CHAPTER III

MULTIVARIATE α-LAPLACE DISTRIBUTIONS

3.1 INTRODUCTION

The univariate α-Laplace distribution has characteristic function $1/(1+|t|^\alpha)$, $0 < \alpha \leq 2$, $t \in \mathbb{R}^1$. Noting that any unimodal probability density in $\mathbb{R}^n$ is a projection of some spherically symmetric probability density in $\mathbb{R}^{n+2}$ (Zolatarev, 1981), the unimodality of the class of α-Laplace distributions suggests the existence of a spherically symmetric distribution in $\mathbb{R}^3$ and the natural extension is the characteristic function

$$f_{n,\alpha}(t) = (1/(1+|t|^\alpha)), \quad 0 < \alpha \leq 2, \quad t = (t_1, t_2, \ldots, t_n)^T \in \mathbb{R}^n, \quad n \geq 1.$$  

Because of the spherical symmetry we can write

$$f_{n,\alpha}(t) = \chi_{n,\alpha}(|t|), \quad t \in \mathbb{R}^n$$  

(3.1.1)

where $\chi_{n,\alpha}(s)$ is a real function defined on the half line $s \geq 0$. Also, the corresponding $n$-dimensional density $g_{n,\alpha}(x), \quad x \in \mathbb{R}^n$ can be written as

$$g_{n,\alpha}(x) = h_{n,\alpha}(x), \quad x = |x|, \quad x \in \mathbb{R}^n$$  

(3.1.2)

where $h_{n,\alpha}(x)$ is a real function defined on the non-negative real line.

An important property of the spherically symmetric distribution (Kelker (1970), Zolatarev (1981)) is that the multidimensional densities
can be obtained from the corresponding one dimensional densities using the recurrence relation

$$h_{n+2,a}(x) = -2 \pi h_{n+1,a}(x)$$

(3.1.3)

and

$$h_{2,a}(x) = -\frac{1}{\pi} \int_1^\infty (t^2 - 1)^{-\frac{1}{2}} h_{1,a}(tx) \, dt$$

(3.1.4)

where \( h_{n,a}(x) \) is the density of \( X \in \mathbb{R}^n \).

Some results on spherical symmetry are discussed in section 2. The density and distribution functions of the multivariate \( a \)-Laplace distribution for some rational values of \( a > 1 \) are given in section 3, in terms of the Meijer's \( G \) - function.

The \( a \)-Laplace random variables admit certain representations in terms of the stable and exponential random variables and Zolatarev (1981) has given explicit formulae for the absolute moments of order \( \delta (0 < \delta < a) \) of multivariate spherically symmetric stable distributions. Using these results, the absolute moments of order \( \delta (0 < \delta < a) \) for the multivariate \( a \)-Laplace distributions of exponent \( a \) are obtained in section 4.

In section 5 we show that the characteristic function of the \( a \)-Laplace distribution is i.i.d. It can be seen that these distributions are normally attracted to the multivariate stable law. In section 6 we claim that the multivariate \( a \)-Laplace distribution is unimodal.
which follows from the fact that this distribution is spherically symmetric for any \( n \geq 1 \).

### 3.2 SPHERICALLY SYMMETRIC DISTRIBUTIONS

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space and \( \mathbf{X} = (X_1, X_2, \ldots, X_n)' \) be an \( nx1 \) random vector with spherical symmetry. Its density will be a function of the norm of \( \mathbf{X} \):

\[
X = |\mathbf{X}| = \sqrt{\sum_{i=1}^{n} X_i^2}
\]

(3.2.1)

It is well known that its characteristic function

\[
f_n(t) = E \left( e^{it \cdot \mathbf{X}} \right) = E \left( e^{itX \cos \theta} \right)
\]

where \( t \) is the norm of the vector \( \mathbf{t} \) and \( \theta \) is the angle between \( \mathbf{X} \) and \( \mathbf{t} \). Since all directions are equally probable, the distribution of the magnitude is independent of \( \theta \). Several authors define \( \mathbf{X} \) has a spherically symmetric distribution if its characteristic function \( f_n(t) \) is a function of \( t = |\mathbf{t}| \) alone, i.e., the characteristic function is itself spherical. Therefore we can write

\[
f_n(t) = \chi_n(t)
\]

(3.2.2)

where \( t = |\mathbf{t}| \), \( \chi_n \) is a real function defined on the half line \( t \geq 0 \).
Let $\mathbf{y} \in \mathbb{R}^n$ has the uniform distribution over the surface of the unit sphere $\{ \mathbf{y} : |\mathbf{y}| = 1, \mathbf{y} \in \mathbb{R}^n \}$ and $R$ be a random variable with positive support and independent of $\mathbf{y}$. Then $\mathbf{x}$ admits the stochastic representation

$$\mathbf{x} \overset{\text{d}}{=} R\mathbf{y}$$

(3.2.3)

Several properties which are known only for the normal distribution are common to all spherical distributions. Let $\mathbf{x}$ has a spherical distribution with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{I}$. Kelker (1970) has proved that if $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, with $\mathbf{x}_1$ and $\mathbf{x}_2$ vectors of $n_1$ and $n_2$ components respectively, then

$$U = \frac{|\mathbf{x}_1|^2}{|\mathbf{x}_1|^2 + |\mathbf{x}_2|^2} = \frac{|\mathbf{x}_1|^2}{|\mathbf{x}|^2}$$

has Beta distribution Beta $(n_1, n_2)$.

It is also well known that a spherical distribution and its projections have the same characteristic function which means that the projections of spherical distributions are again spherical and do not differ in their functional form (Laurent, 1975).

The characteristic function of the spherical distribution

$$f_n(t) = E(e^{it\mathbf{x}} \cos \theta)$$

$$= \int_0^\infty x^{n-1} h_n(x) \, dx \int e^{itx} \cos \theta \, dS_n$$

where $S_n$ is the surface of the unit sphere in $n$-dimensions.
where $S_n$ is the surface of the unit sphere in $n$-dimensions. Using the transformations

$$x_1 = x \cos \theta_1$$
$$x_2 = x \sin \theta_1 \cos \theta_2$$
$$\vdots$$
$$x_{n-1} = x \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1}$$
$$x_n = x \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1}$$

$$\chi_n(t) = 2\pi \frac{n}{2} - \frac{n}{2} + 1 \int_0^\infty x^n J_{\frac{n}{2}-1}(tx)dx$$  \hspace{1cm} (3.2.4)$$

where $J_\nu(.)$ is the Bessel function of the first kind of order $\nu$. The result (3.2.4) enables us to solve the problems involving spherically symmetric distributions by the Hankel transform technique.

3.3 DENSITY AND DISTRIBUTION FUNCTIONS

The characteristic function $f_{n,\alpha}(t) = 1/(1+|t|^\alpha)$, $0 < \alpha \leq 2$, $t \in \mathbb{R}^n$ is a function of $|t|$ alone and therefore it is spherically symmetric. For $n=1$, Pillai (1985) has characterized this class as $\alpha$-Laplace distributions. Since $f_{n,\alpha}(t)$ is spherically symmetric, the Hankel transform technique may be used to derive the densities and distributions.
Here we obtain the densities for \( n < 1 + 2 \alpha \), for rational values of \( \alpha (0 < \alpha \leq 2) \) and are expressed in terms of Meijer's \( G \) function.

**THEOREM 3.3.1** The \( n \)-dimensional density of the \( \alpha \)-Laplace distribution is given by

\[
\begin{align*}
    h_{n, \alpha}(x) &= \frac{2}{\pi} \frac{n - q}{2 - q} x^{\frac{1}{2} - 1} \times G^q_{q,pq} \left( \frac{2p}{x} \right) \\
    &= \frac{\Delta (p, \frac{n}{4} - \frac{n}{2p}) \Delta (q, \frac{(n-2)q}{4p})}{\Delta (p, \frac{n}{4} - \frac{n}{p})}
\end{align*}
\]

where \( \frac{p}{q} = \frac{\alpha}{2} \), \( p \) and \( q \) are positive integers in their lowest orders, \( n < 1 + \frac{4p}{q} \) and \( \Delta (r, a) \) represents the set of \( r \) parameters \( \frac{a}{r}, \frac{a+1}{r}, \ldots, \frac{a+r-1}{r} \).

**Proof.** Let \( g_{n, \alpha}(x) \) denote \( n \)-dimensional density of \( X, X \in \mathbb{R}^n \). We have

\[
    h_{n, \alpha}(x) = h_{n, \alpha}(x) \quad \text{by the Hankel transform of } f_{n, \alpha}(t) = \chi_{n, \alpha}(t),
\]

where \( t = |t| \).
Let \( t = \frac{z}{2} \), then

\[
h_{n,a}(x) \sim \int_0^\infty \frac{z^{n/2 - 1/4}}{1 + z^{a/2}} (xz^{1/2}) \, dz
\]

But

\[
\frac{z^\beta}{1 + az^a} = a^{-\beta/a} G_{1,1}^{1,1}(az^a)^{(\beta/a)}
\]

where

\[
G_{m,n}^{p,q}(z) = \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^m \Gamma(b_j + s) \Gamma(1-a_j - s)}{\prod_{j=m+1}^n \Gamma(1-b_j - s) \Gamma(a_j + s)} z^{-s} \, ds
\]

is the Meijer's G-function (Mathai and Saxena, 1973).

Thus

\[
h_{n,a}(x) \sim \int_0^\infty z^{\frac{n}{4} - \frac{1}{4}} G_{1,1}^{1,1}(z^{1/2}) \, dz
\]

This is the Hankel transform of the G-function. Therefore, on integration we obtain (3.3.1). \( \square \)

**Note 1.** Let \( p_{n,a}(x) \) be the univariate density of \( X = |X| \). We have...
Therefore
\[ p_{n,a}(x) = \frac{n}{2} \times \frac{n-1}{\pi} \frac{1}{\Gamma\left(\frac{n}{2}\right)} h_{n,a}(x) \] (3.3.3)

\[ p_{n,a}(x) = -\frac{n}{2} + 3-q \left(\frac{n}{2}\right)^{-2} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{\pi} \frac{1}{pq} \]
\[ \times G_{q+2p,q}^{q,p} \left(2\frac{p}{x}\right) \Delta(p, \frac{1}{2} - \frac{n}{4}), \Delta(q, \frac{(n-2)q}{4p}), \Delta(p, \frac{n}{2} - \frac{1}{2}) \]
\[ \Delta(q, \frac{(n-2)q}{4p}) \] (3.3.4)

Note 2. A similar expression can be obtained for the distribution function \( P_{n,a}(x) = \int p_{n,a}(u)du \).

\[ p_{n,a}(x) = -\frac{n}{2} + 2-q \left(\frac{n}{2}\right)^{-1} \frac{1}{\pi} \frac{1}{pq} \]
\[ \times G_{q+2p,q}^{q,p} \left(2\frac{p}{x}\right) \Delta(p, \frac{1}{2} - \frac{n}{4}), \Delta(q, \frac{(n-2)q}{4p}), \Delta(p, \frac{n}{4} + \frac{1}{2}) \]
\[ \Delta(q, \frac{(n-2)q}{4p}) \] (3.3.5)

for \( n < 3 + \frac{4p}{q} \).

Note 3. Using the properties of the \( G \) - function, particular cases can be evaluated. Some of them are
(a) \( \alpha = 2 \)

The condition \( n < 1 + \frac{4p}{q} \) restricts our attention to \( n < 5 \).

In this case \( p=q=1 \).

\[
\begin{align*}
   h_{n,0}(x) &= 2 \left( -\frac{n}{2} + \frac{1}{\pi} \right) \left( -\frac{n}{2} + 1 \right) \left( -\frac{n}{2} - \frac{1}{\pi} \right) \\
   &\quad \times G_{3,1}^{1,2} \left( \frac{2}{x} \right)^2 \left( \frac{2}{x} \right)^2 \\
   &\quad \times \Delta \left( 1, \frac{1}{2} - \frac{n}{4} \right), \Delta \left( 1, \frac{n}{4} - \frac{1}{2} \right), \Delta \left( 1, \frac{n}{4} - \frac{1}{2} \right) \\
   &\quad \Delta \left( 1, \frac{n}{4} - \frac{1}{2} \right)
\end{align*}
\]

\[
\begin{align*}
   &= 2 \left( -\frac{n}{2} + \frac{1}{\pi} \right) \left( -\frac{n}{2} + 1 \right) \left( -\frac{n}{2} - \frac{1}{\pi} \right) \\
   &\quad \times G_{3,1}^{1,2} \left( \frac{2}{x} \right)^2 \\
   &\quad \left( \frac{2}{x} \right)^2 \\
   &\quad \frac{3}{2} - \frac{n}{4}, \frac{1}{2} + \frac{n}{4}, \frac{1}{2} + \frac{n}{4} \\
   &\quad \frac{1}{2} + \frac{n}{4}
\end{align*}
\]

\[
\begin{align*}
   &= 2 \left( -\frac{n}{2} + \frac{1}{\pi} \right) \left( -\frac{n}{2} + 1 \right) \left( -\frac{n}{2} - \frac{1}{\pi} \right) \\
   &\quad \times G_{1,3}^{2,1} \left( \frac{x}{2} \right)^2 \\
   &\quad \left( \frac{x}{2} \right)^2 \\
   &\quad \frac{1}{2} - \frac{n}{4} \\
   &\quad \frac{n}{4} - \frac{1}{2}, \frac{1}{2} - \frac{n}{4}, \frac{1}{2} - \frac{n}{4}
\end{align*}
\]

\[
\begin{align*}
   &= 2 \left( -\frac{n}{2} + \frac{1}{\pi} \right) \left( -\frac{n}{2} + 1 \right) \left( -\frac{n}{2} - \frac{1}{\pi} \right) \\
   &\quad \times G_{0,2}^{2,0} \left( \frac{x}{2} \right)^2 \\
   &\quad \left( \frac{x}{2} \right)^2 \\
   &\quad \frac{n}{4} - \frac{1}{2}, \frac{1}{2} - \frac{n}{4}, \frac{1}{2} - \frac{n}{4}
\end{align*}
\]
\[ (x) = 2 - \frac{n}{2} - 1 - \frac{n}{2} + 1 \quad \pi^{-\frac{n}{2}} 2K_{\frac{n}{2}}^{-1} \]

i.e., \( h_{n, 2}(x) = (2\pi)^{-\frac{n}{2}} x^{-\frac{n}{2} + 1} K_{\frac{n}{2}}^{-1}(x) \) \hspace{1cm} (3.3.6)

where \( K(\cdot) \) is the Bessel function of the second kind of order \( v \).

(b) \( n = 1 \)

\[ h_{1, \alpha}(x) = \frac{3}{2} - q - \frac{3}{2} \pi \frac{1}{2} - q \]

\[ = \frac{3}{2} - q - \frac{3}{2} \pi \frac{1}{2} - q \]

\[ \times G_{q, q+2p, q}^{2p, 2p} \left( \frac{2p}{x} \right) \]

\[ \Delta(p, -\frac{q}{4p}), \Delta(q, \frac{q}{4p}), \Delta(p, \frac{q}{4p}) \]

\[ \Delta(q, -\frac{q}{4p}) \] \hspace{1cm} (3.3.7)

3.4 SOME REPRESENTATIONS AND MOMENTS

THEOREM 3.4.1. Let \( U \) be an exponential random variable and let \( Y \) distributed independently of \( U \) be multivariate symmetric stable (with c.f. \( e^{-|t|^{\alpha}} \)). Then

\[ X \overset{D}{=} U^{1/\alpha} Y_{\alpha} \] \hspace{1cm} (3.4.1)

Proof

\[ E(e^{itX}) = E(e^{itU^{1/\alpha} Y_{\alpha}}) \]

\[ = E_U E_{Y_{\alpha}} [e^{itU^{1/\alpha} Y_{\alpha}} | U] \]

\[ = E_U (e^{-|t|^{\alpha} U}) \]

\[ = \frac{1}{1+|t|^{\alpha}}, \]
which is the characteristic function of the $\alpha$-Laplace distribution.

**THEOREM 3.4.2.** If $U$ is a positive valued random variable with Laplace transform $\frac{1}{1+\beta t}$, $0 < \beta \leq 1$, $t \geq 0$ and $Y_\alpha$ is n-dimensional symmetric stable with exponent $\alpha$ distributed independently of $U$,

$$X \overset{D}{=} U^{1/\alpha} Y_\alpha \quad (3.4.2)$$

has n-variate $\alpha$-Laplace distribution with exponent $\alpha \beta$.

**Proof.**

$$E(e^{itX}) = E(e^{itU^{1/\alpha} Y_\alpha})$$

$$= E_U E_{Y_\alpha} (e^{itU^{1/\alpha} Y_\alpha} | U)$$

$$= E_U (e^{-|t|^\alpha U})$$

$$= \frac{1}{1+|t|^\alpha \beta}$$

which is the desired result. \qed

**THEOREM 3.4.3.** Let $F$ be the distribution corresponding to the Laplace transform $\frac{1}{1+t^\beta}$, $0 < \beta \leq 1$, $t \geq 0$. The characteristic function
of the multivariate $\alpha$-Laplace distribution admits the representation

$$X_{n,2\beta}(|t|) = \int_0^\infty e^{-u|t|^2} dF(u)$$  \hspace{1cm} (3.4.3)

Proof. Let $F$ be the distribution function of a positive valued random variable $U$ with Laplace transform \( \frac{1}{1+t^\beta} \), \( 0 < \beta \leq 1 \). Then

$$\int_0^\infty e^{-u|t|^2} dF(u) = \frac{1}{1+|t|^{2\beta}}$$

If we let $\beta = \frac{\alpha}{2}$, the R.H.S. is obviously \( \frac{1}{1+|t|^\alpha} \), \( 0 < \alpha \leq 2 \)

which is the characteristic function of the multivariate $\alpha$-Laplace distribution and the representation (3.4.3) follows. $\square$

By (3.4.1), the multivariate $\alpha$-Laplace random vector $X = U^{1/\alpha} Y_\alpha$

where $Y_\alpha$ has spherically symmetric stable distribution in $\mathbb{R}^n$ with exponent $\alpha (0 < \alpha \leq 2)$ and $U$ distributed independently of $Y_\alpha$ has exponential distribution with parameter 1.

By Zolatarev (1981), for $0 < \delta < \alpha$,

$$E|Y_\alpha|^\delta = \frac{2^{\delta} \Gamma(1-\delta/\alpha) \Gamma((n+\delta)/2)}{\Gamma(1-\delta/2) \Gamma(n/2)}$$  \hspace{1cm} (3.4.4)

Therefore,

$$E|X|^\delta = E|U^{1/\alpha} Y_\alpha|^\delta$$
which proves the following

**THEOREM 3.4.4.** If \( X \) has \( n \)-dimensional \( \alpha \)-Laplace distribution, \( 0 < \alpha \leq 2 \), for \( 0 < \delta < \alpha \)

\[
E|X|^\delta = \frac{2 \delta \Gamma(1+\delta/\alpha) \Gamma(1-\delta/\alpha) \Gamma((n+\delta)/2)}{\Gamma(1-\delta/2) \Gamma(n/2)}
\]  

(3.4.5)

3.5 INFINITE DIVISIBILITY

We show that \( f_{n,\alpha}(t) = 1/(1+c|t|^\alpha), t \in \mathbb{R}^n, 0 < \alpha \leq 2 \) is an infinitely divisible characteristic function for \( c > 0 \). We require the following

**LEMMA 3.5.1.** If \( \phi(t) \) is an i.d. characteristic function then

\[
1/(1 - \log \phi(t)) \text{ is an i.d. characteristic function.}
\]

**Proof.** By Lukacs (1970), if \( \psi(t) \) is any characteristic function, \( (p-1) / (p-\psi(t)) \) is an i.d. characteristic function for \( p > 1 \).

Take \( \psi(t) = \phi^{1/n}(t), p = 1 + 1/n \)

\[
\frac{p-1}{p-\psi(t)} = \frac{1/n}{1+ \frac{1}{n} \phi^{1/n}(t)}
\]

\[
= \frac{1}{1+ \frac{1}{n} \phi^{1/n}(t)} + \frac{1}{1-\log \psi(t)}
\]
as \( n \) tends to \( \infty \), which is an i.d. characteristic function.

Also, we define infinite divisibility for multivariate distributions with characteristic function \( \phi(t_1, t_2, \ldots, t_p) \) if for every \( n \),

\[
\phi(t_1, t_2, \ldots, t_p) = \gamma^n(t_1, t_2, \ldots, t_p)
\]

where \( \gamma(t_1, t_2, \ldots, t_p) \) is the characteristic function of some \( p \)-variate distribution.

**THEOREM 3.5.1.** The characteristic function \( f_{n,a}(t) = 1/(1+c|t|^a) \), \( 0 < a < 2, c > 0 \), \( t \in \mathbb{R}^n \) for \( n \geq 1 \) is infinitely divisible.

**Proof.** The multivariate stable characteristic function \( \exp \{-c|t|^a\} \), \( 0 < a < 2, c > 0, t \in \mathbb{R}^n \) is infinitely divisible. By the above Lemma,

\[
\frac{1}{1-\log \{ \exp \{-c|t|^a\} \}} = \frac{1}{1+c|t|^a}
\]

is also infinitely divisible.

If \( f(t) = \frac{1}{1+c|t|^a} \)

\( \phi(t) = \exp \{ 1 - \frac{1}{f(t)} \} \)

\[= \exp \{ -c|t|^a \} \]

which is the characteristic function of multivariate symmetric stable distribution and hence i.d. This shows that the multivariate \( \alpha \)-Laplace distribution is g.i.d.

The multivariate \( \alpha \)-Laplace distributions are normally attracted to the corresponding multivariate symmetric stable laws. The proof is analogous to that of the univariate case.
3.6 UNIMODALITY

The projections of spherical distributions are again spherical and the necessary and sufficient condition for the unimodality of a probability distribution with characteristic function \( f_n(t) = \chi_n(|t|), \) \( t \in \mathbb{R}^n \) is that \( \chi_{n+2}(|t|), \) \( t \in \mathbb{R}^{n+2} \) should be the characteristic function of some spherically symmetric distribution in \( \mathbb{R}^{n+2} \) (Zolatarev, 1981). To prove the unimodality of the multivariate \( \alpha \)-Laplace distribution, it remains only to show that \( f_{n,\alpha}(t) \) is a spherically symmetric characteristic function for every \( n \).

From the representation (3.4.3) it follows that \( f_{n,\alpha}(t) \) is not only spherically symmetric for every \( n \geq 1 \), but also infinitely divisible (Zolatarev, 1981).