CHAPTER – 5

A MODEL FOR INVENTORY SYSTEM ALLOWING PARTIAL BACKLOGGING FOR PRODUCTS HAVING DETERIORATION WHICH IS NOT INSTANTANEOUS

5.1 INTRODUCTION

Inventory is often where the biggest costs are hidden in businesses. This is also the first reason for choosing inventory management as the subject of this research as discussed during the introduction. Various processes taking place at the manufacturer; for three reasons only inventory will be picked out. First of all stocks are responsible for a large part of the total working capital costs: up to about one third. Inventory costs also represent a significant component of total logistics costs. Consequently the biggest benefits can thus be gained by reducing these costs. Working capital invested in stocks could also have been a very useful resource when it could have been used otherwise. Capital invested in stocks is thus, from a company-perspective, a ‘useless’ waste of money. Cost reductions are required by the market in order to keep offering competitive products and services; reducing the working capital costs using more efficient inventory management is one way to achieve this goal. Secondly, risk may be involved in stock keeping. The products
may be stolen, can be damaged by fire, water etc. or may decay over time. Consequently these events might influence the production process and could even cause it to stop and orders are delivered too late accordingly. If stock levels are lower the related risks will also be reduced. Risks caused by maintaining stocks are again related to costs, because stocks have to be stored secure and have to be protected against these risks, which costs money. One more reason to focus on inventory management is because inventory costs are some of the easiest to identify and reduce when attacking supply chain problems. Budgets are often under pressure and costs have to be reduced to keep up with the competition. In accordance working capital costs will have to be reduced; optimizing internal logistics is a way to do this in a relatively easy manner. Inventory control is presumed to enable stock reductions and thus as a consequence able to reduce related costs and risks. Inventory management aims to control materials and related costs and finance.

In most of the traditional models having decaying products and having deterioration process started at the earliest it is received in the stock. But in realistic worldly situations, the quality is maintained up to a certain period of time and products maintain their quality and originality up to a certain extent of time, and deterioration is not instantaneous.
In real situations, customers wait for the products which have trendy demand and supply and whose product life cycle is smaller. But when the waiting period is longer then backlogging rate is less. Therefore, backlogging exists and should not be ignored.

**Hill (1995)** had been the pioneer in discussing the inventory model having demand rate as ramp type.

**Mandal and Pal (1998)** allowed shortage for decaying items.

**Wu and Ouyang (2000)** also discussed the inventory system having ramp type demand.

**Wu (2001)** proposed inventory model which shows depletion by demand and also by Weibull distribution. In the model, variable rate of backlogging is taken. The model allow the shortages of items but only partial backlogging is there. The backlog is dependent on the next replenishment.

**Giri et al. (2003)** discussed EOQ model for single item. Weibull distribution is there for deteriorating products. The ramp type demand rate is taken into consideration. The shortages are allowed. The model is for n infinite planning period.
5.2 BASIC ASSUMPTIONS & NOTATIONS

1. Demand rate is ramp type:

\[ D(t) = \begin{cases} \frac{a}{e^{bt}}, & t < \mu \\ \frac{a}{e^{b\mu}}, & t \geq \mu \end{cases} \]

2. a, b are not variable but constants

3. \( \lambda \) is a constant.

4. Planning period is prescribed.

5. Shortages will be there & partially backlogged.

6. Deterioration rate is constant.

7. No replacement for decaying products for the cycle considered.

8. Single item system for a fixed period of time.

9. There is no lead time.

Notations

1. \( \mu \) is the start of deterioration.

2. \( \eta \) is the rate of deterioration.

3. \( Q \) is the EPQ.

4. \( A \) is setup cost.

5. \( h \) is holding cost.

6. \( c_1 \) is the production cost.

7. \( c_2 \) is the shortage cost.

8. \( c_4 \) is the lost sales

9. \( e^{-\delta t} \) is backlogging rate.
5.3 FORMULATION OF MODEL

5.3.1 Case 1: 0 < µ < t₁

Figure 5.1: Case 1: (µ < t₁)

The following equations represent the inventory level at time t

\[ I'(t) = (\lambda - 1)ae^{bt} \quad 0 \leq t \leq \mu \]
\[ I'(t) = (\lambda - 1)ae^{b\mu} - \eta I(t) \quad \mu \leq t \leq t₁ \]
\[ I'(t) = -ae^{b\mu} - \eta I(t) \quad t₁ \leq t \leq t₂ \]
\[ I'(t) = -e^{-\delta(t₂-t₁)}ae^{b\mu} \quad t₂ \leq t \leq t₃ \]
\[ I'(t) = (\lambda - 1)ae^{b\mu} \quad t₃ \leq t \leq t₄ \]

I(0)=0, I(μ)=I(μ¹), I(t₂)=0, I(t₄)=0.

Solving equations (1)–(5), we get

\[ I(t) = (\lambda - 1)\frac{a}{b}(e^{bt} - 1) \quad 0 \leq t \leq \mu \]
\[ I(t) = (\lambda - 1)\left[ae^{\mu}\left(t-\mu\right) + \frac{\eta}{2}(t₂-\mu²)(t₂-t₁) + \frac{a}{b}(e^{b\mu} - 1)e^{b\mu}\right]e^{-\gamma} \quad \mu \leq t \leq t₁ \]
\[ I(t) = \left[ ae^{b \mu} \left( (t_2 - t) + \frac{\eta}{2}(t_2^2 - t^2) \right) e^{-\eta t} \right] \quad t_1 \leq t \leq t_2 \]

\[ I(t) = \frac{ae^{b \mu}}{\delta} \left[ e^{-\delta(t_3 - t_2)} - e^{-\delta(t_3 - t)} \right] \quad t_2 \leq t \leq t_3 \]

\[ I(t) = -(\lambda - 1) ae^{b \mu} (t_4 - t) \quad t_3 \leq t \leq t_4 \]

Since inventory level is continuous at \( t_3 \), so we get

\[ \Rightarrow \quad t_3 = f(t_2, t_4) \]

The EPQ is as under:

\[ Q = \int_{0}^{\mu} P(t) \, dt + \int_{t_1}^{t_2} P(t) \, dt + \int_{t_3}^{t_4} P(t) \, dt \]

Set up cost is:

\[ C_R = A \]

Holding cost is:

\[ C_H = \int_{0}^{\mu} h I(t) \, dt + \int_{t_1}^{t_2} h I(t) \, dt + \int_{t_3}^{t_4} h I(t) \, dt \]

\[ = -\frac{h\eta}{2} \left( t_1^2 - \mu^2 \right) + (\lambda - 1) ae^{b \mu} \left[ \frac{h}{2} \left( t_1^2 - \mu^2 \right) - \frac{h\eta}{3} \left( t_1^3 - \mu^3 \right) \right] \]

\[ + \frac{\eta}{2} \left( \frac{h}{3} \left( t_1^3 - \mu^3 \right) - h \mu^2 (t_1 - \mu) - \frac{h\eta}{4} \left( t_1^4 - \mu^4 \right) \right) \]

\[ + ae^{b \mu} \left[ \left( t_2 + \frac{\eta t_2^2}{2} \right) \left( h(t_2 - t_1) - \frac{h\eta}{2} \left( t_2^2 - t_1^2 \right) \right) - \frac{h}{2} (t_2^2 - t_1^2) \right] \]

\[ + \frac{h\eta}{3} \left( t_2^3 - t_1^3 \right) - \frac{\eta}{2} \left[ \frac{h}{3} (t_2^3 - t_1^3) - \frac{h\eta}{4} (t_2^4 - t_1^4) \right] \]

Production cost \( C_P \) is given by
\[ C_p = c_1 Q \]
\[ = c_1 \lambda a \left[ \frac{e^{b \mu} - 1}{b} + e^{b \mu} \left( t_1 - \mu t + t_4 - t_3 \right) \right] \]

Shortage cost is:
\[ C_s = c_3 \left[ \int_{t_2}^{t_3} \{-I(t)\} dt + \int_{t_3}^{t_4} \{-I(t)\} dt \right] \]

Lost sales is:
\[ C_l = c_4 \int_{t_2}^{1} \left(1 - e^{-\delta(t_3-t_2)}\right) ae^{b \mu} dt \]
\[ = c_4 ae^{b \mu} \left[ (t_3 - t_2) - \frac{1}{\delta} \left(1 - e^{-(t_3-t_2)}\right) \right] \]

Now, the total cost of the system can be formulated as
\[ T C (t_1, t_2) = C_r + C_r + C_p + C_s + C_l \]

### 5.4 An Example

a=50, b=0.26, \( \lambda=1.50 \), \( \mu=1.20 \), \( \eta=0.04 \), h=1.10, c_1=2, c_3=2.80, c_4=9, t_4=8, \( \delta=0.50 \)

Then we get \( t_1^* = 1.5682 \), \( t_2^* = 4.3791 \), \( t_3^* = 6.2713 \), Q=478.29 and TC(\( t_1^* \), \( t_2^* \))=6278.57.

### 5.5 Sensitivity Analysis
Table 5.1: Effect of changes in parameters.

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Variation in $t_1$, $t_2$ and $t_3$ w.r.t. 'a'

Variation in $Q^*$ w.r.t. 'a'
Variation in TC w.r.t. 'a'

Variation in $t_1$, $t_2$ and $t_3$ w.r.t. 'b'
Variation in $t_1$, $t_2$ and $t_3$ w.r.t. $\eta$

Variation in $Q^*$ w.r.t. $\eta$
Variation in TC w.r.t. 'η'

Variation in $t_1$, $t_2$ and $t_3$ w.r.t. 'h'
Variation in $t_1$, $t_2$ and $t_3$ w.r.t. $\delta$:

- $t_1^*$
- $t_2^*$
- $t_3^*$

Variation in $Q^*$ w.r.t. $\delta$:
5.3.2 Case 2: $t_1 < \mu < t_2$

Figure 5.2. Case 2: $(t_1 < \mu)$
The following equations show the stock level:

\[ I'(t) = (\lambda - 1) a e^{bt} \quad 0 \leq t \leq t_1 \]

\[ I'(t) = -a e^{bt} \quad t_1 \leq t \leq \mu \]

\[ I'(t) = -a e^{b\mu} - \eta I(t) \quad \mu \leq t \leq t_2 \]

\[ I'(t) = -e^{-\delta(t_3-t_1)} a e^{b\mu} \quad t_2 \leq t \leq t_3 \]

\[ I'(t) = (\lambda - 1) a e^{b\mu} \quad t_3 \leq t \leq t_4 \]

Now, we get

\[ I(t) = (\lambda - 1) \frac{a}{b} \left( e^{bt} - 1 \right) \quad 0 \leq t \leq t_1 \]

\[ I(t) = \frac{a}{b} \left( e^{b\mu} - e^{b\mu} \right) + a e^{b\mu} \left[ (t_2 - \mu) + \frac{\eta}{2}(t_2^2 - \mu^2) \right] e^{-\delta t} \quad t_1 \leq t \leq \mu \]

\[ I(t) = a e^{b\mu} \left[ (t_2 - t) + \frac{\eta}{2}(t_2^2 - t^2) \right] e^{-t} \quad \mu \leq t \leq t_2 \]

\[ I(t) = \frac{a e^{b\mu}}{\delta} \left[ e^{-\delta(t_3-t_2)} - e^{-\delta(t_3-t)} \right] \quad t_2 \leq t \leq t_3 \]

\[ I(t) = -(\lambda - 1) a e^{b\mu} (t_4 - t) \quad t_3 \leq t \leq t_4 \]

Now we get

\[ \frac{a e^{b\mu}}{\delta} \left[ e^{-\delta(t_3-t_2)} - 1 \right](a + b\mu)(t_2 - t_3) = (\lambda - 1) a e^{b\mu} (t_3 - t_4) \]
⇒ \( t_3 = f(t_2, t_4) \)

The EPQ is as follows:

\[
Q = \int_0^{t_1} P(t)dt + \int_{t_1}^{t_2} P(t)dt
\]

Set up cost is:

\[ C_R = \text{A} \]

Holding cost is:

\[
C_H = \int_0^{t_1} h(t) dt + \int_{t_1}^{\mu} h(t) dt + \int_{\mu}^{t_2} h(t) dt
\]

\[
\{ h(\mu - t_1) \} - \frac{a}{b} \left\{ \frac{e^{by}}{b} (h) - \frac{e^{by}}{b} (h) \right\}
\]

\[
+ \beta_2 ae^{by} \left[ \left( t_2 + \frac{\eta t_2^2}{2} \right) \left\{ h(t_2 - \mu) - \frac{\eta}{2} (t_2^2 - \mu^2) \right\} \right]
\]

\[
- \left\{ \frac{h}{2} (t_2^2 - \mu^2) + \frac{\eta}{6} (t_2^3 - \mu^3) \right\}
\]
\[ +\left\{ \frac{h\eta}{3}(t_2^3 - \mu^3) + \frac{h\eta^2}{8}(t_2^4 - \mu^4) \right\} \]

(3) Production cost is:

\[ C_p = c_1 Q \]

\[ = c_1 \lambda a \left[ \frac{(e^{b\mu} - 1)}{b} + e^{b\mu} \{ t_4 - t_3 \} \right] \]

Shortage cost is:

\[ C_s = c_3 \left[ \int_{t_2}^{t_1} \{- I(t)\} \, dt + \int_{t_3}^{t_4} \{- I(t)\} \, dt \right] \]

\[ = c_3 \left[ \frac{a e^{b\mu}}{\delta} \left( e^{-(t_2-t_3)} \left( t_2 - t_3 - \frac{1}{\delta} \right) + \frac{1}{\delta} \right) + \frac{ae^{b\mu}}{2} \left( \lambda - 1 \right) \left( t_3 - t_4 \right)^2 \right] \]

Lost sales cost is:

\[ C_L = c_4 \int_{t_2}^{t_1} \left( 1 - e^{-\delta(t-t_1)} \right) ae^{b\mu} \, dt \]

\[ = c_4 ae^{b\mu} \left[ (t_3 - t_2) - \frac{1}{\delta} \left( 1 - e^{-(t_2-t_1)} \right) \right] \]

Now, the total cost of the system is:

\[ TC(t_1, t_2) = C_R + C_H + C_p + C_s + C_L \]
5.6 An Example

Suppose $A=120$, $a=80$, $b=0.20$, $\lambda=1.40$, $\mu=2.00$, $\eta=0.030$, $h=0.60$, $c_1=3.10$, $c_3=2.2$, $c_4=9$, $t_4=8$ and $\delta=0.04$. Then we get $t_1=1.4172$, $t_2=4.6212$, $t_3=6.2781$, $Q=257.28$ and $TC(t_1, t_2)=4093.12$.

Fig 5.3: Convexity of the cost function
## 5.7 Sensitivity Analysis

### Table 5.2: Parameters changes

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**Variation in t₁, t₂ and t₃ w.r.t. 'a'**

- t₁*
- t₂*
- t₃*
Variation in $t_1$, $t_2$ and $t_3$ w.r.t. 'b'

Variation in TC w.r.t. 'b'
Variation in $Q^*$ w.r.t. 'b'

Variation in $t_1^*, t_2^*, t_3^*$ w.r.t. 'δ'
Variation in $t_1$, $t_2$ and $t_3$ w.r.t. $\eta$:

- $t_1^*$
- $t_2^*$
- $t_3^*$

Variation in $Q^*$ w.r.t. $\eta$:

- $Q^*$
Variation in TC w.r.t. 'η'

Variation in $t_1$, $t_2$ and $t_3$ w.r.t. 'h'
Variation in $Q^*$ w.r.t. 'h'

Variation in TC w.r.t. 'h'
5.8 CONCLUSION

Uncertainties are the most important reason to keep inventories. If for example a specific order is delivered exactly according to plan and on the agreed date and time, but the wrong goods are delivered or the delivery is damaged and can therefore not be used. This example illustrates two possible causes of uncertainty. Although a delivery might be perfectly on time, there might still be something wrong with the stock as well. Uncertainties in delivery times may also form a reason to maintain a safety stock, in case a delivery arrives late. If all processes subsequent to a specific delivery are interrupted as well, it may cause major losses in the end. For this reason a stock is usually kept, to cope with unforeseen events that could otherwise prevent the production from moving on. Another important source of uncertainties is caused on the demand side; the expected orders placed by the clients are hard to predict. For guaranteed deliveries and a certain quality of service to the clients, also a stock is often maintained to cope with uncertainties on the demand side. To summarize, stocks thus allow for variation and uncertainty in both supply and demand, which lets operations continue smoothly when problems arise. A model for inventory system allowing partial backlogging for products having deterioration which is not instantaneous is developed.
The numerical example clearly justify the theory. The sensitivity analysis clearly shows that the model elucidated is well suited and adapts properly.