, D.S. Passman, James A. Cohn, Donald Livingstone, Walter Rudin, Hans Schneider, D.B.Coleman, Sudarshan K. Seghal, Raymond G. Ayoub, Christine Ayuob, I.Hughes, K.R. Pearson, Robert Sandling, R. Keith Dennis, Warren May, Masley, P.F. Smith, Jan Krempa,
K.R. Goodearl, M.M. Parmenter, W.B. Vasantha, Eric Jespers and Nair Alexandre Fernandes

The fundamentals of the theory of circulant matrices have been discussed in chapter – II. This chapter contains the definitions of a Permutation matrix, Fourier matrix, Circulant matrix, Representor of a circulant, Block Circulant matrix and Kronecker Product of matrices and some theorems related to necessary and sufficient conditions on circulant matrices and the formula for computing the inverse of a given circulant matrix. The important properties of Kronecker product of matrices have also been listed.

Chapter – III deals with an introduction to Halidon rings. We define primitive m th root of unity in a ring and a Halidon ring and some theorems on Halidon rings have been proved.

The main theorem, which is an extension of a theorem of G. Higman is proved in Chapter- IV. Also we prove a theorem which has an application to prove Lagrange and Wilson’s theorems and derive a formula for computing the inverse of an element and idempotents in a group algebra of a cyclic over Halidon rings. Another theorem we prove in this chapter is the that under what conditions an invertible element in a subalgebra of
given group algebra has its inverse again in the subalgebra. Then we derive a formula for computing units in an integral group ring of a cyclic group and construct another splitting of $U(\mathbb{Z}G)$ of a cyclic group of order 7.

The Chapter V is pertaining to the group algebras of dihedral, dicyclic, generalised quaternions and semidihedral groups over Halidon rings. In this Chapter we discuss the structure theorems, characterisations of group of units, formulae for computing units and idempotents and a special formula for computing units through respective reductions to group algebras of cyclic groups. The method adopted to have the characterisation of group of units in the integral group ring of dihedral are different from that of Fernandes [9] and Huges and Pearson[12] The characterisation of group of units in the integral group ring of other groups were also discussed. Also we have a theorem which gives the structure of unit group of integral group rings of above groups.

Finally we give another proof of the classical theorems of Lagrange and Wilson in number theory in Chapter VI.

An appendix has been prepared to exhibit the derivation of the matrix of an element under regular representations of different group algebras of groups under
consideration. The construction of idempotents and computation of units are also exhibited.

The appendix succeeds the references and list of symbols used.

LIST OF PUBLICATIONS