CHAPTER II

α-LAPLACE DISTRIBUTIONS

2.1 INTRODUCTION

The function

\[ f_\alpha(t) = \frac{1}{1+|t|^\alpha} \]  

(2.1.1)

0 < \alpha \leq 2, -\infty < t < \infty is a characteristic function (Linnik, 1953) and Laha proved that the corresponding distribution is unimodal (Lukacs, 1970). The distribution with characteristic function (2.1.1) is named α-Laplace since \( \alpha = 2 \) corresponds to the Laplace distribution. The parameter \( \alpha \) is called the exponent of the distribution.

A larger class of distributions called 'semi-α-Laplace distributions' of which, α-Laplace distributions form a subclass was characterized by Pillai (1985). It was subsequently, shown that for \( \alpha > 1 \) \( |f_\alpha(t)| \) is integrable and for \( \alpha \leq 1 \), it is of the Polya-type. Thus the corresponding distributions have absolutely continuous densities.

The densities of the α-Laplace distributions for rational values of \( \alpha \) and the absolute moments of order \( \delta < \alpha \) are obtained in section 2. In sections 3 and 4 we establish the infinite divisibility and the geometric infinite divisibility of these distributions. The attraction to the stable law is established in section 5.
2.2 DENSITY AND MOMENTS

The characteristic function of the $\alpha$-Laplace distribution is a function of $|t|$ alone for fixed $\alpha$ and therefore we can write $f_\alpha(t) = \chi_\alpha(u)$ where $u = |t|, u \geq 0$. Any such function is a characteristic function of a spherically symmetric distribution and the corresponding density can be obtained by the Hankel transform technique. Here we obtain the density of the $\alpha$-Laplace distribution for rational values of $\alpha > 1$ in terms of the Meijer's $G$ - function (Mathai and Saxena (1973), see also Chapter III).

THEOREM 2.2.1. The density of the $\alpha$-Laplace distribution is given by

$$h_\alpha(s) = 2^{\frac{3}{2} - q} s^{\frac{-3}{2}} \pi^{\frac{1}{2} - q} \frac{pq}{\Delta(p, \frac{q}{2}), \Delta(q, -q/4p), \Delta(p, -q/4p)}$$

$$\times G_{q+2p,q}(\frac{2p}{s})^{2p}$$

where $s = |x|$, $\frac{p}{q} = \frac{\alpha}{2}$, $p$ and $q$ are positive integers in their lowest orders and $\Delta(r,a)$ represents the set of $r$ parameters $\frac{a}{r}, \frac{a+1}{r}, \ldots, \frac{a+r-1}{r}$.
Proof. Since the distribution is spherical we let $g_a(x) = h_a(s)$ where $g_a(x)$ is the density of $X$ and $h_a(s)$ is a real function defined on the non-negative real axis as

$$h_a(s) = 2 \pi^{-\frac{3}{2}} s^{\frac{1}{2}} \int_0^\infty \frac{u^{\frac{1}{2}}}{1+u^a} J_{-\frac{1}{2}}(su) \, du$$  \quad (2.2.2)$$

where $J_v(.)$ is the Bessel function of the first kind. This is the Hankel transform of $(1+u^a)^{-1}$ and evaluating (2.2.2), we obtain (2.2.1) (See 3.3 page 28).

We note that for $a = 2$

$$h_2(s) = 2^{\frac{3}{2}} \pi^{-\frac{3}{2}} s^{-\frac{3}{2}} G_{1,2}^{1,2} \left( \frac{s}{2}^2 \right)^{\frac{1}{4}}$$

$$= 2 \left( \frac{s}{2^\frac{1}{2}} \right)^{-\frac{3}{2}} \pi^{-\frac{1}{2}} G_{0,2}^{2,0} \left( \frac{s}{2}^2 \right)^{\frac{1}{4}}$$

$$= \left( \frac{s}{2^\frac{1}{2}} \right)^{-\frac{3}{2}} K_{-\frac{1}{2}} (s)$$

where $K_v(.)$ is the Bessel function of the second kind. But

$$\left( \frac{s}{2^\frac{1}{2}} \right)^{-\frac{3}{2}} K_{-\frac{1}{2}} (s) = \frac{e^{-s}}{\sqrt{s}}$$

and therefore,

$$g_2(x) = \frac{e^{-|x|}}{\sqrt{2}}, \quad -\infty < x < \infty$$  \quad (2.2.3)$$

which is the density of the Laplace distribution.
For the $\alpha$-Laplace distribution with exponent $\alpha$, $E|X|^\delta$ exists for $0 < \delta < \alpha$ (Pillai, 1985). To obtain the moments we note that a random variable with the $\alpha$-Laplace distribution admits the representation

$$X \overset{D}{=} U^{1/\alpha} Y$$

where $U$ is an exponential random variable and $Y$ is symmetric stable with exponent $\alpha$, distributed independently of $U$.

**Theorem 2.2.2.** For the $\alpha$-Laplace distribution

$$E|X|^\delta = \frac{2^\delta}{\sqrt{\pi}} \frac{\Gamma(1+\delta/\alpha) \Gamma(1-\delta/\alpha) \Gamma((1+\delta)/2)}{\Gamma(1-\delta/2)} \quad (2.2.4)$$

**Proof.**

$$E|X|^\delta = E(U^{\delta/\alpha} | Y|^{\delta})$$

$$= E(U^{\delta/\alpha}) E(|Y|^{\delta})$$

$$= \Gamma(1+\delta/\alpha) E(|Y|^{\delta})$$

Since $Y$ is symmetric stable, for $0 < \alpha < \delta$

$$E(|Y|^{\delta}) = \frac{2^\delta \Gamma(1-\delta/\alpha) \Gamma((1+\delta)/2)}{\Gamma(1-\delta/2) \Gamma(1/2)}$$


Therefore,

$$E|X|^\delta = \frac{2^\delta}{\sqrt{\pi}} \frac{\Gamma(1+\delta/\alpha) \Gamma(1-\delta/\alpha) \Gamma((1+\delta)/2)}{\Gamma(1-\delta/2)}$$
2.3 INFINITE DIVISIBILITY

The infinite divisibility of the $\alpha$-Laplace distribution can be seen from the fact it belongs to the class-L. We prove the following

**THEOREM 2.3.1.** The $\alpha$-Laplace distribution belongs to the class-L.

**Proof.** We have

$$f_\alpha(t) = \frac{1}{1+|t|^\alpha}, \quad 0 < \alpha < 2$$

and it suffices to prove that for $0 < c < 1$, $f_\alpha(t)/f_\alpha(ct)$ is a characteristic function. If $g(t)$ is any characteristic function then

$$p^{-1} \frac{p-1}{p-g(t)},$$

for $p > 1$ is the characteristic function of an i.d. distribution. Let

$$g(t) = \frac{1}{1+c^\alpha |t|^\alpha}$$

and choose

$$p = \frac{1}{1-c^\alpha}$$

Then

$$\frac{p-1}{p-g(t)} = \frac{f_\alpha(t)}{f_\alpha(ct)}$$

and hence belongs to the class-L. □
2.4 GEOMETRIC INFINITE DIVISIBILITY

The concept of geometric infinite divisibility of a random variable or its distribution has been recently introduced by Klebanov, et. al. (1984).

DEFINITION 2.4.1. A random variable $X$ is geometrically infinitely divisible (g.i.d.) if for every $p, 0 < p < 1$,

$$X \overset{D}{=} \sum_{j=1}^{\infty} p X_{p}^{(j)}$$

where $\{X_{p}^{(n)}, n \geq 1\}$ is a sequence of i.i.d. random variables and $N_{p}$ is a geometric random variable with mean $1/p$ and is independent of the sequence $\{X_{p}^{(n)}, n \geq 1\}$.

THEOREM (Klebanov, 1984). A random variable $X$ with characteristic function $f(t)$ is g.i.d. iff

$$\phi(t) = \exp \{1 - 1/f(t)\}$$

is an i.d. characteristic function.

It is easily seen that if $f(t) = \frac{1}{1 + |t|^\alpha}$, $\phi(t) = \exp \{-|t|^\alpha\}$ which is the characteristic function of a symmetric stable distribution with exponent $\alpha$. Thus we have proved the following

THEOREM 2.4.1. The $\alpha$-Laplace distribution is g.i.d.
2.5 ATTRACTION TO THE STABLE LAW

It can be shown that the $\alpha$-Laplace distribution with exponent $\alpha$ belongs to the domain of normal attraction of the stable law with exponent $\alpha$.

**THEOREM 2.5.1.** The $\alpha$-Laplace distribution is normally attracted to the stable law with exponent $\alpha$.

**Proof.** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with characteristic function (2.1.1). Let

$$S_n = \frac{1}{n^{1/\alpha}} (X_1 + X_2 + \ldots + X_n)$$

The characteristic function of $S_n = (1 + \frac{|t|^\alpha}{n})^{-n}$ tends to $\exp \{-|t|^\alpha\}$ as $n$ tends to infinity, which proves the assertion. $\square$