CHAPTER II

α - INVERTED STABLE DISTRIBUTION AND RELATED LAWS

2.1 INTRODUCTION

Among the class of stable distributions, the distribution with positive support and totally skewed to the right is considered here. That is the stable distribution with Laplace transform \( \exp(-\lambda^\alpha) \), \( \lambda \geq 0 \), \( 0 < \alpha \leq 1 \) or equivalently the characteristic function \( \exp(-(-i\tau)^\alpha) \). Here we introduce a distribution as a function of the stable random variable having the above cited Laplace transform. We call the distribution as \( \alpha \)-inverted stable. (Samuel et al. (1983)). The Laplace transform of this distribution is a Mittag-Leffler function (Feller (1966), Page 428). Mittag Leffler function and the related distributions are discussed by Pillai (1990a). Fujita (1993) generalized the Mittag Leffler distribution and obtained a characterization of the geometrically infinitely divisible distributions in (0,\( \infty \)).

The half Cauchy distribution is obtained as the product of the stable distribution with Laplace transform \( \exp(-\lambda^\alpha) \) and the \( 1/\alpha \)th power of the \( \alpha \)-inverted stable distribution (or equivalently the
quotient of two identically distributed, not necessarily independent stable random variables with Laplace transform \( \exp(-\lambda^a) \). The Laplace transform of the above product is a Mittag-Leffler function. We have obtained the p.d.f corresponding to this Laplace transform. It is found that which is the p.d.f of the half Cauchy distribution with a special reparametrization. This distribution has considerable importance in statistical literature. Kozubowski (1998b, 1998c, 1998d) used this distribution to obtain mixture representations for strictly geometric stable laws, Linnik and Mittag-Leffler distributions. But the results can be obtained easily by applying the Laplace transform technique without going into deep algebra or any supporting lemmas. The \( \alpha \)-inverted stable distribution has considerable applications in theoretical as well as in applied fields. Some of the applications are discussed in chapter III.

**DEFINITION 2.1.1**

Let \( X \) be the stable random variable with characteristic function

\[
\phi(t) = \exp(-it^a) ; \quad 0 < \alpha \leq 1, \quad -\infty < t < \infty
\]

Then the distribution represented by the random variable

\[
Y = \left( \frac{\sigma}{X} \right)^\alpha, \quad \sigma > 0
\]
is called the $\alpha$-inverted stable distribution, $\sigma$ is the scale parameter. Feller (1966, pp 428) expressed the distribution function of $Y$ in terms of that of $X$ and stated that $Y$ has the Laplace transform, the Mittag-Leffler function

$$\gamma(\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k \sigma^{k\alpha}}{\Gamma(1+k\alpha)}$$

(2.1.1)

Let $G(x)$ and $F(x)$ be the distribution function of $Y$ and $X$ respectively then

$$G_Y(x) = 1 - F_x\left(\frac{\sigma}{x^{1/\alpha}}\right)$$

Here

$$X \sim S_{\alpha}\left(\left(\cos\frac{\pi}{2}\right)^{1/\alpha},1,0\right)$$

This chapter is organized as follows: section 2.2 consists of the elementary properties of the distribution. Section 2.3 deals with a characterization theorem of $\alpha$-inverted stable distribution and some distributional properties which include the inter-relationships among exponential, half Cauchy, Mittag-Leffler, stable and $\alpha$-inverted stable distributions. In section 2.4 we obtained the mixture representation of the p.d.f of the Mittag-Leffler distribution in terms of the p.d.f of the exponential and half Cauchy distributions. In section 2.5 we shown
that the $\alpha$-inverted stable distribution is normally attracted to the positive stable law.

2.2 SOME ELEMENTARY PROPERTIES

Here we discuss some elementary properties of the $\alpha$-inverted stable distribution. Let $F(x)$, $f(x)$ and $G(y)$, $g(y)$ denote the distribution function and density function of the random variables $X$ and $Y$ respectively. (Feller (1966)). Then

$$F(x) = 1 + \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(k\alpha)}{k!} x^{-k\alpha} \sin(\pi k\alpha)$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k! \Gamma(1 - k\alpha)} x^{-k\alpha}$$

$$f(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(k\alpha + 1)}{k!} x^{-(k\alpha + 1)} \sin(\pi k\alpha); \quad x > 0$$

$$0 < \alpha \leq 1$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k! \Gamma(-k\alpha)} x^{-(k\alpha + 1)}$$

and $G(y) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \Gamma(k\alpha) y^k \sigma^{-k\alpha} \sin(\pi k\alpha)$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k! \Gamma(1 - k\alpha)} y^k \sigma^{-k\alpha}$$
\[ g(y) = \frac{1}{\pi \alpha} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \Gamma(k\alpha + 1)}{k!} \sigma^{-k\alpha} y^{k-1} \sin(\pi k\alpha) \]

\[ = \frac{1}{\alpha} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} y^{k-1} \sigma^{-k\alpha}}{k! \Gamma(-k\alpha)} \]

**REMARKS 2.2.1**

(i) \[ g(y) \rightarrow e^{-y}, y > 0 \text{ as } \alpha \downarrow 0 \]

(ii) When \( \alpha = \frac{1}{2} \)

\[ g(y) = \frac{\sigma^{-\alpha}}{\Gamma(1 - \alpha)} \exp \left( -\frac{y^2}{4\sigma} \right); y > 0 \]

That is \( \frac{1}{2} g(y) \) is truncated normal with mean \( \mu = 0 \) and variance \( \sigma/2 \)

(iii) \[ g(y) \rightarrow \frac{\sigma^{-\alpha}}{\Gamma(1 - \alpha)} \text{ as } y \downarrow 0 \]

(iv) \[ y^{-1} G(y) \rightarrow \frac{\sigma^{-\alpha}}{\Gamma(1 - \alpha)} \text{ as } y \downarrow 0 \]

(v) \[ g(y) \approx \frac{y^{2\alpha-1}}{[2\pi(1 - \alpha)\sigma^{\alpha/(1 - \alpha)}]^{1/2}} \exp \left[ -\left(1 - \alpha\right)\alpha^{1-\alpha} \left(\frac{y}{\sigma} \right)^{\frac{1}{1-\alpha}} \right] \]

; \( y > \sigma^\alpha \)

(vi) When \( \alpha = 1/3 \)

\[ yg(y) = \frac{3}{\pi} \sin \left( \frac{\pi}{3} \right) z J_{1/3}(z) \]
where \( z = \left( \frac{y}{3\sigma^{1/3}} \right)^{3/2} \) and \( J_\nu(z) \) is the modified Bessel function of order \( \nu \).

(vii) When \( \alpha = 3/4 \)

\[
g(y) = -\left( \frac{128}{27} \right)^{1/2} \sum_{k=1}^{3} \sin\left( \frac{3\pi k}{4} \right) \omega^k
\]

\[
2F_2 \left( \frac{1}{3} + \frac{n}{4}, \frac{3}{2} + \frac{n}{4}, \frac{1}{2}; \frac{n(n-1)}{8}, \frac{n(7-n)}{8}, -\omega^4 \right)
\]

where \( \omega = \frac{-y}{4} \left( \frac{3}{\sigma} \right)^{3/4} \) and \( 2F_2 \) is the generalized hypergeometric function.

(viii) The characteristic function of \( Y \) is \( \mathbb{E} e^{i\alpha t Y} \), \( \alpha \in \mathbb{R} \).

(ix) If the random variable \( Y \) is \( \alpha \)-inverted stable with parameters \( \alpha \) and \( \sigma \). Then \( Z = \alpha Y \) is \( \alpha \)-inverted stable with parameters \( \alpha \) and \( a^{1/\alpha} \sigma \) (\( a > 0 \), is a constant).

(x) Let \( \mu_r \) be the \( r \)th order moment about zero

\[
\mu_r = \frac{r! \sigma^r}{\Gamma(1 + r\alpha)}
\]
The mean and variance are given by

$$\mu_1 = \frac{\sigma^\alpha}{\Gamma(1 + \alpha)}$$

and variance

$$\mu_2 = \sigma^{2\alpha} \left( \frac{1}{\alpha \Gamma(2\alpha)} - \frac{1}{(\alpha \Gamma(\alpha))^2} \right)$$

The moment measure of skewness

$$\beta_1 = \left[ \frac{2}{\alpha \Gamma(3\alpha)} - \frac{3}{\alpha^2 \Gamma(\alpha) \Gamma(2\alpha)} + \frac{2}{\alpha^3 (\Gamma(\alpha))^3} \right]^2 \frac{1}{\left[ \frac{1}{\alpha \Gamma(2\alpha)} - \frac{1}{(\alpha \Gamma(\alpha))^2} \right]^3}$$

The moment measure of kurtosis

$$\beta_2 = \left[ \frac{6\alpha^3}{\Gamma(4\alpha)} - \frac{8\alpha^2}{\Gamma(\alpha) \Gamma(3\alpha)} + \frac{6\alpha}{\Gamma(\alpha) \Gamma(2\alpha)} - \frac{3}{\Gamma(\alpha)^4} \right] \left[ \frac{\alpha}{\Gamma(2\alpha)} - \frac{1}{(\Gamma(\alpha))^2} \right]^2$$

(xi) The characteristic function of the $\alpha$-inverted stable distribution is analytic in the sense that the moments of all order exists and they are bounded by $r! \eta^r$, where $\eta$ is a positive number.

That is

$$|\mu_r| \leq r! \eta^r$$
The moments obey the recurrence relation

\[ \mu_{r+1} = \frac{(r\sigma^\alpha \Gamma(r\alpha))}{\Gamma((r+1)\alpha)} \mu_r, \quad r = 1, 2, \ldots \]

where \( \mu_r \) is the \( r \)th order moment about the origin zero.

### 2.3 Certain Distributional Properties

In this section we discuss a characterization theorem and some distributional properties of the \( \alpha \)-inverted stable distribution. The distributional properties include the inter-relationships among exponential, half Cauchy (with a special reparametrization), Mittag-Leffler, stable and \( \alpha \)-inverted stable distributions.

**THEOREM 2.3.1**

If \( W \) is exponential with mean one and \( Y \) is \( \alpha \)-inverted stable \((\sigma = 1)\), then

\[ W \overset{d}{=} W^\alpha Y \]

**PROOF**

Let \( \phi_w(t) \) be the characteristic function of \( W^\alpha Y \)

Consider

\[ \phi_w(t) = E(e^{itw^\alpha Y}), \]
\[ \mathbb{E}_W \mathbb{E}(e^{itW^\alpha Y}/W), \]
\[ = \mathbb{E} \sum_{k=0}^{\infty} \frac{(itW^\alpha)^k}{\Gamma(1 + k\alpha)}, \]
\[ = \sum_{k=0}^{\infty} (it)^k, \]
\[ = \frac{1}{1 - it}. \]

The right side is the characteristic function of the exponential distribution with mean one. Hence the result.

**REMARK 2.3.1**

If \( W \) is exponential with mean \( \theta \) (\( \theta > 0 \)) and \( Y \) is \( \alpha \)-inverted stable with parameter \( \sigma \), then \( W^\alpha Y \) is exponential with mean \( (\theta/\sigma)^\alpha \).

**THEOREM 2.3.2**

If \( X \) is distributed as exponential with mean one and \( Y \) is \( \alpha \)-inverted stable, then

\[ Z = \left( \frac{X}{Y} \right)^{1/\alpha}, \]

is Mittag-Leffler.
PROOF

Let \( G(y) \) be the distribution function of the random variable \( Y \). Consider \( P(Z \leq z) = P\left( \left( \frac{X}{Y} \right)^{\frac{1}{\alpha}} \leq z \right) \)

\[
= 1 - P\left( \frac{Y}{X} \leq \left( \frac{1}{z} \right)^{\alpha} \right)
= 1 - \int_0^\infty G\left( \frac{x}{z^\alpha} \right) e^{-x} \, dx
\]

\[
= 1 - \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k\alpha)}{k!} \sigma^{-k\alpha} \sin \pi k\alpha \left( \frac{x}{z^\alpha} \right)^k \, dx
= 1 - \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k\alpha)}{k!} \sigma^{-k\alpha} \sin \pi k\alpha z^{-k\alpha} \Gamma(k+1) \, dx
= 1 - \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (z\sigma)^{-k\alpha}}{\Gamma(1-k\alpha)}
= 1 - \sum_{k=0}^{\infty} \frac{(-1)^k (z\sigma)^{k\alpha}}{\Gamma(1+k\alpha)}
= 1 - E_\alpha(-(z\sigma)^\alpha)
= F_\alpha(z)
\]

where \( F_\alpha(x) \) is the Mittag-Leffler distribution function with parameters \( \alpha \) and \( \sigma \). Hence the theorem.
THEOREM 2.3.3

Let \( Y_1, Y_2, \ldots, Y_n \) be independent random variables with positive support such that for \( 0 < \alpha < 1 \)

\[
\left( \frac{1}{Y_1} \right)^{1/\alpha} + \left( \frac{1}{Y_2} \right)^{1/\alpha} + \ldots + \left( \frac{1}{Y_n} \right)^{1/\alpha}
\]

is stable if and only if \( Y_1, Y_2, \ldots, Y_n \) are \( \alpha \)-inverted stable.

PROOF

Let \( S = \left( \frac{1}{Y_1} \right)^{1/\alpha} + \left( \frac{1}{Y_2} \right)^{1/\alpha} + \ldots + \left( \frac{1}{Y_n} \right)^{1/\alpha} \)

be stable. Since \( S \) is stable, each of the component random variables is stable. By the definition of \( \alpha \)-inverted stable distribution \( Y_1, Y_2, \ldots, Y_n \) are \( \alpha \)-inverted stable.

Conversely suppose that \( Y_1, Y_2, \ldots, Y_n \) are \( \alpha \)-inverted stable, then \( (1/Y_1)^{1/\alpha}, (1/Y_2)^{1/\alpha}, \ldots, (1/Y_n)^{1/\alpha} \) are stable. Hence \( S \) is stable.

THEOREM 2.3.4

If the random variable \( X \) follows stable distribution with Laplace transform \( \exp(-\lambda^\alpha) \) and \( Y \) is Mittag-Leffler with Laplace transform \( \frac{1}{1 + \lambda^\alpha} \), then
PROOF

Let \( \gamma_z(\lambda) \) be the Laplace transform of the random variable \( Z \).

Consider

\[
\gamma_z(\lambda) = \mathbb{E}(e^{-\lambda Z})
\]

\[
= \mathbb{E}_Y \mathbb{E}[e^{-(\lambda Y^{1/\alpha})} \mid Y] 
\]

\[
= \mathbb{E}(e^{x_Y})
\]

\[
= \frac{1}{1 + \lambda \alpha^2}.
\]

Since the Laplace transform uniquely determines a distribution in \((0, \infty)\), \(Z = XY^{1/\alpha}\) is Mittag-Leffler with parameter \(\alpha^2\).

Hence the theorem

THEOREM 2.3.5

If \( X \) is stable with Laplace transform \( \exp(-\lambda^\alpha) \) and \( Y \) is \( \alpha \)-inverted stable, then

\[
Z = XY^{1/\alpha},
\]

is half Cauchy with a special reparametrizaton.
PROOF

First we find the Laplace transform of $Z$, then we derive the density function. Let $g(y)$ be the probability density function of $Y$

$$g(y) = \frac{1}{\pi \alpha} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k\alpha + 1)}{k!} y^{k-1} \sigma^{-k\alpha} \sin \pi k\alpha .$$

Let $h(u)$ be the probability density function of the random variable $U=Y^{1/\alpha}$

$$h(u) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k\alpha + 1)}{k!} u^{k\alpha-1} \sigma^{-k\alpha} \sin \pi k\alpha ,$$

Let $\gamma_u(\lambda)$ be the Laplace transform of $U=Y^{1/\alpha}$

$$\gamma_u(\lambda) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k\alpha + 1)}{k!} \sin \pi k\alpha \frac{\Gamma(k\alpha)}{\lambda^{k\alpha}} \sigma^{-k\alpha} .$$

Let $l_z(\lambda)$ be the Laplace transform of $Z=XY^{1/\alpha}$

By Parseval relation

$$l_z(\lambda) = \int_{0}^{\infty} \gamma'_x(\lambda u)h(u)du,$$

where $\gamma'$ is the Laplace transform of $X$

$$l_z(\lambda) = \int_{0}^{\infty} e^{(-\lambda u)^{\alpha}} \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k\alpha + 1)}{k!} u^{k\alpha-1} \sigma^{-k\alpha} \sin \pi k\alpha .$$

By the substitution $V=U^\alpha$
$$
l_z(\lambda) = \frac{1}{\pi \alpha} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k\alpha + 1)}{k!} \sin\pi k\alpha \frac{\Gamma(k)}{\lambda^{k\alpha}} \sigma^{-k\alpha},
$$

$$
= -\sum_{k=1}^{\infty} (-1)^k \frac{\lambda^{-k\alpha} \sigma^{-k\alpha}}{\Gamma(1 - k\alpha)},
$$

$$
= \sum_{k=0}^{\infty} (-1)^k \frac{(\lambda \sigma)^{k\alpha}}{\Gamma(1 + k\alpha)},
$$

$$
= E_\alpha (-\lambda \sigma^\alpha).
$$

**REMARK 2.3.2**

(i) An alternative proof using Laplace transform technique is given in the appendix.

(ii) Note that $1 - E_\alpha (-\lambda \sigma^\alpha)$ is the Mittag-Leffler distribution with parameters $\alpha$ and $\sigma$.

(iii) The characteristic function $\phi(t)$ of $Z$,

$$
\phi(t) = \sum_{k=0}^{\infty} (-1)^k \frac{(-it\sigma)^{k\alpha}}{\Gamma(1 + k\alpha)}
$$

**THEOREM 2.3.6**

If $\phi(t) = \sum_{k=0}^{\infty} (-1)^k \frac{(-it\sigma)^{k\alpha}}{\Gamma(1 + k\alpha)}$, the corresponding pdf

$$
g(z) = \frac{\sigma^\alpha}{\pi} \frac{z^{\alpha-1} \sin \pi \alpha}{z^{2\alpha} + 2\sigma^\alpha z^\alpha \cos \pi \alpha + \sigma^{2\alpha}}; \quad z > 0,
$$
which is the pdf of the half Cauchy distribution with a special reparametrization.

PROOF

By inversion theorem

\[ g(z) = \text{Re} \frac{1}{\pi} \int_0^\infty e^{-izt} \phi(t) \, dt ; \] where \( \text{Re} \) denotes the real part,

\[
= \text{Re} \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{(-i\sigma)^{\alpha}}{\Gamma(1 + \alpha)} \frac{e^{-izt^{\alpha}}}{t^{\alpha+1}} \, dt \\
= \text{Re} \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \left( \frac{-i}{i} \right)^{\alpha} \frac{\sigma^{\alpha}}{Z^{\alpha+1}} \left( \frac{1}{1} \right) \\
= \text{Re} \frac{-i}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\sigma^{\alpha}}{Z^{\alpha+1}} \left( \cos \pi k \alpha - i \sin \pi k \alpha \right) \\
= \text{Re} \frac{-1}{\pi Z} \sum_{k=0}^{\infty} (-1)^k \frac{\sigma^{\alpha}}{Z^{\alpha+1}} \sin \pi k \alpha \\
= \frac{-\sigma^{\alpha}}{\pi} \frac{z^{\alpha-1} \sin \pi \alpha}{Z^{\alpha} + 2\sigma^{\alpha}Z^{\alpha} \cos \pi \alpha + \sigma^{2\alpha}} ; \quad z > 0 .
\]

REMARK 2.3.3

(i) when \( \sigma = 1 \)
\[ g(z) = \frac{1}{\pi} \frac{z^{a-1}\sin\pi\alpha}{z^{2a} + 2z^a \cos\pi\alpha + 1}; \quad z > 0 \]

(ii) \( Z \) is the ratio of two identically distributed random variables having the Laplace transform \( \exp(-\lambda^a) \), not necessarily independent. Hence \( Z \) and \( 1/Z \) has the same distribution.

(iii) the random variable \( W=Z^\alpha \) has the probability density function

\[ h(w) = \frac{\sigma^\alpha}{\pi\alpha} \frac{\sin\pi\alpha}{w^2 + 2\sigma^a w \cos\pi\alpha + \sigma^{2a}}; \quad w > 0 \]

when \( \sigma = 1 \)

\[ h(w) = \frac{1}{\pi\alpha} \frac{\sin\pi\alpha}{w^2 + 2w \cos\pi\alpha + 1}; \quad w > 0 \]

(iv) The d.f. \( G(z) \) of the r.v. \( Z \) is;

\[ G(z) = \frac{1}{\pi\alpha} \tan^{-1} \left( \frac{\sin\pi\alpha}{\left(\frac{\sigma}{z}\right)^a + \cos\pi\alpha} \right) \]

(v) \[ \int_0^\alpha g(z)dz = \frac{1}{2} \]

Hence \( \sigma \) is the median
Kotz and Ostrovski (1996) proved that the value of the integral for $0 < p < 1, 0 < \alpha < 2$

\[
\frac{t^\alpha}{\pi \sin \pi \rho} \int_0^\infty \frac{\tau^{\alpha-1} d\tau}{(1 + \tau)(\tau^{2\alpha} + \tau^{2\alpha} + 2 \tau^{\alpha} \tau^{\alpha} \cos \pi \rho)} = \frac{1}{1 + t^\alpha}
\]  
(2.3.1)

Therefore

\[
\int_0^\infty \frac{1}{1 + z} \frac{\sigma^\alpha \pi^{\alpha-1} \sin \pi \alpha}{1 + \sigma^\alpha z + 2 \sigma^\alpha \pi \alpha \cos \pi \alpha} dz = \int_0^\infty \frac{1}{1 + z} g(z) dz
\]

That is,

\[
E \left( \frac{1}{1 + Z} \right) = \frac{1}{1 + \sigma^\alpha}
\]

**PROPOSITION 2.3.1**

Let $X \sim S_{\alpha} \left( \cos \frac{\pi \alpha}{z}, 1, 0 \right)$ with $0 < \alpha' < 1$ and let $0 < \alpha < \alpha'$. Let

\[
A \sim S_{\alpha'} \left( \cos \frac{\pi \alpha}{2 \alpha'}, 1, 0 \right).
\]

Then

\[
Z = A^{\alpha'} X \sim S_{\alpha} \left( \cos \frac{\pi \alpha}{2} \right)^{\alpha}, 1, 0).
\]
PROOF

Let $\gamma(\lambda)$ be the Laplace transform of $Z$.

$$\gamma(\lambda) = E(\exp(-\lambda A^{\nu} X)),$$

$$= E(E(\exp(-(\lambda A^{\nu})X)/A)),$$

$$= E(\exp(-\lambda^{\nu} A)),

= \exp(-\lambda^{\nu}).$$

Hence the result.

**THEOREM 2.3.7**

If $V$ and $W$ are independent inverted stable random variables with parameters $(\alpha', \sigma_1)$ and $(\alpha/\alpha', \sigma_2)$, then

$$U = V^{\nu} W,$$

is inverted stable with parameters $(\alpha, \sigma_1^{\nu} \sigma_2^{\nu})$.

**PROOF**

Since $V$ and $W$ are inverted stable random variables with parameters $(\alpha', \sigma_1)$ and $(\alpha/\alpha', \sigma_2)$, for $\sigma_1, \sigma_2 > 0$ there exist stable random variables $X$ and $A$ such that
\[ V = \left( \frac{\sigma_1}{X} \right)^\alpha \quad \text{and} \quad W = \left( \frac{\sigma_2}{A} \right)^\alpha, \]

Where \( S_\alpha \left( \cos \frac{\pi \alpha}{2}, 1, 0 \right) \) and \( A \sim S_\sigma \left( \cos \frac{\pi \alpha}{2}, 1, 0 \right) \)

Consider \( U = V^{\alpha/\lambda} \cdot W \)

\[ = \left( \frac{\sigma_1}{X} \right)^\alpha \left( \frac{\sigma_2}{A} \right)^{\alpha/\lambda} \]
\[ = \left( \frac{\sigma_1 \sigma_2^{\alpha/\lambda}}{X A^{\alpha/\lambda}} \right)^\alpha \]

By proposition 2.3.1, \( Z = A^{\alpha/\lambda} X \sim S_\alpha \left( \cos \frac{\pi \alpha}{2}, 1, 0 \right) \)

Hence \( U = \left( \frac{\sigma_1 \sigma_2^{\alpha/\lambda}}{Z} \right)^\alpha \)
\[ = \left( \frac{\sigma}{Z} \right)^\alpha, \]

where \( \sigma = \sigma_1 \sigma_2^{\alpha/\lambda} \).

By definition \( U \) is \( \alpha \)-inverted stable with parameter \( \sigma \).

This completes the proof.
2.4 MIXTURE REPRESENTATIONS

Kozubowski (1998c) used the pdf of the half Cauchy distribution to obtain mixture representation for Mittag-Leffler and Linnik distribution by evaluating the integral (2.3.1). But the proof can be made simple and easy by using the Laplace transform technique.

THEOREM 2.4.1

If \( T \) is exponential with mean one and \( Z = X Y^{1/\alpha} \) has Laplace transform

\[
\ell_Z(\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{\alpha k}}{\Gamma(1 + k\alpha)}
\]

Then,

\[
M \mid T \overset{d}{=} T Z \overset{d}{=} (T / Z)
\]

where \( M \) is Mittag-Leffler, \( X \) is stable with Laplace transform \( \exp(-\lambda^\alpha) \) and \( Y \) is \( \alpha \)-inverted stable.

PROOF

Let \( \gamma_M(\lambda) \) be the Laplace transform of \( M \)

\[
\gamma_M(\lambda) = \mathbb{E}(e^{\lambda M}),
\]

\[
= \mathbb{E} \left( e^{\lambda X T / T} \right),
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{\alpha k}}{\Gamma(1 + k\alpha)} \mathbb{E}(T^{\alpha k}),
\]

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The right side is the Laplace transform of a Mittag-Leffler distribution with parameter \( \alpha \).

Hence \( M \overset{d}{=} TZ \)

Since \( Z \) and \( \frac{1}{Z} \) has the same distribution, as the quotient of two identical random variable

\[
M \overset{d}{=} TZ \overset{d}{=} \left( \frac{T}{Z} \right)
\]

Using the above relation we can obtain the distribution function \( F_M(.) \) of \( M \) as a mixture of exponential and \( Z \). By Luckas(1970)

\[
F_M(x) = \int (1 - e^{-z}) \, dG(z),
\]

where \( G \) is the distribution function of the random variable \( Z \).

\[
=1-\frac{1}{\pi} \int_0^\infty e^{-zx} \frac{z^{\alpha-1} \sin \pi \alpha}{z^{2\alpha} + 2z^\alpha \cos \pi \alpha + 1} \, dz
\]

Differentiating the above equation with respect to \( x \), we get the density of \( M \).
We can get a similar mixture representation for the Linnik distribution for $0 < \alpha < 1$ as above. If $L$ is a Linnik random variable with parameter $\alpha (0 < \alpha < 1)$, then

$$L \overset{d}{=} T \overset{d}{=} \frac{Z}{T}. $$

2.5 $\alpha$-INVERTED STABLE DISTRIBUTION IS NORMALLY ATTRACTED TO POSITIVE STABLE LAW

Like the $\alpha$-Poisson distribution, the $\alpha$-inverted stable distribution is normally attracted to positive stable law.

PROOF

We have to prove that if $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed $\alpha$-inverted stable random variables, then there exists a sequence $\{a_n, n \geq 1\}$ of reals and a sequence $\{b_n, n \geq 1\}$ of positive reals such that $\frac{1}{b_n} (X_1 + X_2 + \ldots + X_n) - a_n$ converges in distribution to the positive stable law.

Let $S_n = X_1 + X_2 + \ldots + X_n$ and $\phi(\lambda)$ be the Laplace transform of $X_1, X_2, \ldots, X_n$. 

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Consider \( \gamma_{Sn} (\lambda) = \left[ \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k}{n^{\alpha k}} \right]^n \).

\[
\log \phi_{\lambda} (\lambda) = n \log \left[ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k}{n^{\alpha k}} \right] = n \left[ \frac{-\lambda \sigma^\alpha}{n^{\alpha (1 + \alpha)}} + O(1/n^2) \right]
\]

\[
= \frac{-\lambda \sigma^\alpha}{\Gamma(1 + \alpha)} + O(1/n).
\]

Hence \( \lim_{n \to \infty} \gamma_{Sn} (\lambda) = \exp \left( \frac{-\lambda \sigma^\alpha}{\Gamma(1 + \alpha)} \right) \)

Hence the result.

This is an example of the applications of \( \alpha \)-inverted Mittag-Leffler distribution. The first passage time of \( \alpha \)-stable Lévy motion is shown to be the \( \alpha \)-inverted stable distribution. Further areas of applications like in renewal theory, random walk with infinite mean waiting time and in relaxation phenomena are discussed. The number of renewal epochs in \( (\alpha, \alpha) \) follows asymptotically \( \alpha \)-inverted stable distribution when the sum of random variables have infinite expectation. The random walk with infinite mean waiting time (Fractal time random walk) is asymptotically \( \alpha \)-inverted stable. The Laplace transform of the \( \alpha \)-inverted stable distribution is the Mittag-Leffler function \( \gamma(\lambda) \) as in (2.1.1) and \( 1 - \gamma(\lambda) \) is the Mittag-Leffler distribution as defined in Pillai (1990a). Thus both the \( \alpha \)-inverted stable distribution and the Mittag-