CHAPTER 1

INTRODUCTION

1.1 A BRIEF HISTORY

In fuzzy set theory, cardinality, fuzzy entropy and subsethood measure are some basic concepts. Further, fuzzy matrix theory also occupies a very important position in the theory of fuzzy sets.

Cardinality in a crisp set theory represents the number of elements in a given set and due to the graded membership of elements to given sets in fuzzy set theory; it has a different intuitive calculation. Cardinality of fuzzy sets plays a vital role in finding entropy, subsethood and similarity measures of fuzzy sets. There are two approaches to find the cardinality of fuzzy sets - a non fuzzy cardinality and a fuzzy cardinality. The first one contains the so called scalar approach which has a single ordinary cardinal number or a non negative real number as the object of expressing the size of fuzzy set. Fuzzy cardinality produces a fuzzy set as the measure. Here the main interest is in scalar cardinality of fuzzy sets because it is less complicated and gives results that can be compared directly to other measures. A basic definition of scalar cardinality named as the power of finite fuzzy set, was proposed by Luca and Termini [14]. This definition is simple and the power of a finite
fuzzy set A is given by the sum of the membership degrees of the fuzzy set A. In the simple case, the scalar cardinality of a fuzzy set

$$\mu_A: X \rightarrow [0,1] . \hspace{1cm} \text{(1.1.1)}$$

is defined as the sum of the membership degrees of finite fuzzy set i.e \(|A| = \sum_{x \in X} \mu_A(x)\).

Some authors refer to \(|A|\) as the \(\sum \text{count}\) (sigma count). An extension of this definition to p-power of finite fuzzy sets was proposed by Kaufmann [28] and Gottwald [21]. Another definition of scalar cardinality based on FECount, was introduced by Rolescu [58] and properties of scalar cardinalities of fuzzy sets could be found for example in ([58], [72], [73], [74], [80]). An axiomatic approach to the scalar cardinality of fuzzy sets was suggested by Wygralak ([72], [73] & [74]). Zadeh ([78]) introduced the relative measure of scalar cardinality of fuzzy sets.

Subsethood measure describes the degree to which a fuzzy set is a subset of another fuzzy set. At present many researchers have contributed to the area of fuzzy subsethood ([19], [20], [35], [45], [47], [51], [63], [68], [71], [76], [77]). Many other subsethood measures have also been proposed in the literature.

Kosko ([42], [43]) sets out again the measure originally proposed by Sanchez [60] getting it two fold, in a geometric and algebraic way. Several authors take an axiomatic approach to the study of inclusion measure i.e they provide a list of properties (“axioms”) which a reasonable inclusion measure must satisfy and then they examine whether a particular
inclusion measure or a list of inclusion measures possess these properties. A prime example of this approach is Sinha and Dougherty’s [63]. They list nine properties that a reasonable inclusion measure should have and then proceed to introduce inclusion measure which possesses these properties. A somewhat different point of view is taken by Bandler and Kohout [1]. These authors obtain several inclusion measures from fuzzy implication operators. A related approach is that of Willmott [69], where the transitivity of inclusion measure is also studied. Axiomatization and connection to implication operators are combined in number of papers. Young [77] combines the above approaches and also connects inclusion measure to fuzzy entropy. Burillo et al. [12] introduced a particular family of implication operators and show that the inclusion measure obtained from this family (in Bandler and Kohout’s manner), satisfy Sinha and Dougherty’s axioms for inclusion measures. Fan. et. at [20] discussed the connection between inclusion measure, fuzzy entropy and fuzzy implication. They comment on Young’s axioms and proposed their own list of axioms.

Fuzziness a feature of uncertainty results from lack of sharp distinction of being or not being a member of the set. Measure of fuzziness used and cited in the literature is the fuzzy entropy was chosen due to intrinsic similarity with Shannon’s entropy. However the two function measures fundamentally different types of uncertainty. Basically, Shannon’s entropy measures the average uncertainty in bits associated with prediction of outcomes in a random experiments. Fuzzy entropy is the measurement of fuzziness of fuzzy sets and thus has an important position in fuzzy systems such as fuzzy decision making systems, fuzzy control systems, fuzzy neural network systems and fuzzy management information
systems. In other words, measuring fuzziness of a fuzzy set is an important step in fuzzy system. Depending on its context, entropy is used for quantifying the amount of disorder of information, or of fuzziness usually defined within either a statistical or fuzzy framework.

Until now, fuzzy entropy calculus has been studied from different angles and consequently several methods for calculating the entropy was defined. Kaufmann [28] proposed a way of measuring the degree of fuzzification of fuzzy sets using fuzzy distance between the nearest non fuzzy sets and the fuzzy set. Another way of measuring the entropy was proposed by Yager ([75] & [76]) and this is to measure the distance between the fuzzy set and its complement. Another approach is to use the entropy function to measure the entropy.

The purpose of this thesis is to introduce a new definition of entropy of fuzzy numbers and then to give an overview of some of the interesting results related to entropy of fuzzy numbers.

Fuzzy matrices also occupied a very important position in our works. Matrices with entries in \([0, 1]\) and matrix operation defined by fuzzy logical operations are called fuzzy matrices. Fuzzy matrices were introduced first time by Thomason [66]. All fuzzy matrices are matrices but every matrix is not a fuzzy matrix. Fuzzy matrices play an important role in scientific development. Fuzzy matrices play a fundamental role in fuzzy set theory. They provide us with a rich framework within which many problems of practical applications of the theory can be formulated. Fuzzy matrices can be successfully used when fuzzy uncertainty occurs in a problem. These results are extensively used for cluster analysis and
classification problem of static patterns under subjective measure of similarity. On the other hand, fuzzy matrices are generalized Boolean matrices which have been studied for fruitful results. And the theory of Boolean matrices can be back to the theory of matrices with non negative contents, for which most famous classical results were obtained 1907 to 1912 by Parren and Frobenius. So the theory of fuzzy matrices is interesting in its own right. Several authors have presented a number of results on the convergence of power sequence of fuzzy matrices ([23], [40], [66]). Ragab et. al [55] presented some properties on determinant and adjoint of square fuzzy matrices. Kim [31] presented some properties of the determinant of square fuzzy matrices. Two new operators and some applications of fuzzy matrices are given in [64]. The complement of a fuzzy matrix is used to analyze the complement nature of any system considered e.g. if A represent the crowded network of a system at a particular point of time, then its complement $A^c$, stands for the clear network of the system at that point of time. An important connection between fuzzy sets and fuzzy matrices has been recognized and this has led us to define fuzzy matrices in a quite different way. This will inevitably play an important role in any problem area that involves complementation of fuzzy matrices.

1.2 OBJECTIVES OF THE RESEARCH

The main objective of this research is to bring about an attention of the fact that there are some shortcomings in the existing definition of complementation of fuzzy sets and as such the results associated with it should have to be given proper attention. Here we make an attempt to extend some basic concepts and results, particularly the ideas of reference
function to fuzzy settings and to study their properties. This research intends to promote and explore some new definitions and associated properties with particular focus on the complementation of fuzzy sets. It is also intended to study the entropy of fuzzy numbers and their properties with the help of the Randomness-Fuzziness Consistency Principle. Considering these points, the present research has been undertaken with the following objectives:

(i) To implement a new definition of complementation of fuzzy sets on the basis of reference function.
(ii) To define cardinality of fuzzy sets on the basis of reference function.
(iii) To comment on the existing entropy-subsethood relationships and also to replace some existing properties of subsethood measure of fuzzy sets with some new ones to avoid any controversy.
(iv) To propose a new definition of entropy of fuzzy numbers.
(v) To propose some properties of entropy of fuzzy numbers under arithmetic operations.
(vi) To represent fuzzy matrices on the basis of reference function and to define determinant and adjoint of fuzzy matrices accordingly.

In this thesis we are trying to bring out some interesting aspects of superimposition. The main focus of this research is to develop a new methodology to discuss more appropriately the results obtained through conventional methods and to apply the proposed methodology in different areas under considerations.
1.3 CONCEPT OF FUZZINESS

Processing of inexact data is one of the challenging problems for modern engineers and practitioners in different areas. Since Kolmogorov [39] had introduced the axiomatic of modern probability theory in 1933 and even before it was the main tool used by the scientists. In real world, complexity often arises due to complexity in the form of ambiguity. The theory of probability has been an age old and effective tool to handle uncertainty, but it can be applied to situations whose characteristics are based on random processes. Uncertainty arises due to partial information about the problem, or due to information which is not fully reliable. But the apparatus of probability theory cannot cover all kinds of vagueness, thus the necessity of the new approach was obvious. At the beginning of 20th century, philosophers and mathematicians actively discussed impossibility of putting real processes or objects into the strict frames based on the principles of bivalent logic. Thus alternative method was required to fill up the gap. The notion of fuzzy set stems from the observation made by Zadeh [78] that “more often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership”. This observation emphasizes the gap existing between mental representation of reality and usual mathematical representation thereof, which are based on binary logic, precise numbers and the like. Zadeh’s classic paper of 1965 opened up a new area of modern mathematics, namely, Fuzzy Set Theory, even though some ideas presented in the paper were envisioned some 30 years earlier by the American philosopher Black [11]. This is a non-probabilistic measure of uncertainty. Kaufmann [29], Kandel and Lee [26] and Dubois and Prade [16] and many other researchers further contributed to the
development of the Fuzzy Set theory. Zadeh [81] introduced fuzzy set and mathematical
definition of inclusion, union, intersection, compliment relation and convexity in his first
paper ‘Fuzzy Sets’. His work was in the purpose of modelling how people reach at
conclusions when the information available is imprecise, incomplete and not reliable.
According to him a fuzzy set is a class with unsharp boundaries whereas classical set theory
has well defined boundaries, Treadwell [67]. Fuzziness is a type of imprecision that stems
from grouping of elements into classes that do not have sharply defined boundaries. Such
classes are called fuzzy sets. Fuzziness is associated with subjective judgement. Fuzzy set
theory prescribes a calculus for the treatment of uncertainty associated with the
imprecision, Zadeh [79]. Fuzzy set was developed to generalize classical set theory in such
a manner as to allow the possibility of partial membership in a set. The concept of fuzziness
can be found in many areas of daily life where human judgement, evaluation and decision
are important. The aim of the theory of fuzzy set is to develop a methodology for the
formulation and solution of problems that are complex or ill defined, to be susceptible to
analysis by conventional techniques.

1.3.1 THE UNIVERSE OF DISCOURSE

All elements in a set are taken from a universe of discourse or universe set that contains all
the elements that can be taken into consideration when the set is formed. In reality there is
no such thing as a set or a fuzzy set because all sets are subsets of some universe set, even
though the term 'set' is predominantly in use. In the fuzzy case, each element in the universe
set is a member of the fuzzy set to some degree, even zero. The set of elements that have a
non-zero membership is referred to as the *support*. We will use the notation $X$ for the *universe set*.

1.3.2 FUZZY SET

The notion of fuzzy sets is an extension of the most fundamental property of sets. Fuzzy sets allow a grading of *to what extent* an element of a set belongs to that specific set. It is a class of objects with a continuum of grades of membership. Unlike classical or crisp set, where an element either belongs to or does not belong to the set, in a fuzzy set there is a gradual assessment of the membership of elements in a set. Let us have an example of white colour (A) and black colour (B). Then what will be with the grey colour? In classical set theory, grey colour does not belong to any of these two sets and to represent these grey colour we need another set grey (G). But in fuzzy set theory these grey balls belong to set A and B with some graded membership. The darker the grey, the more it tends to be a member of the set black and less the member of the set white. It is full of ambiguity and uncertainty and it is reasonable to use the fuzzy membership assignment, Treadwell [67]. Therefore fuzzy sets support a flexible sense of membership of an element to a set. All the information contained in a fuzzy set is described by its membership function.

If $X$ is a universe of discourse and $x$ is any particular element of $X$, then a fuzzy set $F$, defined on $X$ may be written as a collection of ordered pairs

$$F = \{(x, \mu_{F}(x)); x \in X\} \quad \text{..............................(1.3.2)}$$
where

\( \mu_f: X \rightarrow [0,1] \), is called the *membership function* or *grade of membership* of \( x \) in \( F \) and each pair \((x, \mu_f(x))\) is called a *singleton*.

1.3.3 MEMBERSHIP FUNCTION

All information contained in a fuzzy set is described by its membership function. The membership function of a fuzzy set defines how the grade of membership of an element in the set is determined. It is the graphical representation of the magnitude of participation of each input. It associates a weight with each of the inputs, and ultimately determines an output response. The following two types of notations are commonly used to denote *membership function*.

(i) The membership function of a fuzzy set \( A \) is denoted by \( \mu_A \), i.e. \( \mu_A: X \rightarrow [0,1] \)

(ii) The membership function of the fuzzy set \( A \) has the following form:

\[ A: X \rightarrow [0,1] \]

In this research work, we shall use the notation \( \mu_A: X \rightarrow [0,1] \) to denote membership function.

There are different membership functions associated with each input and output response. Some features of the fuzzy membership functions are

*Shape*: Generally, the membership functions are triangular in shape, but, trapezoidal, bell and exponential shape membership functions are also used.
Core: The core of the membership function of a fuzzy set F is defined as the reason of the universe that is characterized by complete and full membership in the set F, i.e.

\[ core(F) = \{ x \in X | \mu_F(x) = 1 \} \] ................................ (1.3.3)

Support: The support of the membership function of a fuzzy set F is defined as the region of the universe that is characterized by non zero membership in the set F, i.e.

\[ support(F) = \{ x \in X | \mu_F(x) > 0 \} \] ...................................... (1.3.4)

Boundary: The boundaries of the membership function of a fuzzy set F is defined as the region of the universe which contains elements that have non zero membership but not have full membership, i.e.

\[ Boundary(F) = \{ x \in X | 0 < \mu_F(x) < 1 \} \] .......................... (1.3.5)

Height: The height of a fuzzy set F is the maximum value of the membership function, i.e.

\[ hgt(F) = \max \{ \mu_F(x) \} \] ............................................. (1.3.6)

If the \( hgt(F) = 1 \), the fuzzy set is said to be normal fuzzy set and if \( hgt(F) < 1 \), the fuzzy set is said to be subnormal.

1.3.4 BASIC OPERATIONS ON FUZZY SETS

Three basic operations on crisp sets – the complement, intersection and union can be generalized to fuzzy sets in more than one way. However, one particular generalization,
which results in operations that are usually referred to as standard fuzzy set operations, has a specific significance in fuzzy set theory. In the following, we shall note down only the standard operations.

Let us consider two fuzzy sets $A$ and $B$ in the universe $X$ defined with membership functions

$$A = \{(x, \mu_A(x))\}, x \in [0,1],$$
$$B = \{(x, \mu_B(x))\}, x \in [0,1]$$

The operations with $A$ and $B$ are as follows:

(i) Equality: The fuzzy sets $A$ and $B$ are equal denoted by $A = B$ if and only if for every $x \in X$,

$$\mu_A(x) = \mu_B(x)$$

(ii) Containment: The fuzzy set $A$ is contained in fuzzy set $B$ denoted $A \subseteq B$ if for every $x \in U$,

$$\mu_A(x) \leq \mu_B(x)$$

(iii) Complementation: The fuzzy sets $A$ and $A^c$ are complementary if

$$\mu_{A^c}(x) = 1 - \mu_A(x)$$

(iv) Union: The operation intersection of $A$ and $B$ denoted as $A \cup B$ is defined by

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$
(v) Intersection: The operation intersection of A and B denoted as $A \cap B$ is defined by

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

(vi) Product: Algebraic product of the fuzzy sets A and B denoted by AB is defined by

$$\mu_{AB}(x) = \mu_A(x)\mu_B(x)$$

1.3.5 PROPERTIES OF FUZZY SETS

Some important properties involving complementation, intersection and union of fuzzy sets A, B and C are presented below:

(i) Idempotence: $A \cap A = A$ and $A \cup A = A$

(ii) Commutativity: $A \cap B = B \cap A$ and $A \cup B = B \cup A$

(iii) Associativity: $(A \cap B) \cap C = A \cap (B \cap C)$ and $(A \cup B) \cup C = A \cup (B \cup C)$

(iv) Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(v) Double complementation: $(A^c)^c = A$

(vi) DeMorgan’s laws: $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$

(vii) Exclusion: $A \cup A^c \neq X$

(viii) Contradiction: $A \cap A^c \neq \emptyset$

(ix) Absorption: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$

(x) Identity: $A \cup \emptyset = A$, $A \cap X = A$ where $\emptyset$ and X are null set and universal set respectively.
It is important to note here that it is considered for fuzzy sets that the excluded middle laws do not hold for fuzzy sets.

1.3.6 TYPES OF FUZZY SETS

(i) *Normal fuzzy sets*: A fuzzy set is called normal when at least one of its elements attains the maximum possible membership grade. That is a fuzzy set $A$ of a set $X$ is said to be a normal fuzzy set if

$$\mu_A(x) = 1 \text{ for at least one } x \in X.$$ 

(ii) *Convex fuzzy set*: A fuzzy set $A$ of a set $X$ is convex if

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$$

The concept of fuzzy number can be defined from the concept of fuzzy subset with the help of two properties which are convexity and normality of fuzzy subset.

1.3.7 FUZZY NUMBER

The concept of fuzzy numbers has been presented by different authors in different ways. Some important works on the concept of fuzzy numbers can be found in Rodabaugh [56], Dubois and Prade ([15] & [16]), Nahmias [50] and Hutton [24]. A fuzzy number is an extension of ordinary number in the sense that it does not refer to one single value but refers to a set of possible values, where each possible value has its own degree of
membership between 0 and 1. The maximum of presumptions is taken at the level 1 and minimum of presumption is considered at the level 0.

Fuzzy number can be used for uncertainty where the interval of confidence can be defined. The concept of uncertain or fuzzy number can be presented in many ways. Fuzzy numbers are a connector between the fuzzy set theory and the confidence theory.

If a fuzzy set is convex and normalized, and its membership function is defined on $\mathbb{R}$ and piecewise continuous, it is called a fuzzy number. Fuzzy numbers are used in statistics, computer programming, engineering (especially communications) and in experimental sciences. The concept takes into account of the fact that all phenomena in the physical universe have a degree of inherent uncertainty. To deal with fuzzy numbers, the method of $\alpha$-cuts plays a very significant role.

But in this thesis we are trying to bring some interesting aspects of complementation of fuzzy sets defined on the basis of reference function.

1.3.8 TYPES OF FUZZY NUMBERS

The shapes of membership function describing fuzzy numbers are in general arbitrary. However, in practice there are several types of fuzzy numbers that can be described by a suitable set of parameters.
According to shapes different types of fuzzy numbers are – triangular fuzzy number, trapezoidal fuzzy number, bell shaped fuzzy number and exponential fuzzy numbers. Among the various shapes of fuzzy numbers Triangular Fuzzy Number is most popular in use, Lee [46].

**Triangular Fuzzy Number:** A triangular fuzzy number is presented by three points as 
\[ A = [a_1, a_2, a_3], \quad a_1, a_2, a_3 \in \mathbb{R} \] is defined by the membership function

\[
\mu_A(x) = \begin{cases} 
0, & x < a_1 \\
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
\frac{-x+a_3}{a_3-a_2}, & a_2 \leq x \leq a_3 \\
0, & x > a_3
\end{cases}
\]

\[ \text{..........................(1.3.7)} \]
Figure 1.1: Triangular Fuzzy Number

Trapezoidal Fuzzy Number: The ideas of trapezoidal fuzzy numbers come from the fact that there are several points whose membership degrees are maximum.

Trapezoidal fuzzy number is represented as \( A = [a, b, c, d] \) and is defined with membership function as

\[
\mu_A(x) = \begin{cases} 
\frac{x-a}{b-a}, & a \leq x \leq b \\
1, & b \leq x \leq c \\
\frac{x-d}{c-d}, & c \leq x \leq d \\
0, & \text{otherwise} 
\end{cases}
\] .......(1.3.8)
1.3.9 COMPLEMENT OF A FUZZY NUMBER

According to the Zadehian definition, if a normal fuzzy number \( N = [\alpha, \beta, \gamma] \) is associated with a membership function \( \mu_N(x) \), where

\[
\mu_N(x) = \begin{cases} 
\psi_1(x), & \alpha \leq x \leq \beta \\
\psi_2(x), & \beta \leq x \leq \gamma \\
0, & \text{otherwise}
\end{cases}
\]  
(1.3.9)

the complement \( N^c \) will have the membership function \( \mu_N^c(x) \),
where

\[
\mu_N^c(x) = \begin{cases} 
1 - \psi_1(x), & \alpha \leq x \leq \beta \\
1 - \psi_2(x), & \beta \leq x \leq \gamma \\
1, & \text{otherwise}
\end{cases}
\]  \hspace{1cm} \text{...........................(1.3.10)}

Figure 1.3 Complement of a fuzzy number

1.3.10 ALPHA CUT OF FUZZY NUMBERS

An \( \alpha \)-cut of a fuzzy number \( A \) is a crisp set \( A^\alpha \) that contains all the elements in the universal set that have membership value in \( A \) greater than or equal to \( \alpha \) i.e.

\[
A^\alpha = \{ x \mid A(x) \geq \alpha \}
\]  \hspace{1cm} \text{............... (1.3.11)}
1.3.11 ARITHMETIC WITH FUZZY NUMBERS

Operation of fuzzy number can be generalized from that of crisp interval. Let us have a look at the operation of interval. Let, $A = [a_1, a_2], B = [b_1, b_2]$ are expressed in interval and $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

Let $*$ denotes any of the four arithmetic operations on closed intervals. A general property of all arithmetic operation on closed intervals is given by -

$[a_1, a_2] * [b_1, b_2] = [f \ast g | a_1 \leq f \leq a_2, b_1 \leq g \leq b_2]$ where $0 \notin [b_1, b_2]$ in defining multiplication and $[a_1, a_2] / [b_1, b_2]$ is not defined when $0 \in [b_1, b_2]$. The arithmetic operation of closed interval is also closed interval. The basics of interval arithmetic are given next.

The main operations of closed intervals are

i) Addition $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$

ii) Subtraction $[a_1, a_2] - [b_1, b_2] = [a_1 - b_1, a_2 - b_2]$

iii) Multiplication $[a_1, a_2] \cdot [b_1, b_2] = [\min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2), \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)]$

provided $0 \notin [b_1, b_2]$

iv) Division-$\frac{[a_1, a_2]}{[b_1, b_2]} = [\min\left(\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right), \max\left(\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right)]$

provided $0 \notin [b_1, b_2]$
Inverse of an interval is given by 

\[ [a_1, a_2]^{-1} = [\min\left(\frac{1}{a_1}, \frac{1}{a_2}\right), \max\left(\frac{1}{a_1}, \frac{1}{a_2}\right)] \]

Fuzzy arithmetic is a well-formulated theory, Kaufmann and Gupta [30] that allows for precise management of uncertain values e.g. values occurring when choosing among alternatives produced by measurement tools or resulted from expert subjective judgements. It is an extension of standard arithmetic based on two properties of fuzzy numbers. The first one is that each fuzzy set and thus also each fuzzy number can fully and uniquely be represented by its \( \alpha \) cuts. The second property is that \( \alpha \) cut of each fuzzy number are closed interval of real numbers for all \( \alpha \in (0,1] \). These properties enable us to define arithmetic operations on fuzzy number in terms of arithmetic operation on their \( \alpha \) -cuts.

An excellent text on fuzzy arithmetic is the one by Kaufmann and Gupta [30], also Klir and Yuan [36] in their books cover the basis of fuzzy arithmetic rather well. In fuzzy sets subtraction and division are defined as the obvious extension to addition and multiplication respectively.

**Addition:** If \( A = [a_1, a_2, a_3] \) and \( B = [b_1, b_2, b_3] \) are triangular fuzzy numbers then \( A + B \) is also a triangular fuzzy numbers. Let us use the interval of confidence.

\[ A^\alpha = [a_1 + (a_2 - a_1)\alpha, a_3 - (a_3 - a_2)\alpha] \]

and

\[ B^\alpha = [b_1 + (b_2 - b_1)\alpha, b_3 - (b_3 - b_2)\alpha] \]

Then \( A^\alpha + B^\alpha = [a_1 + b_1 + (a_2 + b_2 - a_1 - b_1)\alpha, a_3 + b_3 - (a_3 + b_3 - a_2 - b_2)\alpha] \) ..............(1.3.12)
∀ \( x, a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}, a_1 \leq a_2 \leq a_3, b_1 \leq b_2 \leq b_3 \)

Now, \( x_1 = a_1 + b_1 + (a_2 + b_2 - a_1 - b_1)\alpha \)

and

\[ x_2 = a_3 + b_3 - (a_3 + b_3 - a_2 - b_2)\alpha \]

Solving these above two equations for \( \alpha \) we shall have

\[ \alpha = \frac{x_1 - (a_1 + b_1)}{a_2 + b_2 - a_1 - b_1} \]

............... (1.3.13)

\[ \alpha = \frac{-x_2 + a_3 + b_3}{a_3 + b_3 - a_2 - b_2} \]

..........................(1.3.14)

As \( \alpha \in [0,1] \), from (1.3.13) it is seen that \( \alpha = 0 \) if \( x_1 = a_1 + b_1 \) and \( \alpha = 1 \) if \( x_1 = a_2 + b_2 \)

Again from (1.3.14), it is seen that \( \alpha = 0 \) if \( x_2 = a_3 + b_3 \) and \( \alpha = 1 \) if \( x_2 = a_2 + b_2 \)

Therefore the membership function of \( A^\alpha + B^\alpha \) would be

\[ \mu_{A_1\oplus B_1}(x) = \begin{cases} 
0, & x < a_1 + b_1 \\
\frac{x - (a_1 + b_1)}{a_2 + b_2 - a_1 - b_1}, & a_1 + b_1 \leq x \leq a_2 + b_2 \\
\frac{-x + a_3 + b_3}{a_3 + b_3 - a_2 - b_2}, & a_2 + b_2 \leq x \leq a_3 + b_3 \\
0, & x > a_3 + b_3 
\end{cases} \]

..............................(1.3.15)
Thus for addition operation we are able to write

\[ [a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1 + b_1, a_2 + b_2, a_3 + b_3] \]

**Subtraction**: Subtraction of triangular fuzzy number \( B \) from the triangular fuzzy number \( A \) i.e. \( A - B \) is nothing but the addition of \( A \) and the opposite image of \( B \). The opposite image of \( B \) is \( B^\prime = [b_1, b_2, b_3]^\prime = [-b_3, -b_2, -b_1] \). And the \( \alpha \)-cut interval of this opposite image will be \( B^\alpha = [-b_3 + (b_3 - b_2)\alpha, -b_1 - (b_2 - b_1)\alpha] \)

Thus the \( \alpha \) cut interval of \( A - B \) would be

\[
A^\alpha - B^\alpha = A^\alpha + B^\prime^\alpha
\]

\[
= [a_1 - b_3 + (a_2 - b_2 - a_1 + b_1)\alpha, a_3 - b_1 - (a_3 - b_3 - a_2 + b_2)\alpha] \]

\[ \forall x, a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}, a_1 \leq a_2 \leq a_3, b_1 \leq b_2 \leq b_3 \]

The fuzzy membership function of \( A^\alpha - B^\alpha \) would be

\[
\mu_{A-B}(x) = \begin{cases} 
0, x < a_1 - b_3 \\
\frac{x - (a_1 - b_3)}{a_2 - b_2 - a_1 + b_1}, a_1 - b_3 \leq x \leq a_2 - b_2 \\
\frac{-x + a_3 - b_1}{a_3 - b_1 - a_2 + b_2}, a_2 - b_2 \leq x \leq a_3 - b_1 \\
0, x > a_3 - b_1 
\end{cases} \]

\[ \text{...... (1.3.17)} \]

Hence, if \( A \) and \( B \) are triangular fuzzy numbers then \( A^\prime, B^\prime, A - B \) are also triangular fuzzy numbers and thus for subtraction operation we are able to write
\[ [a_1, a_2, a_3] - [b_1, b_2, b_3] = [a_1 - b_3, a_2 - b_2, a_3 - b_1] \]

**Multiplication:** If a triangular fuzzy number is multiplied by \( k, k \in \mathbb{R} \) then \( k.A \) is also a triangular fuzzy numbers

\[
k.A = [k.a_1, k.a_2, k.a_3], k > 0
\]

\[
= [k.a_1, k.a_2, k.a_3], k < 0
\]

\[
= [0, k, 0, 0], k = 0
\]

Let us consider the multiplication of two triangular fuzzy numbers \( A \) and \( B \). Then the \( \alpha \) cut interval of \( A \cdot B \) would be

\[
A^\alpha B^\alpha = [a_1^\alpha, a_2^\alpha, a_3^\alpha][b_1^\alpha, b_2^\alpha, b_3^\alpha] \quad \text{..................(1.3.18)}
\]

**Division:** The division of two triangular fuzzy numbers \( A \) and \( B \) is defined as

\[
A^\alpha / B^\alpha = \left[ \min \left( \frac{a_1^\alpha}{b_1^\alpha}, \frac{a_2^\alpha}{b_2^\alpha}, \frac{a_3^\alpha}{b_3^\alpha} \right), \max \left( \frac{a_1^\alpha}{b_1^\alpha}, \frac{a_2^\alpha}{b_2^\alpha}, \frac{a_3^\alpha}{b_3^\alpha} \right) \right]
\]

\[
\quad \text{.......................... (1.3.19)}
\]

1.4 CONTRVERSIES OVER THE FUZZY SET THEORY

There is a controversy over the application of fuzzy theory to real world events. Fuzzy theory is not the panacea for dealing with the world of uncertainty in certain terms, but it is not a contender. Fuzzification is a natural way to insert uncertainty and imprecision in
modeling. Uncertainty can be explained in many ways, by confidence interval, by the Laplace law of equiprobability etc. Uncertainty can be in the form of fuzzy, vague or ambiguous. In fuzzy set theory one can apply several kind of structural uncertainty. Uncertainty involved in any problem-solving situation is a result of some information deficiency. Information may be incomplete, imprecise, fragmentary, not fully reliable, contradictory or deficient in some other ways. Uncertainty is an epistemological problem. The fuzzy set provides us with an intuitively pleasing method of representing a form of uncertainty. Fuzzy set handles uncertainty as deterministic where Bayesian theory see probabilities and fuzzy theory see different amount of membership in events that are not probable but as real event, Treadwell [67]. This process may not always be capable of deciding upon which alternative an organization should select. That is why the application of the fuzzy set decision model may not always have to conclusive result. It has been said that fuzzy theory and its mathematical application are not as easily applied as the more commonly used concept. Some of the papers are mentioned below to show how the researchers in the field of the theory of fuzzy sets criticized some concepts of the theory whereas some of them made an attempt to redefine it from their standpoints which can be observed from the following works:

Shimoda [61] presented a new and natural interpretation of fuzzy sets and fuzzy relations, but still did not change the fact that it could not satisfy all formulas of the classical set system.
Piega [53] presented a new definition of the fuzzy set: a fuzzy set \( A \) of the elements \( x \) is a collection of the elements which possess a specific property \( pA \) of the set and are qualified in the set by a qualifier \( QA \) using a qualification algorithm \( QA_{lg}A \). But nothing about essential shortcomings and mistakes of Zadeh's fuzzy set theory and how to overcome them completely was discussed in it.

Shi Gao, Yu Gao and Yue Hu [22] found that there is some mistakes Zadeh's fuzzy sets and found that it is incorrect to define the set complement as , because it can be proved that set complement may not exist in Zadeh's fuzzy set theory. According to them it leads to logical confusion, and is seriously mistaken to believe that logics of fuzzy sets necessarily go against classical and normal thinking, logic, and concepts. Since they found some shortcomings in the Zadeh's fuzzy set theory, they wanted to move away from it and worked towards removing the shortcomings which according to them debarred fuzzy sets to satisfy all the properties of classical sets. They introduced a new fuzzy set theory, called C-fuzzy set theory which satisfies all the formulas of the classical set theory. The C-fuzzy set theory proposed by them was shown to overcome all of the errors and shortcomings, and more reasonably reflects fuzzy phenomenon in the natural world. It satisfies all relations, formulas, and operations of the classical set theory.

Baruah ([4], [5] & [6]) realized that there are some shortcomings in the theory of fuzzy sets. It was observed that the complementation of fuzzy sets and the probability-possibility consistency principles are not well defined. He defined the complementation of fuzzy sets on the basis of reference function to make it realistic. Further a new probability-possibility
consistency principle called the \textit{Randomness-Fuzziness Consistency Principle} was derived on the basis of \textit{superimposition} of sets to bridge the gap between the two that currently exists.

1.5 IMPORTANCE OF FUZZY SET THEORY

In real world, complexity generally arises from uncertainty in the form of ambiguity. Uncertainty arises due to lack of information about a problem or due to information which is not fully reliable or due to inherent imprecision in language with which the problem is defined or due to receipt of information from more than one source. Fuzzy set theory is an excellent mathematical tool to handle uncertainty arising due to vagueness, Zadeh [78]. Since inception, the theory of fuzzy sets has advanced in a variety of ways and in many disciplines. The fuzzy set theory — a theory of graded concepts, a theory in which everything is a matter of degree — can be considered as a generalization of classical set theory. Because of this fuzzy set theory has a wider scope of applicability than classical set theory in solving various problems. Since the inception of fuzzy sets, the application of this theory has mushroomed. Applications appear in computer science, artificial intelligence, decision analysis, information science, engineering, expert systems, pattern recognition, management science, operation research and robotics. Theoretical mathematics has also been touched by the concept of fuzziness. Applications of it in different areas have been classified by Zimmermann [85] as follows:

(i) Applications to mathematics- generalization of traditional mathematics such as topography, graph theory, algebra, logic and so on.
(ii) Applications to algorithms- clustering methods, control algorithms, mathematical programming and so on.

(iii) Applications to standard models such as transportation model, inventory model, maintenance model and so on.

(iv) Applications to real world problem of different kinds.

Besides these areas, the mathematics of fuzziness has the most promising areas of applications in the field of pattern recognition which extends into areas of image processing, medical diagnosis, speech and hand writing recognitions and many others.

1.6 ORGANIZATION OF THE THESIS

For better understanding of the subject matter, this thesis has been organized into six chapters each consisting of several subsections.

Chapter 1: Introduction.

Chapter 2: Methodology.

Chapter 3: Cardinality of fuzzy sets.

Chapter 4: Subsethood and entropy relationship of fuzzy sets.

Chapter 5: Entropy of fuzzy numbers.

Chapter 6: Square Fuzzy Matrices.
Now brief discussions of our research topics are as follows:

Chapter 1 which is an introductory chapter presents some basic concepts of fuzzy set theory and its importance in various fields. In this chapter, a literature survey of the various approaches such as cardinality, subsethood, entropy and also the fuzzy matrices are given. It also deals with the structure of the thesis.

Chapter 2 presents the proposed methodologies in this thesis. In this chapter, a new definition of complementation of fuzzy sets on the basis of reference function is discussed. Further, a theoretical background of constructing the membership function of a fuzzy number is also discussed. This theory is based on a set operation called superimposition. The Randomness-Fuzziness Consistency Principle is used in explaining the construction of a normal fuzzy number. These are the methods on which the works on the subsequent chapters are carried out.

Chapter 3 considers the cardinality of fuzzy sets. Here we have proposed a new definition of cardinality of fuzzy sets. This new definition of cardinality of fuzzy sets is based on the result that a fuzzy set should be defined with the help of two functions, Baruah [5]. Further some properties of cardinality of fuzzy sets so defined are also considered. The process is discussed with the help of numerical examples.

Chapter 4 presents the relationship between subsethood and entropy of fuzzy sets. Thereafter, these relationships are studied on the basis of the new definition of complementation of fuzzy sets. Also some new properties of subsethood are added to the
existing ones which are derived from the standpoints of the new definition of complementation.

Chapter 5 is about the entropy of fuzzy numbers. In this chapter a new definition of entropy of fuzzy numbers is proposed. This new definition of entropy has been derived by using Shannon’s Entropy Index and the Randomness-Fuzziness Consistency Principle introduced by Baruah ([4], [5] & [7]). In this work we are dealing with triangular fuzzy number and trapezoidal fuzzy number. It is not because they are easy to handle, but because they are the simplest possible normal fuzzy numbers arising out of the simplest possible probability law. Further some of the properties of entropy of triangular as well as trapezoidal fuzzy numbers are discussed in this chapter. The processes discussed are illustrated with the help of numerical examples.

Chapter 6 deals with fuzzy matrix represented on the basis of reference function. A new approach to find the sum and product of fuzzy matrices is introduced and some properties are discussed. Further, trace of a fuzzy matrix is defined along with some properties. Thereafter, the process of finding the determinant and adjoint of fuzzy matrices on the basis of reference function are introduced in this chapter. Further, some properties of determinant and adjoint of fuzzy matrices are studied and in the process it is observed that the properties are analogous to the properties of usual matrices. Numerical examples are provided to make the matter clear and simple in all of the above cases.

A detailed reference of the relevant literature is given at the end.