CHAPTER 4

Energy of signed digraphs

In this Chapter, we extend the concept of energy to signed digraphs and we obtain Coulson’s integral formula for the energy of signed digraphs. We characterize unicyclic signed digraphs with minimal and maximal energy. We extend the concept of non complete extended p sum (or briefly, NEPS) to signed digraphs. We construct pairs of non cospectral equienergetic signed digraphs. Moreover, we extend McClelland’s inequality to signed digraphs and also obtain sharp upper bound for energy of signed digraph in terms of the number of arcs.

4.1 Introduction

A signed digraph is defined to be a pair $S = (D, \sigma)$ where $D = (V, \mathcal{A})$ is the underlying digraph and $\sigma : \mathcal{A} \to \{-1, 1\}$ is the signing function. The sets of positive and negative arcs of $S$ are respectively denoted by $\mathcal{A}^+$ and $\mathcal{A}^-$. Thus $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-$. A signed digraph is said to be homogeneous if all of its arcs have either positive sign or negative sign and heterogeneous, otherwise.

Two vertices are adjacent if they are connected by an arc. A path of length $n - 1$ ($n \geq 2$), denoted by $P_n$, is a signed digraph on $n$ vertices $v_1, v_2, \cdots, v_n$ with $n - 1$ signed arcs $(v_i, v_{i+1})$, $i = 1, 2, \ldots, n - 1$. A cycle of length $n$ is a signed digraph having vertices $v_1, v_2, \cdots, v_n$ and signed arcs $(v_i, v_{i+1})$, $i = 1, 2, \cdots, n - 1$ and $(v_n, v_1)$. The sign of a signed digraph is defined as the product of signs of its arcs. A signed digraph is said to be positive (respectively, negative) if its sign is positive (respectively, negative) i.e., it contains an even (respectively, odd) number of negative arcs. A signed digraph is said to be all-positive (respectively, all-negative) if all its arcs are positive (respectively, negative). A signed digraph is said to be cycle balanced if each of its cycles is positive, otherwise non cycle balanced. Throughout this Chapter, we call cycle balanced cycle a positive cycle and non cycle balanced cycle a negative cycle and respectively denote them by $C_n$ and $C_n^-$, where $n$ is the number of vertices. Further dotted arcs denote the negative arcs and bold arcs denote the positive arcs. A linear signed subdigraph of a signed digraph is a subdigraph with indegree and outdegree of each vertex
equal to one.

The adjacency matrix of a signed digraph \( S \) whose vertices are \( v_1, v_2, \ldots, v_n \) is the \( n \times n \) matrix \( A(S) = (a_{ij}) \), where

\[
a_{ij} = \begin{cases} 
\sigma(v_i, v_j), & \text{if there is an arc from } v_i \text{ to } v_j, \\
0, & \text{otherwise.} 
\end{cases}
\]

The characteristic polynomial \( |xI - A(S)| \) of the adjacency matrix \( A(S) \) of signed digraph \( S \) is called the characteristic polynomial of \( S \) and is denoted by \( \phi_S(x) \). The eigenvalues of \( A(S) \) are called the eigenvalues of \( S \).

A signed digraph is said to be symmetric if \((u,v) \in A^+ \) or \( A^- \), then \((v,u) \in A^+ \) or \( A^- \), where \( u, v \in V \). A one to one correspondence between signed graphs and symmetric signed digraphs is given by \( S \rightsquigarrow \overrightarrow{S} \), where \( \overrightarrow{S} \) has the same vertex set as that of signed graph \( S \), and each signed edge \((u,v)\) is replaced by a pair of symmetric arcs \((u,v)\) and \((v,u)\) both with same sign as that of edge \((u,v)\). Under this correspondence a signed graph can be identified with a symmetric signed digraph. A signed digraph is said to be skew symmetric if its adjacency matrix is skew symmetric. We denote a skew symmetric signed digraph of order \( n \) by \( S_n \).

The weighted directed graph \( S \) of an \( n \times n \) matrix \( M = (m_{ij}) \) of reals consists of \( n \) vertices with vertex \( i \) joined to vertex \( j \) by a directed arc with weight \( m_{ij} \) if and only if \( m_{ij} \) is non-zero. In case the matrix consists of entries \(-1, 0, 1\), then we get a signed digraph. Thus there is a one to one correspondence between the set of integral \((-1, 0, 1)\)-matrices of order \( n \) and the set of signed digraphs of order \( n \).

The following is the coefficient Theorem for signed digraphs [2].

**Theorem 4.1.1.** If \( S \) is a signed digraph with characteristic polynomial

\[
\phi_S(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,
\]

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then
\[ a_j = \sum_{L \in \mathcal{L}_j} (-1)^{p(L)} \prod_{Z \in c(L)} s(Z), \]
for all \( j = 1, 2, \ldots, n \), where \( \mathcal{L}_j \) is the set of all linear signed subdigraphs \( L \) of \( S \) of order \( j \), \( p(L) \) denotes number of components of \( L \) and \( c(L) \) denotes the set of all cycles of \( L \) and \( s(Z) \) the sign of cycle \( Z \).

**Remark 4.1.2.** For undirected signed graph (when considered as symmetric signed digraph) Theorem 4.1.1 takes the form of Theorem 3.1.1.

The spectral criterion for cycle balance of signed digraphs given by Acharya [1] is as follows.

**Theorem 4.1.3.** A signed digraph is cycle balanced if and only if it is cospectral with the underlying unsigned digraph.

### 4.2 Energy of signed digraphs

In this section, we extend the concept of energy to signed digraphs in a similar way as graph energy has been extended to energy of digraphs in [62]. Unlike signed graphs the adjacency matrix of a signed digraph need not be real symmetric, so eigenvalues can be complex numbers.

**Definition 4.2.1.** Let \( S \) be a signed digraph of order \( n \) having eigenvalues \( z_1, z_2, \ldots, z_n \). The energy of \( S \) is defined as
\[ E(S) = \sum_{j=1}^{n} |\Re z_j|, \]
where \( \Re z_j \) denotes the real part of complex number \( z_j \).

If \( S \) is a signed graph and \( \overrightarrow{\tilde{S}} \) be its symmetric signed digraph, then clearly \( A(S) = A(\overrightarrow{\tilde{S}}) \) and so \( E(S) = E(\overrightarrow{\tilde{S}}) \). In this way, definition 4.2.1 generalizes the concept of energy of undirected signed graphs.
Example 4.2.2. Let $S$ be a signed digraph shown in Figure 4.1. Clearly, $S$ is non cycle balanced signed digraph. By Theorem 4.1.1, the characteristic polynomial of $S$ is $\phi_S(x) = x^{10} + x^7 = x^7(x^3 + 1)$. The spectrum of $S$ is $\text{spec}(S) = \{-1, 0, \frac{1-\sqrt{3}i}{2}, \frac{1+\sqrt{3}i}{2}\}$, where $i = \sqrt{-1}$, so $E(S) = 2$.

Example 4.2.3. Let $S$ be an acyclic signed digraph. Then by Theorem 4.1.1, the characteristic polynomial of $S$ is $\phi_S(x) = x^n$, so that $\text{spec}(S) = \{0^n\}$ and hence $E(S) = 0$.

Example 4.2.4. Consider $S_n$, the skew symmetric signed digraph on $n \geq 2$ vertices, then eigenvalues are of the form $\pm i\alpha$, where $\alpha \in \mathbb{R}$ and therefore $E(S) = 0$.

Example 4.2.5. If $S$ is the signed directed cycle on $n$ vertices, then the characteristic polynomial of $S$ is $\phi_S(x) = x^n + (-1)^s$, where the symbol $[s]$ is defined as $[s] = 1$ or 0 according as $S$ is positive or negative. If $S = C_n$, then $\text{spec}(S) = \{e^{\frac{2j\pi}{n}}, j = 0, 1, \cdots, n-1\}$ so that $E(S) = \sum_{j=0}^{n-1} |\cos(\frac{2j\pi}{n})|$. If $S = C_4$, then $\text{spec}(S) = \{1, -1, -\frac{1-\sqrt{2}}{\sqrt{2}}, -\frac{1+\sqrt{2}}{\sqrt{2}}\}$ and $E(S) = 2\sqrt{2}$.

Example 4.2.6. Let $S$ be a signed digraph having $n$ vertices and unique cycle of length $r$, where $2 \leq r \leq n$. Then by Theorem 4.1.1, $\phi_S(x) = x^n + (-1)^s x^{n-r} = x^{n-r}(x^r + (-1)^s)$, where the symbol $[s]$ is defined as $[s] = 1$ or 0 according as $S$ is cycle balanced or non cycle balanced. Clearly, energy equals to the energy of the unique cycle.
Given $t$ signed digraphs $S_1, S_2, \ldots, S_t$, their direct sum denoted by $S_1 \oplus S_2 \oplus \cdots \oplus S_t$ is the signed digraph with $V(S_1 \oplus S_2 \oplus \cdots \oplus S_t) = \bigcup_{j=1}^{t} V(S_j)$ and arc set $A(S_1 \oplus S_2 \oplus \cdots \oplus S_t) = \bigcup_{j=1}^{t} A(S_j)$.

Now we have the following result.

**Theorem 4.2.7.** Let $S$ be a signed digraph on $n$ vertices and $S_1, S_2, \ldots, S_k$ be its strong components. Then $E(S) = \sum_{j=1}^{k} E(S_j)$.

**Proof.** Let $Y = \{a \in A : a \notin c(S)\}$, where $c(S)$ is the set of all cycles of $S$. By Theorem 4.1.1, $\phi_S(x) = \phi_{S-Y}(x)$, where $S-Y$ is the signed digraph obtained from $S$ by deleting the non-cyclic arcs. Clearly, $S-Y = S_1 \oplus S_2 \oplus \cdots \oplus S_k$ and adjacency matrix of signed digraph $S-Y$ is in block diagonal form with diagonal blocks as the adjacency matrices of strong components (isolated vertex is considered as strong component of order one). Therefore $\phi_{S-Y}(x) = \phi_{S_1}(x)\phi_{S_2}(x)\cdots\phi_{S_k}(x)$ and so $E(S) = \sum_{j=1}^{k} E(S_j)$.

**Remark 4.2.8.** From Theorem 4.1.1, $a_j = \sum_{L \in \mathcal{L}_j} (-1)^{p(L)} s(L)$, for $j = 1, 2, \ldots, n$, where $s(L) = \prod_{Z \in c(L)} s(Z)$. Clearly, this sum contains positive and negative ones.

Clearly
+1 arises if and only if
(a) Number of components of $L \in \mathcal{L}_j$ is odd and $s(L) < 0$. We call such linear signed digraphs as type $a$ linear signed digraphs.
(b) Number of components of $L \in \mathcal{L}_j$ is even and $s(L) > 0$. We call such linear signed digraphs as type $b$.

$-1$ will occur if and only if
(c) Number of components of $L \in \mathcal{L}_j$ is odd and $s(L) > 0$. We call such linear signed digraphs as type $c$.
(d) Number of components of $L \in \mathcal{L}_j$ is even and $s(L) < 0$. We call such linear signed digraphs as type $d$.

From the above remark, we observe that $a_j = 0$ if and only if either $S$ is
acyclic or in $S$, for each $j$, number of linear signed digraphs of order $j$ of type $a$ or type $b$ or both types is equal to the number of linear signed digraphs of order $j$ of type $c$ or type $d$ or both types.

An immediate consequence of the Remark 4.2.8 is the following Lemma.

**Lemma 4.2.9.** An integral $(-1,0,1)$-matrix is nilpotent if and only if its underlying signed digraph $S$ is either acyclic or in $S$, for each $j = 1, 2, \cdots, n$, the number of linear signed digraphs of order $j$ of type $a$ or type $b$ or both types is equal to number of linear signed digraphs of order $j$ of type $c$ or type $d$ or both types.

Unlike unsigned strong component, energy of a signed directed strong component can be zero, for example, signed digraph $S_1$ in Figure 4.2. Now we have the following result.

**Theorem 4.2.10.** Let $S$ be a signed digraph of order $n$. Then $E(S) = 0$ if $S$ satisfies one of the following conditions (i) $S$ is acyclic or (ii) each strong component of $S$ is skew symmetric or (iii) for each $j = 1, 2, \cdots, n$, the number of linear signed digraphs of order $j$ of type $a$ or type $b$ or both types is equal to number of linear signed digraphs of order $j$ of type $c$ or type $d$ or both types.

**Proof.** Let $S$ be a signed digraph of order $n$. If $S$ is acyclic or satisfies (iii), then by Lemma 4.2.9, $\phi_S(x) = x^n$ and so $E(S) = 0$. If $S$ satisfies (ii), then the eigenvalues of $S$ are of the form $\pm i\alpha$, where $\alpha \in \mathbb{R}$, therefore $E(S) = 0$.

Here we note that Lemma 4.2.9 characterizes signed digraphs with zero as the only eigenvalue. Skew-symmetric signed digraphs have eigenvalues of the form $\pm i\alpha$, where $\alpha \in \mathbb{R}$. But there are signed digraphs with eigenvalues of the form $\pm i\alpha$, where $\alpha \in \mathbb{R}$, which are not skew symmetric. For example, consider the signed digraph $S$ obtained by joining two copies of $S_2$, a skew symmetric signed digraph of order two, by an arc (sign being immaterial). The spectrum of $S$ is $\text{spec}(S) = \{i^{(2)}, -i^{(2)}\}$, where $i = \sqrt{-1}$. But $S$ is not skew symmetric digraph as $A(S)$ is not a skew symmetric matrix. Therefore characterization of signed digraphs with energy zero reduces to the problem of characterizing signed digraphs with eigenvalues of the form $\pm i\alpha$, where $\alpha \in \mathbb{R}$.
4.3 Computation of energy of signed directed cycles

We first give energy formulae for positive cycles. Let $C_n$ be a positive cycle on $n \geq 2$ vertices. The characteristic polynomial of $C_n$ is $\phi_{C_n}(x) = x^n - 1$, so that $\text{spec}(C_n) = \{ e^{\pm \frac{2\pi j}{n}}, j = 0, 1, \ldots, n-1 \}$, where $\iota = \sqrt{-1}$. Consequently energy of $C_n$ is

$$E(C_n) = \sum_{j=0}^{n-1} |\cos \frac{2j\pi}{n}|.$$

Given a positive integer $n$, it has one of the forms $4k$, or $2k + 1$, or $4k + 2$, where $k \geq 0$.

If $n = 4k$, then

$$E(C_n) = \sum_{j=0}^{4k-1} |\cos \frac{2j\pi}{4k}| = \sum_{j=0}^{4k-1} |\cos \frac{j\pi}{2k}| = 2 \sum_{j=0}^{2k-1} |\cos \frac{j\pi}{2k}|$$

$$= 2 + 4 \sum_{j=1}^{k-1} \cos \frac{j\pi}{2k} = 2 + 4 \{ -\frac{1}{2} + \frac{\sin \frac{(k-\frac{1}{2})\pi}{2k}}{2\sin \frac{\pi}{4k}} \} = 2 \cot \frac{\pi}{n}.$$

If $n = 2k + 1$, then

$$E(C_n) = \sum_{j=0}^{2k} |\cos \frac{2j\pi}{2k+1}| = 1 + 2 \sum_{j=1}^{k} |\cos \frac{2j\pi}{2k+1}| = 1 + 2 \sum_{j=1}^{k} \cos \frac{j\pi}{2k+1}$$

$$= 1 + 2 \{ -\frac{1}{2} + \frac{\sin \frac{(k+\frac{1}{2})\pi}{2k+1}}{2\sin \frac{\pi}{2(2k+1)}} \} = \csc \frac{\pi}{2n}.$$

If $n = 4k + 2$, then

$$E(C_n) = \sum_{j=0}^{4k+1} |\cos \frac{2j\pi}{4k+2}| = \sum_{j=0}^{4k+1} |\cos \frac{j\pi}{2k+1}| = 2 \sum_{j=0}^{2k} |\cos \frac{j\pi}{2k+1}|$$

$$= 2 + 4 \sum_{j=1}^{k} \cos \frac{j\pi}{2k+1} = 2 + 4 \{ -\frac{1}{2} + \frac{\sin \frac{(k+\frac{1}{2})\pi}{2k+1}}{2\sin \frac{\pi}{4k+2}} \} = 2 \csc \frac{\pi}{n}.$$

We now give exact formulae for the energy of negative cycles of length $n$. Let $C_n$ denote the negative cycle with $n$ vertices. Then $\phi_{C_n}(x) = x^n + 1$ and so
$Spec(C_n) = \{ e^{\frac{(2j+1)\pi}{n}}, j = 0, 1, \cdots, n-1 \}$, where $\iota = \sqrt{-1}$. Therefore the energy is given by

$$E(C_n) = \sum_{j=0}^{n-1} |\cos \frac{(2j + 1)\pi}{n}|.$$

If $n = 4k$, then

$$E(C_n) = \sum_{j=0}^{4k-1} |\cos \frac{(2j + 1)\pi}{4k}| = 2 \sum_{j=0}^{2k-1} |\cos \frac{(2j + 1)\pi}{4k}| = 4 \sum_{j=0}^{k-1} \cos \frac{(2j + 1)\pi}{4k}$$

$$= 4\{\cos \frac{\pi}{4k} + \cos \frac{3\pi}{4k} + \cdots + \cos \frac{(2k - 1)\pi}{4k}\}$$

$$= 4 \frac{\cos(\frac{\pi}{4k} + \frac{k-1}{2} \frac{2\pi}{4k}) \sin \frac{k\pi}{8k}}{\sin \frac{\pi}{8k}} = 2 \csc \frac{\pi}{n}.$$

If $n = 4k + 2$, then

$$E(C_n) = \sum_{j=0}^{4k+1} |\cos \frac{(2j + 1)\pi}{4k+2}| = 4 \sum_{j=0}^{k-1} \cos \frac{(2j + 1)\pi}{4k+2}$$

$$= 4\{\cos \frac{\pi}{4k+2} + \cos \frac{3\pi}{4k+2} + \cdots + \cos \frac{(2k - 1)\pi}{4k+2}\} = 2 \cot \frac{\pi}{n}.$$

If $n = 2k + 1$, then since $-1$ is the eigenvalue of $C_n$, we have $spec(C_n) = -spec(C_n)$, and so $E(C_n) = E(C_n)$.

Summarizing, all the above cases can be written as follows:

$$E(C_n) = \begin{cases} 
2 \cot \frac{\pi}{n}, & \text{if } n = 4k, \\
2 \csc \frac{\pi}{n}, & \text{if } n = 4k + 2, \\
\csc \frac{\pi}{2n}, & \text{if } n = 2k + 1.
\end{cases}$$

and

$$E(C_n) = \begin{cases} 
2 \csc \frac{\pi}{n}, & \text{if } n = 4k, \\
2 \cot \frac{\pi}{n}, & \text{if } n = 4k + 2, \\
\csc \frac{\pi}{2n}, & \text{if } n = 2k + 1.
\end{cases}$$
Pêna and Rada [62] proved that the energy of directed unsigned cycles increases monotonically with respect to order \( n \geq 4 \). From energy formulae for positive and negative signed directed cycles, the following two results are immediate.

**Theorem 4.3.1.** Energy of negative cycles increases monotonically with respect to the order. Among all non cycle balanced unicyclic signed digraphs on \( n \) vertices, the cycle has the largest energy. Moreover, the minimal energy is attained in unicyclic signed digraph with unique cycle \( C_2 \).

**Theorem 4.3.2.** Energy of positive and negative cycles satisfy the following:

(i) Energy of positive cycle of odd order equals energy of negative cycle of same order.

(ii) Energy of negative cycle of even order is greater than energy of positive cycle of same order if and only if \( n = 4k \).

(iii) Energy of negative cycle of even order is less than energy of positive cycle of same order if and only if \( n = 4k + 2 \).

We now obtain Coulson’s integral formula for energy of signed digraphs.

**Theorem 4.3.3.** Let \( S \) be a signed digraph with \( n \) vertices having characteristic polynomial \( \phi_S(x) \). Then

\[
E(S) = \sum_{j=1}^{n} |\Re z_j| = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( n - \frac{ix\phi'_S(ix)}{\phi_S(ix)} \right) dx,
\]

where \( z_1, z_2, \ldots, z_n \) are the eigenvalues of signed digraph \( S \) and \( \int_{-\infty}^{\infty} F(x)dx \) denotes principle value of the respective integral.

**Proof.** Consider the function

\[
f(z) = n - \frac{z\phi'_S(z)}{\phi_S(z)},
\]

where \( \phi_S(z) \) is the characteristic polynomial of \( S \) and \( z_j = a_j + ib_j, j = 1, 2, \ldots, n \) are its zeros.
Then
\[ f(z) = n - \sum_{j=1}^{n} \frac{z - z_j}{z - z_j} = \sum_{j=1}^{n} \frac{z_j}{z - z_j} \]
implying that
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \left( n - \frac{\imath y \phi'_S(\imath y)}{\phi_S(\imath y)} \right) dy = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\imath y) dy = \frac{1}{\pi} \sum_{j=1}^{n} \int_{-\infty}^{\infty} \frac{z_j}{z_j - \imath y} dy. \]

Using the integrals \( \int_{-\infty}^{\infty} \frac{a}{(y-b)^2+a^2} dy = \pi \text{sgn}(a) \) and \( \int_{-\infty}^{\infty} \frac{y-b}{(y-b)^2+a^2} dy = 0 \), where \( a \) and \( b \) are real numbers and \( \text{sgn}(a) \) denotes the sign of real number \( a \), we have
\[ \int_{-\infty}^{\infty} \frac{z_j}{z_j - \imath y} dy = \int_{-\infty}^{\infty} \frac{a_j + \imath b_j}{a_j - \imath(y-b_j)} dy \]
\[ = \int_{-\infty}^{\infty} \frac{a_j^2 - b_j(y-b_j) + [a_j(y-b_j) + a_j b_j] \imath}{(y-b_j)^2 + a_j^2} dy \]
\[ = \pi a_j \text{sgn}(a_j) + \pi b_j \text{sgn}(a_j) \imath = \pi \text{sgn}(a_j)(a_j + \imath b_j) \]
\[ = \pi \text{sgn}(a_j) z_j. \]

Therefore
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \left( n - \frac{\imath y \phi'_S(\imath y)}{\phi_S(\imath y)} \right) dy = \frac{1}{\pi} \sum_{j=1}^{n} \pi \text{sgn}(a_j) z_j = \sum_{j=1}^{n} \text{sgn}(\Re z_j) z_j = \sum_{j=1}^{n} |\Re z_j| = E(S). \]

The Coulson’s integral formula given above is another motivation to define the energy of a signed digraph as the sum of absolute values of real parts of eigenvalues.

**Example 4.3.4.** Consider the cycle \( C_4 \), the characteristic polynomial is \( \phi_{C_4}(x) = x^4 + 1 \) and hence
\[ E(C_4) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ 4 - \frac{4 \imath x (\imath x)^3}{(\imath x)^4 + 1} \right] dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{4}{x^4 + 1} dx = \frac{4}{\pi} \frac{\pi}{2 \sin \frac{\pi}{4}} = 2\sqrt{2}, \]
as calculated in example 4.2.5.

An immediate consequence of Coulson’s integral formula is the following observation, the proof being similar to the proof of Theorem 3.3.2 for signed graphs.

**Theorem 4.3.5.** If \( S \) is a signed digraph on \( n \) vertices, then

\[
E(S) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log |x^n \phi_S(\frac{t}{x})|dx.
\]

### 4.4 NEPS in signed digraphs

We recall that [41] Kronecker product of two matrices \( A = (a_{ij})_{r \times s} \) and \( B = (b_{ij})_{t \times u} \) denoted by \( A \otimes B \) is a matrix of order \( rt \times su \) obtained by replacing each entry \( a_{ij} \) of \( A \) by a block \( a_{ij}B \). Thus \( A \otimes B \) consists of all \( rt \times su \) possible products of an entry of \( A \) with an entry of \( B \). The Kronecker product is a component wise operation, i.e., \( (A \otimes B)(C \otimes D) = (AC) \otimes (BD) \), provided the products \( AC \) and \( BD \) exist. This operation is also associative, so we can define the multiple product \( A_1 \otimes A_2 \otimes \cdots \otimes A_m \). Let order of \( A_i \) be \( r_i \times s_i \). We index elements of \( A_i \) by \( a_{i,j} \); and those of multiple product by a pair of \( m \)-tuples, a row index \( j = (j_1, j_2, \cdots, j_m) \) and a column index \( k = (k_1, k_2, \cdots, k_m) \), where \( 1 \leq j_i \leq r_i \) and \( 1 \leq k_i \leq s_i \). The element \( a_{jk} \) of the product matrix is

\[
a_{jk} = a_{1,j_1,k_1} a_{2,j_2,k_2} \cdots a_{m,j_m,k_m}.
\]

**Lemma 4.4.1.** [23]. Let \( A_i \), for \( i = 1, 2, \cdots, m \), be a square matrix of order \( n_i \) and \( \xi_{ij} \), for \( j = 1, 2, \cdots, n_i \) be its eigenvalues. If \( k_1, k_2, \cdots, k_m \) are non-negative integers, then the \( n_1 n_2 \cdots n_m \) eigenvalues of the matrix \( A_{1}^{k_1} \otimes \cdots \otimes A_{m}^{k_m} \) are \( \xi_{j_1 j_2 \cdots j_m} = \xi_{1j_1}^{k_1} \cdots \xi_{mj_m}^{k_m} \) for \( 1 \leq j_i \leq n_i \). Let \( k_p = (k_{p1}, k_{p2}, \cdots, k_{pm}) \), for \( p = 1, 2, \cdots, q \), be vectors of non-negative integers. Then the \( n_1 n_2 \cdots n_m \) eigenvalues of \( \sum_{p=1}^{q} A_{1}^{k_{p1}} \otimes \cdots \otimes A_{m}^{k_{pm}} \) are \( \xi_{j_1 j_2 \cdots j_m} = \sum_{p=1}^{q} \xi_{1j_1}^{k_{p1}} \cdots \xi_{mj_m}^{k_{pm}} \).

For NEPS in graphs see [14]. The following definition extends this concept to signed digraphs.
**Definition 4.4.2.** Let $\mathcal{B}$ be a set of binary $n$-tuples called basis for the product. The non-complete extended $p$-sum (or simply called NEPS) of signed digraphs $S_1, S_2, \ldots, S_n$ with basis $\mathcal{B}$ denoted by NEPS($S_1, S_2, \ldots, S_n; \mathcal{B}$) is a signed digraph with vertex set $V(S_1) \times V(S_2) \times \cdots \times V(S_n)$. There is an arc from vertex $(u_1, u_2, \ldots, u_n)$ to $(v_1, v_2, \ldots, v_n)$ if and only if there exists $(\beta_1, \beta_2, \ldots, \beta_n) \in \mathcal{B}$ such that $(u_i, v_i) \in A(S_i)$ whenever $\beta_i = 1$ and $u_i = v_i$ whenever $\beta_i = 0$. The sign of the arc is given by

$$
\sigma((u_1, u_2, \ldots, u_n), (v_1, v_2, \ldots, v_n)) = \prod_{i=1}^{n} \sigma_i(u_i, v_i)^{\beta_i} = \prod_{i; \beta_i=1} \sigma_i(u_i, v_i).
$$

Assume that the basis $\mathcal{B}$ has $r \geq 1$ elements, i.e., $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_r\} \subseteq \{0, 1\}^n \setminus \{(0, 0, \ldots, 0)\}$, we define

$$
\text{NEPS}(S_1, S_2, \ldots, S_n; \mathcal{B}) = \bigcup_{\beta \in \mathcal{B}} \text{NEPS}(S_1, S_2, \ldots, S_n; \beta).
$$

**Example 4.4.3.** The Kronecker product $S_1 \otimes S_2 \otimes \cdots \otimes S_n$ of signed digraphs $S_1, S_2, \ldots, S_n$ is the NEPS of these signed digraphs with basis $\mathcal{B} = \{(1, 1, \cdots, 1)\}$; the Cartesian product $S_1 \times S_2 \times \cdots \times S_n$ is NEPS with basis $\mathcal{B} = \{e_i\}$, $i = 1, 2, \cdots, n$, where $e_i$ is $n$-tuple with 1 at $i$th position and 0 otherwise.

The following result shows that two different basis vectors give disjoint arc sets. Proof is similar to signed graphs [23].

**Lemma 4.4.4.** If $S = \text{NEPS}(S_1, S_2, \cdots, S_n; \beta)$ and $S' = \text{NEPS}(S_1, S_2, \cdots, S_n; \beta')$, $\beta \neq \beta'$, then $\mathcal{A}(S) \cap \mathcal{A}(S') = \emptyset$.

The following result gives adjacency matrix and spectra of NEPS in terms of the constituent factor signed digraphs.

**Theorem 4.4.5.** If $S = \text{NEPS}(S_1, S_2, \cdots, S_n; \mathcal{B})$, then the adjacency matrix is given by $A(S) = \sum_{\beta \in \mathcal{B}} A_1^{\beta_1} \otimes \cdots \otimes A_n^{\beta_n}$, and eigenvalues are given by $z_{j_1j_2\cdots j_n} = \sum_{\beta \in \mathcal{B}} z_{1j_1}^{\beta_1} \cdots z_{nj_n}^{\beta_n}$, where $1 \leq j_i \leq |V(S_i)|$, $i = 1, 2, \cdots, n$.

**Proof.** Let $u = (u_{1j_1}, u_{2j_2}, \ldots, u_{nj_n})$ and $v = (v_{1k_1}, v_{2k_2}, \ldots, v_{nk_n})$, where $1 \leq
\[ j_i, k_i \leq |V(S_i)|, \text{ for } i = 1, 2, \ldots, n, \text{ be any two vertices of } S. \text{ Then} \]

\[
[A(S)]_{uv} = \sum_{\beta \in \mathcal{B}} (A^1_{u1j_1} v_{1k_1} (A^2_{u2j_2} v_{2k_2}) (A^n_{u nj_n} v_{nk_n})
\]

\[= \sigma_1 (u_{1j_1}, v_{1k_1}) \sigma_2 (u_{2j_2}, v_{2k_2}) \cdots \sigma_n (u_{nj_n}, v_{nk_n})
\]

\[= a_{1j_1k_1} a_{2j_2k_2} \cdots a_{nj_nk_n} = [\sum_{\beta \in \mathcal{B}} A^1_{u1j_1} \otimes \cdots \otimes A^n_{u nj_n}]_{uv}. \]

The second part of the result follows by Lemma 4.4.1. \qed

We note two special cases of Theorem 4.4.5.

(i) The Kronecker product \( S_1 \otimes S_2 \otimes \cdots \otimes S_n \) has eigenvalues \( z_{j_1j_2\cdots j_n} = z_{1j_1} z_{2j_2} \cdots z_{nj_n}, \) for \( 1 \leq j_i \leq |V(S_i)|, i = 1, 2, \ldots, n. \)

(ii) The Cartesian product \( S_1 \times S_2 \times \cdots \times S_n \) has eigenvalues \( z_{j_1j_2\cdots j_n} = z_{1j_1} + z_{2j_2} + \cdots + z_{nj_n}, \) for \( 1 \leq j_i \leq |V(S_i)|, i = 1, 2, \ldots, n. \)

Germina, Hameed and Zaslavsky \cite{23} considered the problem of balance in NEPS of signed graphs. It is natural to consider the problem of cycle balance for signed digraphs. The next result gives sufficient but not necessary condition for cycle balance of NEPS and the proof follows on same lines as that in undirected case.

**Theorem 4.4.6.** NEPS\((S_1, S_2, \ldots, S_n; \mathcal{B})\) is balanced if \( S_1, S_2, \ldots, S_n \) are cycle balanced.

**Remark 4.4.7.** (i) Theorem 4.4.6 does not have a general converse. A counter example is \( S = \text{NEPS}(\neg C_3, \neg K_2, \{1, 1\}) \), where \( \neg C_3 \) denotes all negative directed cycle of order 3 and \( \neg K_2 \) is symmetric signed digraph of order 2 with both arcs negative. \( S \) is all positive and hence cycle balanced. However \( \neg C_3 \) is non cycle balanced.

(ii) In view of Theorem 4.1.3, the converse of Theorem 4.4.6 is always true if basis \( \mathcal{B} = \{e_i\}, i = 1, 2, \ldots, n \).

Now we have the following result.

**Theorem 4.4.8.** The following statements are equivalent about Cartesian product \( S = S_1 \times S_2 \times \cdots \times S_n \).
(i) $S$ is cycle balanced.
(ii) All of $S_1, S_2, \ldots, S_n$ are cycle balanced.
(iii) $S$ and $S^n$ are cospectral.

Proof. Theorem 4.1.3 implies equivalence of (i) and (iii). Also (ii) of Remark 4.4.7 implies equivalence of (i) and (ii).

4.5 Upper bounds for the energy of signed digraphs

Let $S$ be a signed digraph of order $n$ with adjacency matrix $A(S) = (a_{ij})$. The powers of $A(S)$ count the number of walks in signed manner. Let $w^+_{ij}(l)$ and $w^-_{ij}(l)$ respectively denote the number of positive and negative walks of length $l$ from $v_i$ to $v_j$. The following result relates the integral powers of the adjacency matrix with the number of positive and negative walks.

**Theorem 4.5.1.** If $A$ is an adjacency matrix of a signed digraph on $n$ vertices, then $[A^l]_{ij} = w^+_{ij}(l) - w^-_{ij}(l)$.

**Proof.** We prove the result by induction on $l$. For $l = 1$, the result is vacuously true. For $l = 2$, let $n^+_{ij}$ denote the number of positive neighbours of distinct vertices $v_i$ and $v_j$, and let $n^-_{ij}$ the number of their common negative neighbours and $n^\pm_{ij}$ be the number of neighbours that are positive to one vertex and negative to other. The $(i,i)$ entry of $A^2$ equals $w^+_{ii}(2) - w^-_{ii}(2)$. For $(i,j), i \neq j$, $n^+_{ij} + n^-_{ij} = w^+_{ij}(2)$ and $n^+_{ij} = w^-_{ij}(2)$, so that $(i,j)$th entry$= w^+_{ij}(2) - w^-_{ij}(2)$. Now assume the result to be true for $l = m$.

We have, $[A^{m+1}]_{ij} = [A^mA]_{ij} = \sum_{k=1}^{n} [A^m]_{ik}[A]_{kj} = w^+_{ij}(m+1) - w^-_{ij}(m+1)$, by induction hypothesis. Therefore, the result follows.

In the signed digraph $S$, let $c^+_m$ denote the number of positive closed walks of length $m$ and $c^-_m$ the number of negative closed walks of length $m$. In view of the fact that sum of eigenvalues of a matrix equals to its trace, we have the following observation.

**Corollary 4.5.2.** If $z_1, z_2, \ldots, z_n$ are the eigenvalues of a signed digraph $S$, then $\sum_{j=1}^{n} z_j^m = c^+_m - c^-_m$.

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Now we extend the results of [76] to signed digraphs.

**Lemma 4.5.3.** Let $S$ be a signed digraph having $n$ vertices and $a$ arcs and let $z_1, z_2, \cdots, z_n$ be its eigenvalues. Then

(i) $\sum_{j=1}^{n} (\mathfrak{R}z_j)^2 - \sum_{j=1}^{n} (\mathfrak{I}z_j)^2 = c_2^+ - c_2^-$, (ii) $\sum_{j=1}^{n} (\mathfrak{R}z_j)^2 + \sum_{j=1}^{n} (\mathfrak{I}z_j)^2 \leq a = a^+ + a^-.$

**Proof.** By Corollary 4.5.2, we have

$$c_2^+ - c_2^- = \sum_{j=1}^{n} z_j^2 = \sum_{j=1}^{n} (\mathfrak{R}z_j)^2 - \sum_{j=1}^{n} (\mathfrak{I}z_j)^2 + 2i \sum_{j=1}^{n} \mathfrak{R}z_j \mathfrak{I}z_j.$$

Equating real and imaginary parts proves (i).

By Schur’s unitary triangularization, there exists a unitary matrix $U$ such that the adjacency matrix $A$ of the signed digraph $S$ is unitarily similar to an upper triangular matrix $T = (t_{jk})$ with $t_{jj} = z_j$ for each $j = 1, 2, \cdots, n$. Then

$$\sum_{j,k=1}^{n} |a_{jk}|^2 = \sum_{j,k=1}^{n} |t_{jk}|^2.$$

As $A$ is $(-1, 0, 1)-$matrix, we have

$$a = \sum_{j,k=1}^{n} |\sigma(v_j, v_k)| = \sum_{j,k=1}^{n} |a_{jk}| = \sum_{j,k=1}^{n} |a_{jk}|^2 = \sum_{j,k=1}^{n} |t_{jk}|^2 \geq \sum_{j=1}^{n} |t_{jj}|^2$$

$$= \sum_{j=1}^{n} |z_j|^2 = \sum_{j=1}^{n} \mathfrak{R}z_j^2 + \sum_{j=1}^{n} \mathfrak{I}z_j^2.$$

thereby proving (ii). \(\square\)

**Theorem 4.5.4.** Let $S$ be a signed digraph with $n$ vertices and $a = a^+ + a^-$ arcs, and let $z_1, z_2, \cdots, z_n$ be its eigenvalues. Then $E(S) \leq \sqrt{\frac{1}{2} n(a + c_2^+ - c_2^-)}$.

**Proof.** Subtracting part (i) of Lemma 4.5.3 from (ii), we see that

$$\sum_{j=1}^{n} (\mathfrak{I}z_j)^2 \leq \frac{1}{2} (a - (c_2^+ - c_2^-)).$$

Applying Cauchy-Schwarz inequality to vectors $(|\mathfrak{R}z_1|, |\mathfrak{R}z_2|, \cdots, |\mathfrak{R}z_n|)$ and $(1, 1, \cdots, 1)$, we have
\[ E(S) = \sum_{j=1}^{n} |\Re z_j| \leq \sqrt{n} \left( \sum_{j=1}^{n} (\Re z_j)^2 + \sum_{j=1}^{n} (\Im z_j)^2 \right) \]
\[ \leq \sqrt{n} \sqrt{(c_2^+ - c_2^-) + \frac{1}{2}(a - (c_2^+ - c_2^-))} = \sqrt{\frac{1}{2} n(a + c_2^+ - c_2^-)}. \]

\[ \square \]

**Remark 4.5.5.** (i). The upper bound in Theorem 4.5.4 is attained by signed digraphs \( S_1 = (\frac{n}{2}K_2, +) \), \( S_2 = (\frac{n}{2}K_2, -) \), (where \( \frac{n}{2}K_2, + \) and \( \frac{n}{2}K_2, - \) respectively denote symmetric digraphs obtained from \( +K_2 \) and \( -K_2 \)) and skew symmetric signed digraph of order \( n \). Note that \( \text{spec}(S_1) = \text{spec}(S_2) = \{-1^{(n/2)}, +1^{(n/2)}\} \) and eigenvalues of skew symmetric signed digraph of order \( n \) are of the form \( \pm \iota \alpha \), where \( \alpha \in \mathbb{R} \).

(ii). The above result extends McClleland’s inequality for signed graphs [35] which states that \( E(S) \leq \sqrt{2pq} \), holds for every signed graph with \( p \) vertices and \( q \) edges. Let \( \overrightarrow{S} \) be the symmetric signed digraph of signed graph \( S \), then in \( \overrightarrow{S} \), \( a = 2q = c_2^+ = c_2^- = c_2 \). By Theorem 4.5.4, \( E(S) = E(\overrightarrow{S}) \leq \sqrt{\frac{1}{2}p(2q + 2q)} = \sqrt{2pq} \).

Note that if \( S \) is strongly connected, then \( d_{v_j}^+ \geq 1 \)(respectively, \( d_{v_j}^- \geq 1 \)) for all \( j = 1, 2, \ldots, n \). Therefore, \( a = \sum_{j=1}^{n} d_{v_j}^+ \geq n \). Also, for any signed digraph \( S \), \( a \geq c_2^+ - c_2^- \).

The following result gives the sharp upper bound of energy of signed digraphs in terms of the number of arcs.

**Theorem 4.5.6.** Let \( S \) be a signed digraph with \( a \) arcs. Then \( E(S) \leq a \) with equality if and only if \( S = (\frac{n}{2}K_2, +) \) or \( S = (\frac{n}{2}K_2, -) \) plus some isolated vertices.

**Proof.** If \( S \) is acyclic, then the result is obvious. Assume \( S \) is strongly connected, then by Theorem 4.5.4, we have
\[ E(S) \leq \sqrt{\frac{1}{2} n(a + c_2^+ - c_2^-)} \leq \sqrt{na} \leq \sqrt{a^2} = a. \]
In general, let \( S_1, S_2, \ldots, S_k \) be strong components of \( S \) and let the number of vertices and arcs of \( S_j \) respectively be \( n_j \) and \( a_j \). By Theorem 4.2.7, we have

\[
E(S) = \sum_{j=1}^{k} E(S_j) \leq \sum_{j=1}^{k} a_j \leq a. \tag{4.2}
\]

It is easy to see that if \( S = (\frac{q}{2}K_2, +) \) or \( (\frac{q}{2}K_2, -) \) plus some isolated vertices, then \( E(S) = a \). Conversely, if \( S \) is strongly connected and \( E(S) = a \), then all inequalities in (4.1) are equalities. From (4.1), \( na = a^2 \), which gives \( a = 0 \) or \( a = n \). If \( a = 0 \), then \( S \) is a vertex, otherwise \( a = n \). Also from (4.1), \( \frac{1}{2}n(a+c_2^+-c_2^-) = na \), which gives \( c_2^+ - c_2^- = a = n \), which is possible only if \( S = (K_2, +) \) or \( (K_2, -) \).

In general case, all inequalities in (4.2) are equalities. Also, from \( E(S_j) \leq a_j \), we conclude that \( E(S_j) = a_j \) for \( j = 1, 2, \ldots, k \). Then as earlier \( S_j = (K_2, +) \) or \( (K_2, -) \) or a vertex. So, in this case \( S = (\frac{q}{2}K_2, +) \) or \( (\frac{q}{2}K_2, -) \) plus some isolated vertices.

\[\square\]

**Remark 4.5.7.** Theorem 4.5.6 extends the result for signed graphs [35], which states that \( E(S) \leq 2q \) for every signed graph with \( q \) edges with equality if and only if \( S = (\frac{q}{2}K_2, +) \) or \( (\frac{q}{2}K_2, -) \) plus some isolated vertices.

### 4.6 Equienergetic signed digraphs

Two signed digraphs are said to be isomorphic if their underlying digraphs are isomorphic such that the signs are preserved. Any two isomorphic signed digraphs are obviously cospectral. There exist non isomorphic signed digraphs which are cospectral, e.g., consider the signed digraphs \( S_1 \) and \( S_2 \) shown in Figure 4.2.

Clearly, \( S_1 \) and \( S_2 \) are nonisomorphic, but \( \text{spec } S_1 = \{0^{(6)}\} = \text{spec } S_2 \).

![Figure 4.2](image)

Two nonisomorphic signed digraphs \( S_1 \) and \( S_2 \) of same order are said to be...
equienergetic if $E(S_1) = E(S_2)$. Rada [76] proved the existence of pairs of non-symmetric and non cospectral equienergetic digraphs. Cospectral signed digraphs are obviously equienergetic, therefore the problem of equienergetic signed digraphs reduces to the problem of construction of non cospectral pairs of equienergetic signed digraphs such that for every pair not both signed digraphs are cycle balanced.

We have the following result.

**Theorem 4.6.1.** Let $S$ be a signed digraph of order $n$ having eigenvalues $z_1, z_2, \cdots, z_n$ such that $|\Re z_j| \leq 1$ for every $j = 1, 2, \cdots, n$. Then $E(S \times \overrightarrow{K_2}) = 2n$.

**Proof.** Let $z_1, z_2, \cdots, z_t$ be eigenvalues with nonnegative real part and $z_{t+1}, \cdots, z_n$ be those with negative real part. Eigenvalues of Cartesian product $S \times \overrightarrow{K_2}$ are $z_1 \pm 1, z_2 \pm 1, \cdots, z_t \pm 1, z_{t+1} \pm 1, \cdots, z_n \pm 1$. Therefore

$$E(S \times \overrightarrow{K_2}) = \sum_{j=1}^{t} (|\Re z_j + 1| + |\Re z_j - 1|) + \sum_{j=t+1}^{n} (|\Re z_j + 1| + |\Re z_j - 1|).$$

As $|\Re z_j| \leq 1$, for all $i = 1, 2, \cdots, n$, it follows that

$$E(S \times \overrightarrow{K_2}) = \sum_{j=1}^{t} (|\Re z_j + 1 + 1 - \Re z_j|) + \sum_{j=t+1}^{n} (|\Re z_j + 1 - \Re z_j + 1|) = 2t + 2(n-t) = 2n.$$

Now we have the following consequence.

**Corollary 4.6.2.** For $n \geq 2$, $E(C_n \times \overrightarrow{K_2}) = E(C_n \times \overrightarrow{K_2}) = 2n$. Moreover, $C_n \times \overrightarrow{K_2}$ and $C_n \times \overrightarrow{K_2}$ are non cospectral signed digraphs with $2n$ vertices.

**Proof.** We know the eigenvalues of $C_n$ are $e^{i(2j+1)n\pi}, j = 0, 1, \cdots, n-1$ and those of $C_n$ are $e^{2jn\pi}, j = 0, 1, \cdots, n-1$. Clearly, eigenvalues of $C_n$ and $C_n$ meet the requirement of Theorem 4.6.1, so $E(C_n \times \overrightarrow{K_2}) = E(C_n \times \overrightarrow{K_2}) = 2n$. Moreover, $2 \notin \spec(C_n \times \overrightarrow{K_2})$, but $2 \in \spec(C_n \times \overrightarrow{K_2})$ implying that $C_n \times \overrightarrow{K_2}$ and $C_n \times \overrightarrow{K_2}$ are non cospectral. The number of vertices in both signed digraphs is $2n$ which follows by the definition of Cartesian product. In view of Remark 4.4.7 (ii), $C_n \times \overrightarrow{K_2}$ is
non cycle balanced, whereas $C_n \times \overrightarrow{K}_2$ is cycle balanced.

**Example 4.6.3.** For each odd $n$, $C_n$ and $C_n$ is a non cospectral pair of equienergetic signed digraphs, because $\text{spec}(C_n) = -\text{spec}(C_n)$ and $1 \notin \text{spec}(C_n)$ but $1 \in \text{spec}(C_n)$.

From Corollary 4.6.2 and Example 4.6.3, we see for each positive integer $n \geq 3$, there exits a pair of non cospectral signed digraphs with one signed digraph cycle balanced and another non cycle balanced. Now we construct pairs of non cospectral equienergetic signed digraphs of order $2n$, $n \geq 5$ with both constituents non cycle balanced. Let $P^l_n$ ($n \geq l + 1$) be a signed digraph obtained by identifying one pendant vertex of the path $P_{n-l+1}$ with any vertex of $C_l$. Sign of non cyclic arcs is immaterial.

**Theorem 4.6.4.** For each $n \geq 5$, $P^3_n \times \overrightarrow{K}_2$ and $P^4_n \times \overrightarrow{K}_2$ is a pair of non cospectral equienergetic signed digraphs of order and energy equal to $2n$.

**Proof.** Using the fact that $\phi_{P^l_n}(x) = x^{n-l} \phi_{C_l}(x)$ and Theorem 4.6.1, it follows that $E(P^3_n \times \overrightarrow{K}_2) = E(P^4_n \times \overrightarrow{K}_2) = 2n$. Now $1$ is an eigenvalue of $P^3_n \times \overrightarrow{K}_2$ with multiplicity $n-3$ but $1$ is an eigenvalue of $P^4_n \times \overrightarrow{K}_2$ with multiplicity $n-4$, therefore these two signed digraphs are non cospectral. The order of both signed digraphs equals to $2n$ follows by the definition of Cartesian product. In view of Remark 4.4.7 (ii), it follows that both $P^3_n \times \overrightarrow{K}_2$ and $P^4_n \times \overrightarrow{K}_2$ are non cycle balanced.

4.7 Conclusion

We conclude with the following open problems.

**Problem 4.7.1.** Characterize signed digraphs with energy equal to the number of vertices.

**Problem 4.7.2.** Determine bases other than $\{e_i\}$ for which converse of Theorem 4.4.6 holds.