Chapter-1

PROLUSION

1.1 Introduction

The subject of fractional calculus deals with the investigations of integrals and derivatives of an arbitrary real or complex order, which unifies and extends the notions of integer order derivative and \( n \)-fold integral. It has gained importance and popularity during the last four decades or so, mainly due to its vast potential of applications in various seemingly diverse fields of science and engineering, like fluid flow, rheology, diffusion, oscillation, anomalous diffusion, reaction-diffusion, turbulence, diffusive transport, electric networks, polymer physics, chemical physics, electrochemistry of corrosion, reflection processes in complex system, dynamical process in self similar and porous structures and others. To get a complete picture of these applications we can visit Dzherbashyan [31], Caputo [19], Oldham and Spanier [123], Miller and Ross [113], Mainardi [103], Podlubny [131], Debnath [26], Metzler et al. [112] and Kilbas et al. [77].

The growing interest of researchers in this field can be estimated by the fact that during last three decades, three international conferences dedicated exclusively to fractional calculus and its applications were held in the University of New Haven in 1974, University of Staratchlyde, Glasgow in 1884 and in Nihon University, Tokyo in 1889 respectively. In these conferences a huge number of researchers presented their investigations dealing with the theory and application of fractional calculus. The contributions by Ross [138], McBride and Roach [108],
Srivastva and Saigo [166], Nishimoto [122] and Rusev et al. [139] gave further impetus to the research work on fractional calculus.

A comprehensive account of fractional calculus and applications is found in the literature penned by Oldham and Spanier [123], Ross [138], McBride and Roach [108], Miller and Ross [113], Kiryakova [81] and Hifler [69]. Also the treatise in the form of five volumes published by Nishimoto [122] contains a remarkable account of the theory and applications of fractional calculus in variety of areas of mathematical analysis, like ordinary and partial differential equations, summation of series, special functions etc.

The purpose of present research work is to study various fractional calculus operators available in the literature for the solution of fractional diffusion equations arising in the field of mathematical physics, engineering and applied sciences. The standard techniques like successive approximations, use of integral transforms, numerical solutions etc. are utilized and extended for this purpose. In addition, pertinence of recent techniques like Mellin’s transform, Laplace transform, Fourier transform and Sumudu transform have been explored.

1.2 Historical Development and Review of Research Work

Historically, the origin of fractional calculus is attributed to a question posed by L’Hospital in 1695 to Leibniz, in which he wanted to know the meaning of $d^\frac{n}{2}y/dx^n$ if $n = 1/2$. Leibniz replied “This is an apparent paradox from which, one day, useful consequences will be drawn”. This prophecy proved true after
almost two centuries. Now fractional calculus has been established as an exciting new mathematical method of solution of diverse problems in mathematics, science, and engineering [26].

Although the name “fractional calculus” is indeed a misnomer, and the designation “integration and differentiation of arbitrary order” would have been more appropriate, still we usually stick to the terminology “fractional calculus”, as it is in vogue since the days of L’Hospital.

Lacroix [87] was the first mathematician to mention a derivative of arbitrary order in a text book of 1819. Thus for \( y = x^a, \ a \in \mathbb{R}_+ \), he showed that

\[
\frac{d^{1/2}y}{dx^{1/2}} = \frac{\Gamma(a + 1)}{\Gamma(a + 1/2)} x^{a-1/2} \tag{1.2.1}
\]

In particular he showed that \((d/dx)^{1/2} = 2\sqrt{x/\pi}\)

Fourier [50] in 1822 derived an integral representation for

\[
f(x) = \frac{1}{2\pi} \int_R f(\alpha) d\alpha \int_R \cos p(x - \alpha) dp \tag{1.2.2}
\]

and obtained formally the derivative version

\[
\frac{d^v}{dx^v} f(x) = \frac{1}{2\pi} \int_R f(\alpha) d\alpha \int_R p^v \cos \left\{ p(x - \alpha) + \frac{\nu\pi}{2} \right\} dp \tag{1.2.3}
\]

where “the number \( \nu \) was regarded as any quantity whatever, positive or negative”.

In 1823 Abel [1] made an attempt to solve the fractional integral equation arising from brachistochrone problem, namely
and arrived at the solution

\[
g(x) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_0^x \frac{f(u)}{(x-u)^\alpha} \, du
\]  

(1.2.5)

Lutzen [99] briefly summarized the work of Abel in this direction. However Liouville [92], provided a complete solution to the integral equation (1.2.4) in 1832.

It is generally claimed that the first serious commitment to provide a logical definition of a fractional derivative is due to Liouville. He published nine papers on the subject between 1832 and 1837 and continued work in this field till 1855.

Starting with the well known result \( D^n e^{ax} = a^n e^{ax} \), where \( D = \frac{d}{dx} \), \( n \in \mathbb{N} \), Liouville extended it at first in the particular case \( \nu = 1/2, a = 2 \) and then to arbitrary order \( \nu \in \mathbb{R}^+ \) by defining

\[ D^\nu e^{ax} = a^\nu e^{ax} \]  

(1.2.6)

Further assuming the series representation for \( f(x) \) as

\[ f(x) = \sum_{k=0}^{\infty} c_k e^{a_k x} \]
he defined the derivative of arbitrary order \( \nu \) by

\[
D^{\nu}f(x) = \sum_{k=0}^{\infty} c_k (a_k)^{\nu} e^{a_k x} \tag{1.2.7}
\]

Liouville followed another approach by considering the integral

\[
I = \int_{0}^{\infty} u^{a-1} e^{-xu} du.
\]

Substituting \( xu = t \) he obtained the result

\[
I = x^{-a} \int_{0}^{\infty} t^{a-1} e^{-t} dt = x^{-a} \Gamma(a) \text{ for } (\text{Re}(a) > 0)
\]

Operating both sides of \( x^{-a} = I/\Gamma(a) \) with \( D^{\nu} \), and using

\[
D^{\nu} e^{-xu} = (-1)^{\nu} u^{\nu} e^{-xu}
\]

he obtained the definition

\[
D^{\nu}x^{-a} = (-1)^{\nu} \frac{\Gamma(a + \nu)}{\Gamma(a)} x^{-a-\nu} \tag{1.2.8}
\]

which was used later in his investigation of potential theory [69].

In a paper published posthumously in 1847, Riemann defined fractional integration seeking generalization of a Taylor series expansion by adding a complementary function in his definition. Nowadays, this definition is in common use as a definition for fractional integration but with the complimentary function taken to be identically zero. In 1848 Hargreave [63] generalized Leibnitz rule for
the $n^{th}$ derivative of a product to $\alpha^{th}$ derivative, where $\alpha$ was arbitrary. He resolved

the controversy over two systems of fractional operations given by Peacock

\[
\left( \frac{d}{dx} \right)^\theta x^m = \frac{\Gamma(m+1)}{\Gamma(m-\theta+1)} x^{m-\theta}, \theta > 0
\]

and by Liouville

\[
\left( \frac{d}{dx} \right)^\theta x^{-m} = \frac{(-1)^\theta \Gamma(m+\theta)}{\Gamma(m)} x^{-m-\theta}, \theta > 0, \quad m + \theta > 0.
\]

In 1859 Greer [58], making use of Liouville’s definition obtained a formula for the semi-derivatives of $\sin x$ and $\cos x$ and introduced finite difference operator of order $\frac{1}{2}$. In 1868 Letnikov [89] proved the semi group property for the derivatives of arbitrary order showing that

\[
[D^\alpha D^\beta f(x)]_x^{x_0} = [D^{\alpha+\beta} f(x)]_x^{x_0}
\]

and solved certain differential equations by means of operators having fractional exponent.

In 1884 Laurent [88] generalized Cauchy’s integral formula by making use of the generalized product rule of Leibniz. In 1922 Hardy [62] investigated the properties of integrals of fractional order, in particular, theorem of continuity and summability, seeking analogies to properties valid for integer order.
Another approach for developing the theory of fractional calculus employed complex integral transformation. Prominent mathematicians, using this approach are Cauchy (1789 – 1857) and Goursat (1858 - 1936). Further Sonin in 1869 wrote a paper entitled “On the differentiation with arbitrary index” from which the present definition of Riemann-Liouville operator appears to follow. Letnikov [90] further extended Sonin’s work in his four papers by presenting an explanation of the main concepts of the theory of differentiation of an arbitrary index.

The twentieth century witnessed an increased interest of mathematicians in fractional calculus operators. In 1927 Davis [27] employed fractional operators to the class of Volterra integral equations and compared various notations to define fractional operators. He also introduced the following notation for the fractional integro-differential operator

\[ _cD_x^{-\nu}f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x - t)^{\nu-1} f(t)dt \]

In 1936 Fabian [43] derived some properties of the fractional integrals using a method of summability of series and integrals. He extended Riemann’s definition which enabled him to perform fractional integration along any simple curve in the complex plane in contrast to Hardey and Littlewood who integrated along straight lines in the complex plane. In 1939 Erdelyi [38] studied the transformation of hypergeometric integrals. Erdelyi and Kober derived several properties of Reimann-Liouville and Weyl fractional integrals. Subsequently in 1940 Kober [84]
generalized some results of Hardy and Littlewood and established a uniqueness theorem for solution of the equation

\[ g(x) = \int_a^x (x - t)^{\alpha-1} f(t) \, dt \]

by means of Mellin Transform.

Subsequently a large number of studies related to hypergeometric functions and integro-differential equations in the context of fractional calculus cropped up. In 1941, Widder [174] endeavored to obtain Laplace transform of various fractional integrals. In 1949, Reisz [136] applied the fractional integral operators to a variety of areas including the theory of potentials, Lorenz space, relativistic theory, wave equation and Riemann equation.

Stuloff [169] defined difference operators of fractional order in 1950. Later in 1953, Kuttner [86] considered the relation between the integrals

\[ \frac{d^n}{dx^n} \frac{1}{\Gamma(n-k)} \int_0^x (x - t)^{n-k-1} f(t) \, dt \]

and \[ (-1)^n \frac{d^n}{dx^n} \frac{1}{\Gamma(n-k)} \int_x^1 (x - t)^{n-k-1} f(t) \, dt \]

transforms of various fractional operators. This list expedited application of fractional integral operators to various problems in physical and engineering sciences.

In 1970’s the activities in this area gained further momentum. Oldhman and Spanier [123] replaced Fick’s laws by the formulation of new laws involving semi-differentiation. They also discovered a novel method for problems arising in electrochemical kinetics by applying the operator of order $\frac{1}{2}$ for relating the concentration of an electro-active species at the surface of the electrode directly to the faradic current density. Olser [126] introduced certain generalizations of the Leibnitz rule for the derivative of the product of two functions and used them to generate several infinite series expansions involving special functions. During the same year, Love [98] studied the fractional derivatives of imaginary order. In 1972 Oldhman and Spanier [124] gave a general solution of the diffusion equation for semi infinite geometries. Diaz and Olser [30] defined differences of fractional order and gave new analogues to Leibniz rule for the fractional difference of the product of two functions in 1974.

After 1974, a huge surge was witnessed in research activities related to fractional calculus. A testimony to this surge is the organization of a series of conferences devoted to this subject. Manglik [100] obtained the fractional differences of some special functions. In 1980, Raina and Koul [133] made an attempt to connect Laplace transform with fractional integrals. Following this in
1984 Raina [134] discussed multiple Weyl fractional integral of a general system of polynomials. In 1987 Arora and Koul [7] obtained a number of new results concerning certain useful special functions as application of fractional calculus. Katsuyuki Nishimoto [122] and Rusev, Dimovski and Kiryakova [139] made significant contributions to this field. Later on Srivastava and Saigo [166] studied the solution of various boundary value problems involving Euler-Darboux equation and considered the multiplication of certain classes of operators of fractional calculus.

In 1990 Saxena and Ram [146] defined multi-dimensional Kober operators and established certain theorems for these operators. Also Axtell and Bise [9] applied fractional calculus operators to a control system. Nishimoto and Saxena [121] provided the unification of certain functional relations by means of Riemann-Liouville operator. In 1992 Goyal, Jain and Gaur [55] studied fractional integral operators involving a product of generalized hypergeometric function and a general class of polynomials. Furthermore, Tuan and Saigo [172] gave some new multi-dimensional operators of fractional calculus based on the spaces of generalized functions and also applied these operators to elementary and generalized hypergeometric functions of several variables. In 1994, Saxena, Kriyakova and Dove [147] presented a unified approach to extend several results on fractional operators. In 1995 Rutman [140] developed physical interpretation of fractional integration and differentiation. In 1998 Boyedjiev, Dobner and Kalla [17] gave an analytical and numerical treatment of a fractional integro-differential equation of
Volterra type. In 1999 Lorenzo and Hartley [97] introduced generalized functions for the fractional calculus. In the same year Jain and Jain [73] discussed solution of certain dual integral equations involving I-function.

[72] gave numerical approximation of a space fractional diffusion equation with non-homogeneous boundary conditions.

In 2006 Momani [118] investigated the general solution for the space and time fractional diffusion wave equation and Kilbas et.al [78] studied theory and application of fractional differential equations.


In 2008 Mittal and Nigam [114] studied solution of a fractional integro-differential equation by a domain decomposition method. This method gave wider applicability of reduction in the size of computational work. Ahmed [4] studied some fractional integro-differential equations and obtained a mild solution in
arbitrary Banach space. Goyal and Bansal [57] studied fractional kinetic equation and fractional reaction-diffusion equation. Kiryakova [82] authored a review article giving a brief history of the operators of the generalized fractional calculus and showed that all known fractional integral and differential operators in various areas of analysis happen to fall in the scheme of the generalized fractional calculus.


In 2010 Gupta, Sharma and Kilicman [60] gave a note on fractional Sumudu transform and established a relationship between fractional Laplace and Sumudu

In 2011 Loonker and Banerji [96] gave the solution of distributional Abel integral equation by distributional Sumudu transform. In the same year Chaurasia and Singh [21] employed Sumudu transform for solution of fractional kinetic equation.

In 2012 Manohar, Sharma and Garg [102] gave solution of generalized space-time fractional telegraph equation with composite and Riesz-Feller

In 2014, Brunner, Han and Yin [18] considered numerical solution of the time fractional diffusion–wave equation on a two dimensional unbounded spatial domain. They introduced artificial boundaries in order to find exact and approximate boundary conditions.

1.3 A Brief Sketch of Some Special Functions

The development of fractional calculus operators followed closely the researches in the field of special functions. Several fractional operators involve some special functions in their kernels. Also the fractional calculus operators have found applications generally useful to special functions. Therefore in this section we present a brief survey of special functions employed in the present thesis. These include the Meijer’s G-function, Fox's H-function, I-function, Appell’s function, Fox-Wright function, Mittag - Leffler function and a more generalized function of fractional calculus named as R-function.

1.3.1 Meijer’s G-Function

The G-function was introduced by Cornels Simon Meijer (1936) as a very general function intended to include most of the known special functions as particular cases. This was not the only attempt of its kind. The generalized hypergeometric function and Macrobert E-function had the same aim, but Meijer’s
G-function was able to include these as particular cases as well. The majority of the special functions can be represented in terms of the G-functions.

The first definition was made by Meijer using a series, but later on it was replaced by a more general definition in terms of Mellin-Barnes type integral. Meijer’s G-functions provided an interpretation of the symbol pFq when p > q + 1.

The Meijer’s G-function is defined as [111]

\[
G_{p,q}^{m,n} \left[ \begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q \end{array} \right] = G_{p,q}^{m,n} \left[ \begin{array}{c} \alpha_p \\ \beta_q \end{array} \right] = G_{p,q}^{m,n} (z) =
\]

\[
\frac{1}{2\pi i} \int_{L} \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1-a_j + s)z^s ds}{\prod_{j=m+1}^{q} \Gamma(1-b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)}
\]

where an empty product is interpreted as 1. In above equation, 0 < m < q, 0 < n < p, and the parameters are such that no pole of \( \Gamma(b_j - s), j = 1,2,3, \ldots, m \) coincides with any pole of \( \Gamma(1 - a_k + s), k = 1,2,3, \ldots, n \). There are three different paths L of integration:

(i) L runs from \(-i\infty\) to \(+i\infty\) so that all the poles of \( \Gamma(b_j - s), j = 1,2,3, \ldots, m \) are to the right, and all the poles of \( \Gamma(1 - a_k + s), k = 1,2,3, \ldots, n \), are to the left of L. The integral converges if \((p + q) < 2(m + n)\) and \( |\arg z| < \left| m + n - \frac{p}{2} - \frac{q}{2} \right| \pi. \)
(ii) \( L \) is a loop starting and ending at \( +\infty \) and encircling all poles of \( \Gamma(b_j - s), j = 1,2,3, \ldots, m \), once in negative direction but none of the poles of \( \Gamma(1 - a_k + s), k = 1,2,3, \ldots, n \). The integral converges if \( q \geq 1 \) and either \( p < q \) or \( p = q \) and \( |z| < 1 \).

(iii) \( L \) is a loop starting and ending at \( -\infty \) and encircling all poles of \( \Gamma(1 - a_k + s), k = 1,2,3, \ldots, n \), once in the positive direction, but none of the poles of \( \Gamma(b_j - s), j = 1,2,3, \ldots, m \). The integral converges if \( p \geq 1 \) and either \( p > q \) or \( p = q \) and \( |z| > 1 \). It is always assumed that the values of parameters and of the variable \( z \) are such that at least one of the three definitions makes sense. In cases, when more than one of these definitions makes sense, they lead to the same result. Thus no ambiguity arises.

1.3.2 Fox's H–Function

Although G-functions are quite general in character yet a number of special functions, like Wright’s generalized hypergeometric functions do not form their special cases. Therefore Charles Fox introduced and studied a more general function, known as Fox’s H-function. This function contains all the aforementioned functions, including G-function, as its special cases. Fox defined H-function in terms of a general Mellin-Barnes type integral. He also investigated the most general Fourier kernel associated with the H- function and obtained the asymptotic expansions of the kernel for large values of the argument. Fox has also
derived theorems about the H-function as asymmetric Fourier kernel and established certain operational properties for this function.

The H-function is defined by Fox [47, 48] as follows

\[
H(z) = \mathcal{H}_{p,q}^{m,n} \left[ \left( z \left| \begin{array}{c} (a_1, a_1, \ldots, (a_p, a_p)) \\ (b_1, b_1, \ldots, (b_q, b_q)) \end{array} \right. \right) \right] \\
= \frac{1}{2\pi i} \int_{L} \varphi(s) z^s ds \tag{1.3.2}
\]

where \( \varphi(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1-a_i + \alpha_i s)}{\prod_{j=m+1}^{p} \Gamma(1-b_j + \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_i - \alpha_i s)} \)

An empty product is always interpreted as unity; \( m, n, p, q \in \mathbb{N} \) with \( 0 \leq n \leq p, 1 \leq m \leq q, \alpha_i, \beta_j \in \mathbb{R}_+, a_i, b_j \in \mathbb{R} \) or \( C, i = 1,2 \ldots, p; j = 1,2 \ldots, q \). \( L \) is a suitable contour separating the poles

\[
\zeta_{jv} = -\left( \frac{b_j + v}{\beta_j} \right), \quad j = 1,2 \ldots, m, \quad v = 0,1,2 \ldots
\]

of the gamma function \( \Gamma(b_j - \beta_j s) \) from the poles

\[
\omega_{\lambda k} = \left( \frac{1-a_{\lambda} + k}{a_{\lambda}} \right), \quad \lambda = 1,2 \ldots, n, \quad k = 0,1,2 \ldots
\]

of the gamma function \( \Gamma(1-a_{\lambda} + \alpha_{\lambda} s) \), that is

\[
a_{\lambda} (b_j + v) \neq \beta_j (a_{\lambda} - k - 1), j = 1,2 \ldots, m, \lambda = 1,2 \ldots, n,
\]
The contour \( L \) exists on account of above equation. The contour \( L \) is either \( L_{-\infty} \), \( L_{+\infty} \) or \( L_{lyx} \). The following are the definitions of these contours [106].

(i) \( L=L_{-\infty} \) is loop beginning and ending at \(-\infty\) and encircling all the poles of \( \Gamma(b_j - \beta_j s), j = 1,2, ..., m \) once in the positive direction but none of the poles of \( \Gamma(1 - \alpha_\lambda + \alpha_\lambda s), \lambda = 1,2, ..., n \). The integral converges for all \( z \) if \( \mu > 0 \) and \( z \neq 0 \); or \( \mu = 0 \) and \( 0 < |z| < B \). The integral also converges if \( \mu = 0 \) and \( |z| = B \) and \( \Re(\delta) < -1 \), where

\[
B = \left\{ \prod_{j=1}^{p} (\alpha_j)^{(-\alpha_j)} \right\} \left\{ \prod_{j=1}^{q} (\beta_j)^{(\beta_j)} \right\}
\]

\[
\mu = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j , \text{ and}
\]

\[
\delta = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p - q}{2}.
\]

(ii) \( L=L_{+\infty} \) is loop beginning and ending at \(+\infty\) and encircling all the poles of \( \Gamma(1 - \alpha_\lambda + \alpha_\lambda s), \lambda = 1,2, ..., n \) once in the negative direction but none of the poles of \( \Gamma(b_j - \beta_j s), j = 1,2, ..., m \). The integral converges for all \( z \) if \( \mu < 0 \) and \( z \neq 0 \); or \( \mu = 0 \) and \( |z| > B \). The integral also converges if \( \mu = 0 \) and \( |z| = B \) and \( \Re(\delta) < -1 \),
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(iii) \( L = L_{\gamma R} \) is a contour starting at the point \( \gamma - i\infty \) and going to \( \gamma + i\infty \) where \( \gamma \in R \) such that all the poles of \( \Gamma(b_j - \beta_j s), j = 1,2, \ldots, m \) are separated from those of \( \Gamma(1 - a_\lambda + \alpha_\lambda s), \lambda = 1,2, \ldots, n \). The integral converges if

\[
A > 0, \text{arg} z < \frac{1}{2\pi} a, a \neq 0
\]

The integral also converges if \( A = 0, \gamma \mu + \Re(\delta) < -1, \text{arg} z = 0 \) and \( z \neq 0 \) where

\[
A = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j
\]

1.3.3 \( I \) – Function

The generalization in the field of special functions did not end with Fox’s \( H \)-function. The \( H \)-function was also generalized into a new type of function in which the denominator parameters are in the summation form of products of gamma functions. This was named as the \( I \)-function.

The \( I \)-function was introduced by Saxena [159] in connection with the solution of dual integral equations involving sum of \( H \)-functions as kernels. It is defined as

\[
I(z) = I_{p,q,r}^{m,n} \left[ \begin{array}{c} \left( a_{ij}, a_{ij} \right)_{1,n} (a_{ij}, a_{ij})_{n+1,p_i,r} \\ \left( b_{ij}, b_{ij} \right)_{1,m} (b_{ij}, b_{ij})_{m+1,q_i,r} \end{array} \right] = \frac{1}{2\pi i} \int_{L} \varphi(s) z^s ds \tag{1.3.3}
\]
where

\[ \varphi(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} s) \right\}} \]

\( p_i (i= 1, 2, 3, \ldots, r) \), \( q_i (i= 1, 2, 3, \ldots, r) \), m and n are integers satisfying \( 0 < n < p_i \) and \( 0 < m < q_i \), r is finite and \( \alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} \), are complex numbers.

For I – function, there are three different paths L of integration

(i) L is a contour which runs from \( \sigma - i\infty \) to \( \sigma + i\infty \) (\( \sigma \) is real), so that all poles of \( \Gamma(b_j - \beta_j s) \), \( j=1, 2, 3 \ldots, m \) are to the right and all poles of \( \Gamma(1 - a_j + \alpha_j s) \), \( j = 1, 2, 3 \ldots, n \) are to the left of L. The integral converges if \( (p + q) < 2(m + n) \) and \( |\arg z| < \left| m + n - \frac{p_i}{2} - \frac{q_i}{2} \right| \) for all \( i = 1, 2 \ldots r \)

(ii) L is a loop starting and ending at \( \sigma + i\infty \) and encircling all the poles of \( \Gamma(b_j - \beta_j s) \), \( j=1, 2, 3 \ldots, m \). Once in the negative direction but none of the pole of \( \Gamma(1 - a_j + \alpha_j s) \), \( j = 1, 2, 3, \ldots, n \). The integral converges if \( q_i > 1 \) and either \( p_i < q_i \) or \( p_i = q_i \) and \( |z| < 1 \) \( i = 1, 2, \ldots, r \).

(iii) L is a loop starting and ending at \( \sigma + i\infty \) and encircling all the poles of \( \Gamma(1 - a_j + \alpha_j s) \), once in positive direction, but none of the poles of \( \Gamma(b_j - \beta_j s) \), \( j = 1, 2, 3, \ldots, m \). The integral converges if \( p_i \geq 1 \) and either \( p_i < q_i \) or \( p_i = q_i \) and \( |z| > 1 \) for all \( i = 1, 2, \ldots, r \).
On specializing the parameters in I-function we can arrive at G and H functions. Thus G and H functions are particular cases of I-function.

1.3.4 Aleph -Function

The Aleph-function is defined in terms of the Mellin-Barnes type integral in the following manner [155,156]

\[ \Phi(z) = \left[ \frac{1}{2\pi i} \int_L \frac{1}{\Omega^{m,n}_{p_i,q_i,r_i} x} (s) t^s \, ds \right] \]

\[ \Omega^{m,n}_{p_i,q_i,r_i} (s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1-a_j - A_j s)}{\sum_{i=1}^{r} \tau_i \prod_{j=n+1}^{p_j} \Gamma(1-b_{ji} - B_{ji} s) \prod_{j=n+1}^{q_j} \Gamma(a_{ji} + A_{ji} s)} . \]

where \( z \neq 0, \, i = \sqrt{-1} . \)

An account of the convergence conditions for the defining integral can be found in the papers by Saxena and Pogány [155,156].

1.3.5 Mittag-Leffer function

The Mittag-Leffer function is a vital function that finds widespread use in the world of fractional calculus. The Mittag-Leffler function arises naturally in the solution of fractional order integral and differential equations, particularly in the investigations of the fractional generalization of the kinetic equation, random walks, Levy flights, super diffusive transport and in the study of complex systems.
The ordinary and generalized Mittag-Leffler functions interpolate between a purely exponential law and power-law-like behavior of phenomena governed by ordinary kinetic equations and their fractional counterparts. During the various developments of fractional calculus in the last four decades this function has gained importance and popularity on account of its vast applications in the fields of science and engineering.

The Mittag-Leffler [115] function is named after a Swedish mathematician who defined and studied it. This function is a direct generalization of the exponential function, $e^x$ and plays a major role in fractional calculus. Mittag-Leffler function can be defined in terms of a power series as [115]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \alpha k)}, \quad z \in \mathbb{C}, \quad Re(\alpha) > 0 \quad (1.3.4)$$

where $\alpha \in \mathbb{C}, R(\alpha) > 0, z \in \mathbb{C}$.

The generalization of (1.3.4) in the form

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad (\alpha, \beta \in \mathbb{C}, Re(\alpha) > 0) \quad (1.3.5)$$

where $\alpha, \beta \in \mathbb{C}, R(\alpha) > 0, R(\beta) > 0, z \in \mathbb{C}$ is defined and studied by Wiman [175].

The generalization of (1.3.5) was introduced in term of series representation by Prabhakar [132] as

$$E_{\alpha, \beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_nz^n}{\Gamma(n\alpha + \beta)n!}, \quad (\alpha, \beta, \gamma \in \mathbb{C}, Re(\alpha) > 0) \quad (1.3.6)$$
where

\[(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = \begin{cases} 1 & n = 0 \\ \gamma(\gamma + 1)(\gamma + 2) \ldots (\gamma + n - 1), n \in \mathbb{N} \end{cases} \]

For \(\gamma = 1\), (1.3.6) coincides with (1.3.5), while for \(\gamma = \beta = 1\), it reduces to (1.3.4).

Thus

\[E_{\alpha,\beta}^1(z) = E_{\alpha,\beta}(z)\text{ and } E_{\alpha,1}^1(z) = E_{\alpha}(z). \tag{1.3.7}\]

We also have

\[\phi(\mu, \gamma; z) = {}_1F_1(\mu, \gamma; z) = \Gamma(\gamma)E_{1,\beta}^\mu(z) \tag{1.3.8}\]

where \(\phi(\mu, \gamma; z)\) is Kummer’s confluent hypergeometric function.

The Mellin-Barnes integral representation for this function follows from the integral

\[E_{\alpha,\beta}^\gamma(z) = \frac{1}{\Gamma(\gamma)} \frac{1}{2\pi i} \int_{\Omega} \frac{\Gamma(-\xi)\Gamma(\gamma + \xi)(-z)^\xi}{\Gamma(\beta + \xi\alpha)} d\xi \tag{1.3.9}\]

where \(\omega = (-1)^{1/2}\). The contour \(\Omega\) is straight line parallel to the imaginary axis at a distance ‘c’ from the origin which separates the poles of \(\Gamma(-\xi)\) given by \(\xi = \nu(\nu = 0,1,2 \ldots)\) from those of \(\Gamma(\gamma + \xi)\) given by the points \(\xi = -\gamma - \nu(\nu = 0,1,2 \ldots)\). If we calculate the residues at the poles of \(\Gamma(\gamma + \xi)\) at the points \(\xi = -\gamma - \nu(\nu = 0,1,2 \ldots)\) then it gives the analytic continuation formula of this function in the form [153]
From (1.3.10) it follows that for large $z$ its behavior is given by

$$E_{\alpha,\beta}^{\gamma}(z) \sim O(|z|^{-\gamma}), |z| > 1$$  \hspace{1cm} (1.3.11)

The generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma}(z)$ can be represented in terms of the $H$-function in the form

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1} \left[ -z \begin{pmatrix} (0,1) \\ (1-\gamma,1) \end{pmatrix} (0,1)(1-\beta,\alpha) \right] (\alpha, \beta, \gamma \in C, Re(\alpha) > 0) \hspace{1cm} (1.3.12)$$

Putting $\gamma = 1$ in (1.3.12)

$$E_{\alpha,\beta}(z) = H_{1,2}^{1,1} \left[ -z \begin{pmatrix} (0,1) \\ (0,1)(1-\beta,\alpha) \end{pmatrix} \right] \hspace{1cm} (1.3.13)$$

If we further take $\beta = 1$ in (1.3.13) we get

$$E_{\alpha}(z) = H_{1,2}^{1,1} \left[ -z \begin{pmatrix} (0,1) \\ (0,1)(0,\alpha) \end{pmatrix} \right] \hspace{1cm} (1.3.14)$$

The generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ can be represented in terms of the Fox-Wright generalized hypergeometric function in the form

$$E_{\alpha,\beta}^{\gamma,n}(z) = \frac{1}{\Gamma(\gamma)} \Phi_1 \left[ z \begin{pmatrix} (\gamma,1) \\ (\beta,\alpha) \end{pmatrix} (\alpha, \beta, \gamma \in C, Re(\alpha) > 0) \right] \hspace{1cm} (1.3.15)$$

Srivastava and Tomovski [168] introduced and investigated a further generalization of (1.3.5) in 2009 which is defined in the following way
This generalization can also be represented in terms of the Fox-Wright generalized hypergeometric function in the form

\[ E_{\alpha,\beta}^{\gamma,n}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)^{nk}}{\Gamma(\beta + \alpha k)} \frac{z^k}{k!} \quad (1.3.16) \]

\[ z, \beta, \gamma \in \mathbb{C}, \text{Re}(\alpha) > \max\{0, \text{Re}(n) - 1; \text{Re}(n) > 0\} \]

1.4 Fractional calculus operators

Since Leibniz’s historical comments, several attempts were made over the centuries, to formalize a definition which fulfils the criteria of analyticity of the function, possesses similarity with ordinary calculus (integral and differential), observes linearity and satisfies the laws of exponent of integration and differentiation. First such significant definition was obtained by Liouville, incorporating the definition given by Riemann. Named in honour of Riemann and Liouville this definition is given as

\[ aD_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x - t)^{-\alpha - 1} f(t) \, dt; \quad \text{Re}(\alpha) < 0 \quad (1.4.1) \]

This definition is useful for integration of arbitrary order. For differentiation of arbitrary order it cannot be used directly.
Another definition of fractional calculus given by Hermann Weyl is as follows:

\[ xW_\alpha^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^x (t - x)^{\alpha-1} f(t)dt ; \quad Re(\alpha) < 0 \quad (1.4.2) \]

The significant difference between this definition and Riemann-Liouville definition lies in the terminals of integration and the kernel function.

Besides these classical definitions, several new fractional operators were defined by many mathematicians from time to time. These include Caputo operator, Kober operators, Taun and Saigo’s operators, Saigo’s operators and Saigo-Maeda operators.

We now present these definitions in detail alongwith presently employed modified versions of the Riemann-Liouville and Weyl operators.

1.4.1 Riemann-Liouville Operator

The most widely accepted definition of fractional calculus of order \( \alpha \) has been obtained by extending the Cauchy formula for \( n \)-fold integral to \( \alpha \), where \( \alpha \) is real or complex.

Riemann-Liouville defined four different kind of fractional calculus operators as follows:
(i) **Riemann-Liouville left-sided fractional integral**: Let

\[ f(x) \in L(a, b), \alpha \in \mathbb{C}, Re(\alpha) > 0 \text{ then} \]

\[ aI_x^\alpha f(x) = aD_x^{-\alpha} f(x) = I_x^\alpha f(x) \]

\[ = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) dt ; \ x > a \]  \hspace{1cm} (1.4.1)

is called Riemann-Liouville left-sided fractional integral of order \( \alpha \).

(ii) **Riemann-Liouville right-sided fractional integral**: Let

\[ f(x) \in L(a, b), \alpha \in \mathbb{C}, Re(\alpha) > 0 \text{ then} \]

\[ xI_b^\alpha f(x) = xD_b^{-\alpha} f(x) = I_b^\alpha f(x) \]

\[ = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t) dt ; \ x < b \]  \hspace{1cm} (1.4.2)

is called Riemann-Liouville right-sided fractional integral of order \( \alpha \).

(iii) **Riemann-Liouville left-sided fractional derivative**: The left-sided Riemann-Liouville fractional derivative of order \( \alpha > 0 \) is defined as

\[ aD_x^\alpha f(x) = D_x^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d^n}{dx^n} \right) \left\{ \int_a^x \frac{f(t)}{(x - t)^{a-n+1}} dt \right\}, n \]

\[ = [\alpha] + 1 \]  \hspace{1cm} (1.4.3)

where \([\alpha]\) denotes the integral part of number \( \alpha \).

(iv) **Riemann-Liouville right-sided fractional derivative**: The right-sided Riemann-Liouville fractional derivative of order \( \alpha > 0 \) is defined as
\[ b D_x^\alpha f(x) = D_b^\alpha f(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \left( \frac{d}{dx} \right)^n \left( \int_x^b \frac{f(t)}{(x - t)^{\alpha+n+1}} dt \right), \]
\[ n = [\alpha] + 1 \quad (1.4.4) \]

where \([\alpha]\) denotes the integral part of number \(\alpha\).

1.4.2 Basic Properties of Riemann-Liouville Operator:

Fractional integrals have the following properties.

Fractional integrals obey the semi group property which is as under

\[ a I_x^\alpha x I_x^\beta f = a I_x^{\alpha+\beta} f = x I_b^\alpha a I_x^\beta f = x I_b^{\alpha+\beta} f \]
\[ = x I_b^\beta x I_b^\alpha \quad (1.4.5) \]

The integration by parts of fractional integral is given as

\[ \int_a^b f(x) \left( a I_x^\alpha g(x) \right) dx = \int_a^b g(x) \left( x I_b^\alpha f(x) \right) dx \quad (1.4.6) \]

1.4.3 The Weyl Fractional Operator

The Weyl fractional integral is defined in the two ways.

(i) \[ x W_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t - x)^{\alpha-1} f(t) dt ; -\infty < x < \infty \quad (1.4.7) \]

and

(ii) \[ -\infty W_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x - t)^{\alpha-1} f(t) dt ; -\infty < x < \infty \quad (1.4.8) \]

where \(\alpha\) is complex and \(\text{Re}(\alpha) > 0\).
The Weyl fractional derivative of order $\alpha$, is

\[ (iii) \quad x D_\alpha^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n \int_x^\infty \frac{f(t)}{(t-x)^{1+\alpha-n}} \, dt; \quad -\infty < x < \infty \quad (1.4.9) \]

where $n-1 \leq \alpha < n, n \in N, \alpha \in \mathbb{C}$

### 1.4.4 Basic Properties of Weyl Fractional Operator

(i) Weyl fractional integrals obey the semi group property which is as under

\[ x W_\alpha^\alpha x W_\beta^\beta f = x W_{\alpha+\beta}^\beta f = x W_\beta^\beta x W_\alpha^\alpha \quad (1.4.10) \]

(ii) The Weyl fractional integrals satisfy the Parseval identity which is also called fractional integration by parts

\[ \int_0^\infty f(x) (aD_x^\alpha g(x)) \, dx = \int_0^\infty (x W_\alpha^\alpha f(x))g(x) \, dx \quad (1.4.11) \]

The familiar differ-integral operator $a D_x^\alpha$ is defined by Oldham and Spanier [125] as follows:

\[ a D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_a^x (x-t)^{-\alpha-1} f(t) \, dt; & \text{Re}(\alpha) < 0 \\ \frac{d^m}{dx^m} a D_x^{\alpha-m} f(x); & 0 < \text{Re}(\alpha) < 0 \end{cases} \]

where ‘m’ is positive integer.

This definition includes the Riemann-Liouville operator and the Weyl operator.
1.4.5 The Caputo Fractional derivative

The Caputo fractional derivative of a causal function \( f(t) \) (that is \( f(t) = 0 \) for \( t < 0 \)) of order \( \alpha > 0 \), was defined by Caputo in 1969 as follows:

\[
\frac{cD^\alpha}{a} f(x) = aI_x^n - \frac{d^n}{dx^n} f(x)
\]

\[
= \frac{1}{\Gamma(n - \alpha)} \int_a^x \frac{f^n(t)dt}{(x - t)^{\alpha+1-n}} ;
\]

(1.4.12)

where \( n - 1 \leq \alpha < n, n \in N, \alpha \in C \)

1.4.6 The Kober Operators

The Kober operators are the generalization of Riemann-Liouville and Weyl operators. These operators have been used by many authors in deriving the solution of single, dual and triple integral equations involving different special functions as their kernels. The operator

\[
I[\alpha, \eta, f(x)] = I_{\alpha, \eta} f(x) = I_\alpha^n f(x)
\]

\[
= \frac{(x)^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\eta f(t)dt ; \ Re(\alpha) > 0
\]

(1.4.13)

is called Kober operator of first kind, and

\[
R[\alpha, \zeta, f(x)] = R_{\alpha, \zeta} f(x) = I_\zeta^\alpha f(x)
\]

\[
= \frac{(x)^\zeta}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\zeta-\alpha} f(t)dt ; \ Re(\alpha) > 0
\]

(1.4.14)

is called Kober operator of second kind. Equations (1.4.13) and (1.4.14) hold true under following conditions:
When $\bar{\zeta} = 0$, (1.4.13) reduces to Riemann-Liouville operator, that is
\[
I_x^{0,\alpha} f(x) = x^{-\alpha} \int_0^x f(t) dt
\]
and for $\zeta = 0$, (1.4.14) reduces to Weyl operator, that is
\[
K_x^{0,\alpha} F(x) = \int_0^x t^{-\alpha} f(t) dt
\]

### 1.4.7 The Modified Erdelyi-Kober Operators

Further generalizations of Kober operators are given as follows [71]
\[
I(\alpha, \eta; m) f(x) = I(f(x); \alpha, \eta, m)
\]
\[
= \frac{m x^{\eta - m \alpha + m + 1}}{\Gamma(\alpha)} \left\{ \int_0^x (x^m - t^m)^{\alpha - 1} t^{-\eta} f(t) dt \right\};
\]
\[
Re(\alpha) > 0
\]
and
\[
R(\alpha, \zeta; m) f(x) = R(f(x); \alpha, \zeta, m)
\]
\[
= \frac{m x^\zeta}{\Gamma(\alpha)} \left\{ \int_x^\infty (t^m - x^m)^{\alpha - 1} t^{-\zeta - m \alpha + m + 1} f(t) dt \right\}
\]
\[
Re(\alpha) > 0
\]
where
\[
f \in L_p(0, \infty), Re(\alpha) > 0, Re(\eta) > -\frac{1}{q}, Re(\zeta) > -\frac{1}{p}, \quad \frac{1}{p} + \frac{1}{q} = 1, p \geq 1
\]
1.4.8 Taun and Saigo’s Multidimensional Modified Fractional Calculus Operators

The multidimensional modified fractional integrals of order \( \alpha \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \) of a function \( f: \mathbb{R}^n_+ \rightarrow \mathbb{C} \) are defined by

\[
X_+^\alpha f(x) = \frac{1}{\Gamma(\alpha+1)} \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \int_{\mathbb{R}^n_+} \text{min} \left\{ \frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n} \right\}^\alpha f(t) dt \tag{1.4.19}
\]

\[
X_-^\alpha f(x) = \frac{(-1)^n}{\Gamma(\alpha+1)} \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \int_{\mathbb{R}^n_+} \left[ 1 - \text{max} \left\{ \frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n} \right\} \right]^\alpha f(t) dt \tag{1.4.20}
\]

1.4.9 The Saigo’s Operators

In 1978, Saigo defined four different kinds of fractional calculus operators as follows [144].

(i) \( (I_{0+}^{\alpha,\beta,\eta} f)(x) \)

\[
= \left( x^{-\alpha-\beta} \right) \frac{1}{\Gamma(\alpha)} \left\{ \int_0^x (x - t)^{\alpha-1} \frac{\partial^k}{\partial t^k} \left[ (t^{-\alpha+k,\beta-k,\eta-k} f)(x) \right] \right\}; \quad \text{Re}(\alpha) > 0 \quad (1.4.21)
\]

\[
(I_{0+}^{\alpha,\beta,\eta} f)(x) = \left( \frac{d}{dx} \right)^k \left( I_{0+}^{\alpha+k,\beta-k,\eta-k} f \right)(x) \quad \text{Re}(\alpha) \leq 0 \quad k
\]

\[
= \left[ -\text{Re}(\alpha) + 1 \right] \quad (1.4.22)
\]

(ii) \( (I_{0-}^{\alpha,\beta,\eta} f)(x) \)

\[
= \left( x^{-\alpha-\beta} \right) \frac{1}{\Gamma(\alpha)} \left\{ \int_x^\infty (t - x)^{\alpha-1} t^{-\alpha-\beta} \frac{\partial^k}{\partial t^k} \left[ (t^{-\alpha+k,\beta-k,\eta-k} f)(x) \right] \right\}; \quad \text{Re}(\alpha) > 0 \quad (1.4.23)
\]
(iii) \[
\left( I_{0^+}^{\alpha,\beta,\eta} f \right)(x) = \left( -\frac{d}{dx} \right)^k \left( I_{0^+}^{\alpha+k,\beta-k,\eta} f \right)(x)
\]
\[
Re(\alpha) \leq 0, \; k = [-Re(\alpha) + 1] \quad (1.4.24)
\]

(iv) \[
\left( D_{0^+}^{\alpha,\beta,\eta} f \right)(x) = \left( I_{0^+}^{-\alpha,-\beta,\alpha+\eta} f \right)(x), \; Re(\alpha) > 0
\]
\[
= \left( \frac{d}{dx} \right)^k \left( I_{0^+}^{-\alpha+k,-\beta-k,\alpha+\eta-k} f \right)(x), \\
Re(\alpha) > 0, \; k = [Re(\alpha) + 1] \quad (1.4.25)
\]

(v) \[
\left( D_{0^-}^{\alpha,\beta,\eta} f \right)(x) = \left( I_{0^-}^{-\alpha,-\beta,\alpha+\eta} f \right)(x), \; Re(\alpha) > 0
\]
\[
= \left( -\frac{d}{dx} \right)^k \left( I_{0^-}^{-\alpha+k,-\beta-k,\alpha+\eta} f \right)(x)
\]
\[
Re(\alpha) > 0, \; k = [Re(\alpha) + 1] \quad (1.4.26)
\]

1.4.10 The Saigo-Maeda Operator

In 1996 Saigo-Maeda [144] extended all the four fractional calculus operators defined by Saigo as follows:

(i) \[
\left( I_{0^+}^{\alpha,\alpha',\beta',\gamma} f \right)(x)
\]
\[
= \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3(\alpha, \alpha', \beta, \beta'; \gamma; 1-t/x, 1-x/t, f(t)dt
\]
\[
(1.4.27)
\]

where

\[
\alpha, \alpha' > 0, \; Re(\gamma) > 0, \; \beta, \beta' \text{being the real number.}
\]
\[
\left( I_{0^+}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) = \left( \frac{d}{dx} \right)^k \left( I_{0^+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k} f \right)(x) \quad Re(\gamma) \leq 0 \quad k = [-Re(\gamma) + 1] \\
\]

\[(1.4.28)\]

(ii) \[
\left( I_{0^+}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3(\alpha,\alpha',\beta,\beta'; \gamma; 1-x/t, 1-t/x, ) f(t) dt \\
\]

\[(1.4.29)\]

where

\[
\alpha, \alpha' > 0 \quad Re(\gamma) > 0, \quad \beta, \beta' \text{ being the real number}. \\
\]

(iii) \[
\left( I_{0^-}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) = \left( - \frac{d}{dx} \right)^k \left( I_{0^-}^{\alpha,\alpha',\beta+k,\beta',\gamma+k} f \right)(x) \\
\]

\[(1.4.30)\]

\[(i\text{v})\] \[
\left( D_{0^+}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) = \left( I_{0^+}^{-\alpha,-\alpha',-\beta,-\beta',-\gamma} f \right)(x) \\
\]

\[(1.4.31)\]

\[(i\text{vi})\] \[
\left( D_{0^-}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) = \left( - \frac{d}{dx} \right)^k \left( I_{0^-}^{-\alpha,-\alpha',-\beta,-\beta'+k,-\gamma+k} f \right)(x) \\
\]

\[(1.4.32)\] 

\[(1.4.32)\]
1.5 Fractional Calculus Operators and Their Transforms

1.5.1 Laplace Transform

We recall that a function $y(t)$ defined on some domain $J'$ is said to be exponential order $\alpha$ if there exists constant $M, T > 0$ such that $e^{-\alpha t}|y(t)| \leq M$ for all $t \geq T$. If $y(t)$ is of exponential order $\alpha$, then $\int_0^\infty y(t)e^{-st} \, dt$ exists for all $Re(s) > \alpha$. The Laplace transform of $y(t)$ is then defined as [12]

$$y(s) = L\{y(t)\} = \int_0^\infty e^{-st} \, y(t) \, dt \quad (1.5.1)$$

We say that $y(t) = L^{-1}\{y(s)\}$ is the (unique) inverse Laplace transform of $y(s)$.

Laplace Transform of the fractional integral

We have

$$\mathcal{0}I_x^{-v}f(x) = \frac{1}{\Gamma(v)} \int_0^x (x - t)^{v-1} f(t) \, dt \quad (1.5.2)$$

Application of Laplace transform gives

$$L\left\{ \mathcal{0}I_x^{-v}f(x); s \right\} = L\left\{ \frac{t^{v-1}}{\Gamma(v)} \right\} L\{f(t); s\} = s^{-v}F(s) \quad (1.5.3)$$

where $\Re(s) > 0, \Re(v) > 0$.

Laplace Transform of the Fractional Derivative

If $n \in \mathbb{N}$, then by the theory of Laplace Transform, we know that

$$L\left\{ \frac{d^n}{dx^n}f; s \right\} = s^nF(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0+)$$
\[ s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0^+), (n - 1 \leq \alpha < n) \]

(1.5.4)

where \( \Re(s) > 0 \) and \( F(s) \) is the Laplace Transform of \( f(t) \).

**Laplace Transform of Caputo Derivative**

The Caputo derivative of a causal function \( f(t) \) (that is \( f(t) = 0 \) for \( t < 0 \)) with \( \alpha > 0 \) was defined by Caputo [19] in the form

\[
\mathcal{C}_0^\alpha D_x^\alpha f(x) = \mathcal{I}^{n-\alpha}_x \frac{d^n}{dx^n} f(x) = \mathcal{D}_t^{-(n-\alpha)} f^n(t)
\]

\[= \frac{1}{\Gamma(n - \alpha)} \int_a^x (x - t)^{n-\alpha-1} f^n(t), (n - 1 < \alpha < n)\]

(1.5.5)

where \( n \in \mathbb{N} \).

From (1.5.5) and (1.5.3), it follows that

\[ L \{ \mathcal{C}_0^\alpha D_x^\alpha f(t); s \} = s^{-(n-\alpha)} L\{ f^n(t) \} \]

On using (1.5.4), we see that

\[ L \{ \mathcal{C}_0^\alpha D_x^\alpha f(t); s \} = s^{-(n-\alpha)} \left[ s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0^+) \right] \]

\[= s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0^+)(n - 1 < \alpha \leq n) \]

where \( \Re(s) > 0 \) and \( \Re(\alpha) > 0 \).

**1.5.2 Fourier Transform**
The Fourier transform is a generalization of complex Fourier series. The Fourier transform of \( f(x) \) is denoted by \( F(s) \) and is defined as

\[
f(s) = \int_{-\infty}^{\infty} F(x) e^{-isx} \, dx
\]  

(1.5.6)

The inverse formula for Fourier transform is given by

\[
F(x) = F^{-1}\{f(s)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{isx} \, ds
\]  

(1.5.7)

**Fourier Transform of Convolution**

The convolution of two integral functions \( h(t) \) and \( g(t) \) is defined by

\[
h(t) * g(t) = \int_{-\infty}^{\infty} h(t - \xi) g(\xi) \, d\xi = \int_{-\infty}^{\infty} h(\xi) g(t - \xi) \, d\xi
\]

The Fourier transform of the convolution is given by

\[
F\{h(t) * g(t); s\} = H(s) G(s)
\]  

(1.5.8)

**Fourier transform if integer order derivatives**

If \( h(t), h'(t), \ldots, h^{(n-1)}(t) \) vanish for \( t \to \pm \infty \) then

\[
F\{h^n(t); s\} = (-is)^n H(s)
\]  

(1.5.9)

**Fourier transform of the R-L integral**

If

\[
h(t) = \begin{cases} 
  \frac{t^{\alpha-1}}{\Gamma(\alpha)} & (t > 0) \\
  0 & (t \leq 0)
\end{cases}
\]

then

\[
F\{h(t); s\} = (-is)^{-\alpha} H(s)
\]
Using the transform of a convolution, we get

\[ F \left\{ \int_{-\infty}^{t} g(\xi) \, d\xi \right\} = (-is)^{-\alpha}G(s) \]  

(1.5.10)

**Fourier transform of fractional derivatives**

Taking \( b = -\infty \) in (1.4.4) we get

\[ D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{t} \frac{g^n(\xi) \, d\xi}{(t-\xi)^{\alpha+n-1}} = D^\alpha g(t), \tag{1.5.11} \]

\( n - 1 < \alpha < n \)

Then we obtain the following expression for Fourier transform

\[ F \{ D^\alpha g(t); s \} = (-is)^{-n}F \{ g^n(t); s \} \]

\[ = (-is)^{-n}(-is)^n G(s) \]

\[ = (-is)^{\alpha} G(s). \]

**1.5.3 Mellin Transform**

The Mellin transform of a function \( f(x) \), denoted by \( f^*(s) \) is defined as

\[ f^*(s) = M\{f(x); s\} = \int_{0}^{\infty} x^{s-1} f(x) \, dx, \quad x > 0 \tag{1.5.12} \]

The inverse Mellin transform is given by the contour integral

\[ f(x) = M^{-1}\{f^*(s); x\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s) \, x^{-s} \, ds, \tag{1.5.13} \]

where \( \gamma \) is real.
Mellin Transform of the fractional integral

The following results can be obtained by using definition (1.5.12)

\[
(i) \quad M\left(0^{\alpha}I_{\gamma}^{t}f\right)(s) = \frac{\Gamma(1 - \alpha - s)}{\Gamma(1 - \alpha)} f^*(s + \alpha) \tag{1.5.14}
\]

where $\Re(\alpha) > 0$ and $\Re(\alpha + s) < 1$.

\[
(ii) \quad M\left(\alpha^{\alpha}I_{\gamma}^{t}f\right)(s) = \frac{\Gamma(s)}{\Gamma(s + \alpha)} f^*(s + \alpha) \tag{1.5.15}
\]

where $\Re(\alpha) > 0$ and $\Re(s) < 1$.

Mellin transform of the fractional derivative

If $n \in \mathbb{N}$, then

\[
(i) \quad M\{f^{(n)}(t); (s)\} = (-1)^n \frac{\Gamma(s)}{\Gamma(s - n)} M\{f(t); s - n\} \tag{1.5.16}
\]

where $\Re(s) > 0$ and $\Re(s - n) > 0$.

\[
(ii) \quad M(0^{\alpha}D_{\gamma}^{\alpha}f)(s) = (-1)^n \frac{\Gamma(s)}{\Gamma(s - n)} M\{0^{\alpha}I_{\gamma}^{s - \alpha}f\}(s - n),
\]

\[
(n - 1 \leq \Re(\alpha) < n)
\]

\[
= (-1)^n \frac{\Gamma(s)}{\Gamma(s - n)} \frac{\Gamma(1 - (s - \alpha))}{\Gamma(1 - s + n)} M\{f(t); s - \alpha\}, \tag{1.5.17}
\]

where $\Re(s) > 0$ and $\Re(s) < 1 + \Re(\alpha)$.

1.6 Fractional Diffusion
The mathematical description of diffusion has a long history with many alternative formulations including phenomenological models based on conservation of mass and constitutive laws; probabilistic models based on random walks and central limit theorems; microscopic stochastic models based on Brownian motion and Langevin equations; and macroscopic stochastic models based on Fokker-Planck equations. A fundamental result common to the different approaches is that the mean square displacement of a diffusing particle scales linearly with time. However, there have been numerous experimental measurements in which the mean square displacement of diffusing particles scales as a fractional order power law in time. In recent years, a great deal of progress has been made in extending the different models for diffusion to incorporate this fractional diffusion. The techniques of fractional calculus have been found to be very useful in dealing with such models, involving fractional constitutive laws, continuous time random walks and fractional Brownian motions etc [68].

In recent years, there has been growing interest in diffusion in various fields of physics, chemistry and related sciences. It is well known that the fundamental solution (or Green function) of the classical diffusion (or heat) equation can be interpreted as a Gaussian normal probability density function in space evolving in time.

The pioneering work in the study of diffusion in fluids was undertaken by Robert Brown in 1827. He noticed that small particles suspended in fluids perform
peculiarly erratic movements. This phenomenon, which can also be observed in gases, is referred to as Brownian motion. Later on, by kinetic theory of matter it was shown that Brownian motion was an outward manifestation of the molecular motion postulated. In 1905 Albert Einstein developed a satissfactory theory to explain this phenomenon. Einstein considered the case of the free particle that is, a particle on which no forces other than those due to the molecules of the surrounding medium are acting. This motion was modeled by the classical diffusion equation [173]

$$\frac{\partial u(x,t)}{\partial t} = a \frac{\partial^2 u(x,t)}{\partial x^2} , \alpha > 0 , x \in \mathbb{R} , t > 0 . \quad (1.6.1)$$

Einstein’s theory was strengthened and made more rigorous by Smoluchowski, Fokker, Planck, Burger, Wiener and others.

1.6.1 Fractional diffusion-wave equation

A fractional diffusion-wave equation is a linear partial integro-differential equation obtained from the classical diffusion or wave equation by replacing the first or second-order time derivative term by a fractional derivative of order $\alpha , 0 < \alpha \leq 2$, and the second space derivative by a fractional derivative of order, $\beta , 1 < \beta \leq 2$. There has been a great deal of interest in fractional diffusion equations. These equations arise in continuous-time random walks [137], modeling of anomalous diffusive and subdiffusive systems [66], unification of diffusion and wave propagation phenomenon [103] etc. The nature of the diffusion
is characterized by the temporal scaling of the mean-square displacement \( <r^2(t)> \sim t^\alpha \). For standard diffusion \( \alpha = 1 \), whereas in anomalous sub-diffusion \( \alpha < 1 \), and in anomalous super-diffusion \( \alpha > 1 \). Both types of anomalous diffusion have been unified in continuous time random walk models with spatial and temporal memories [66, 83].

Oldham and Spanier [125] considered a fractional diffusion equation that contains first order derivative in space and half order derivative in time. Nigmatullin [120] pointed out that many of the universal electromagnetic and mechanical responses can be modeled accurately using the fractional diffusion-wave equations.

Fujita [52] presented the existence and uniqueness of the solution of the Cauchy problem of the following type

\[
\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = a \frac{\partial^\beta u(x, t)}{\partial x^\beta}, 1 \leq \alpha, \beta \leq 2 \quad (1.6.2)
\]

The results offered an interpretation to a phenomena which lies between the heat equation (\( \alpha = 1, \beta = 2 \)) and the wave equation \( \alpha = \beta = 2 \). In [53, 54] Fujita considered integro-differential equations which exhibit heat diffusion and wave propagation properties.

Mainardi [104, 105] presented analytical investigation of the time-fractional diffusion-wave equations. Using Laplace transform method, he obtained the
fundamental solutions of the basic Cauchy and signaling problems and expressed them in terms of an auxiliary function. He further showed that such a function is an entire function of Wright type. Mainardi [103] provided a comprehensive review of research on the application of calculus in continuum and statistical mechanics including research on fractional diffusion-wave solutions.

Agarwal [2] presented a general solution for a time-fractional diffusion-wave equation defined in a bounded space domain. His solution employed the finite sine transform technique to convert fractional diffusion-wave equation from a space domain to a wave number domain. Subsequently the Laplace transform was used to reduce the resulting equation to an ordinary algebraic equation and finally, the inverse Laplace and inverse sine transforms were used to obtain the desired solutions. Agarwal [3] further used the same technique to obtain a general solution for a fourth-order fractional diffusion-wave equation.

Al-Khaled and Momani [6] used the decomposition method to obtain an approximate solution for the generalized time-fractional diffusion-wave equation. Their results showed the transition from a pure diffusion process (\(\alpha = 1\)) to a pure wave process (\(\alpha = 2\)).

### 1.6.2 The Space-Time fractional diffusion equation

In the standard diffusion equation
where $u = u(x,t)$ is the (real) field variable, replacing the second order space derivative and the first order time derivative by suitable integro-differential operators, we obtain a generalized diffusion equation which may be referred to as the space–time fractional diffusion equation. We write this equation as

$$
\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \quad -\infty < x < \infty, \quad t \geq 0 \quad (1.6.3)
$$

where $\frac{\partial}{\partial t} u(x,t) = D^\beta u(x,t)$ is the space-fractional Riesz-Feller derivative of order $\alpha$ and skewness $\theta$ and $iD^\beta u(x,t)$ is the time-fractional Caputo derivative of order $\beta$.

1.7 Conclusion

This chapter is of introductory nature. Here we throw light on the origin and historical development of fractional integrals and differential operators like Riemann-Liouville operator, Weyl operator, Caputo fractional derivative, Kober operator, Modified Erdelyi Kober operator, Saigo operator, Saigo-Maeda operator etc. Fractional calculus and its elements have been discussed extensively. Moreover this chapter also gives details of Laplace transform, Fourier transform and Mellin transform of various fractional derivatives and integrals. The purpose of studying theories is to apply them to real world problems. Over the last few years, mathematicians harnessed the techniques of fractional calculus to several
applied fields of engineering, science and economics. The author believes that the research work contained in present thesis will continue to grow in the forthcoming years and that it will constitute a modest contribution in the scientific progress of mankind.