Chapter 4

‘NTV’ Metric based Entropies of Interval Valued Intuitionistic Fuzzy Sets and their Applications in Decision Making

4.1 Introduction

In many real-world decision problems the values of the membership function and the non-membership function in an IFS are difficult to be expressed as exact numbers. Instead, the ranges of their values can usually be specified. In such cases, Atanassov and Gargov (1989) generalized the concept of Intuitionistic Fuzzy Set (IFS) to Interval Valued Intuitionistic Fuzzy Set (IVIFS) and studied its various properties. It may be noted that the entropy and similarity measures are two important concepts in the field of fuzzy set theory and are widely investigated by many researchers from different point of view. The similarity measure of IFSs indicates the degree of similarity between two IFSs and plays a significant role in many applications such as pattern recognition, approximate reasoning and
Vlachos and Sergiadis (2007) extended the De Luca and Termini’s (1972) non-probabilistic entropy for fuzzy sets in the study of the intuitionistic fuzzy information measure. Burillo and Bustince (1996a) introduced the notions of entropy of IFSs and interval-valued fuzzy sets (IVFS) to measure the degree of intuitionism of an IFS and IVFS, respectively. Hung and Yang (2006) gave their axiomatic definitions and characterization of entropy of IFSs and IVFSs with the help of probability theory. Dengfeng and Chuntian (2002) proposed some similarity measures on IFSs and applied them in pattern recognition problems. Further, Liang and Shi (2003) pointed out the drawbacks of Li and Cheng (2002) methods and to overcome them, they proposed several new similarity measures and also discussed relationships between these measures. Further, Szmidt and Kacprzyk (2005) defined a similarity measure using distance measure of IFSs and applied these measures in group decision making problems and medical diagnostic reasoning. Xu (2007a) defined some similarity measures for IVIFSs and applied these similarity measures in pattern recognitions. Hung and Yang (2004) presented a similarity measure of IFSs based on Hausdorff metric and applied it to pattern recognition problems. In the study of fuzzy sets, Wang (1997) defined two similarity measures and Pappis and Karacapilidis (1993) defined three kinds of similarity measures. Hung and Yang (2008) extend these similarity measures from the fuzzy sets to IFSs. Further, Xu and Chen (2008) generalized some formulas of similarity measures of IFSs to IVIFSs. Zeng and Guo (2008) proved that some similarity measures and entropies of IVFSs can be deduced by normalized distances of IVFSs based on their axiomatic definitions. Zeng and Li (2006), Zhang et al. (2009) showed that similarity measures and entropies of IVFSs can be obtained by the transformation from each other. Zeng et al. (2009) put straight forward some entropy formulas of IFSs according to the relationship between entropies and similarity measures of IFSs. Later on, Wei et al. (2011) proposed the entropy for the IVIFSs and obtained the similarity measure for the IVIFSs on the basis of proposed entropy.
Xu and Yager (2006) developed some geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator and the intuitionistic fuzzy hybrid geometric (IFHG) operator, and gave an application of the IFHG operator to multi-criteria decision-making problems with intuitionistic fuzzy information. Xu (2007b) developed some arithmetic aggregation operators, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and the intuitionistic fuzzy hybrid aggregation (IFHA) operator. Xu (2007c) defined the concept of interval-valued intuitionistic fuzzy number (IVIFN), and gave some basic operational laws of IVIFNs. He gave an interval-valued intuitionistic fuzzy weighted averaging operator and an interval-valued intuitionistic fuzzy weighted geometric operator and defines the score function and the accuracy function of IVIFNs. Xu and Chen (2007) developed some arithmetic aggregation operators, such as the interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator, the interval-valued intuitionistic fuzzy ordered weighted averaging (IIFOWA) operator and the interval-valued intuitionistic fuzzy hybrid aggregation (IIFHA) operator, and gave an application of the IIFHA operator to multi-criteria decision making problems with interval-valued intuitionistic fuzzy information by using the score function and accuracy function of interval-valued intuitionistic fuzzy numbers.

In this chapter, we study some basic definitions related to the intuitionistic fuzzy sets and the interval-valued intuitionistic fuzzy sets in section 4.2. New similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on ‘NTV’ metric along with their weighted form have been proposed in section 4.3. The proposed similarity measures have also been analogously extended to obtain new intuitionistic fuzzy entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets with the proof of their validity in section 4.4. Further, a new algorithm for multi-criteria group decision making has been provided using the proposed weighted similarity measures in which the weights have been calculated using the proposed entropies in section
4.5. Numerical example by taking interval-valued intuitionistic fuzzy sets has been illustrated in section 4.6.

4.2 Preliminaries

In this section, we present some axiomatic definitions of the similarity measure, entropy measure for intuitionistic fuzzy set and interval-valued intuitionistic fuzzy set which are well known in literature.

**Similarity Measure on IFSs:**
Hung and Yang (2004) proposed that a real-valued function \( S : IFS(X) \times IFS(X) \to [0, 1] \), is called the similarity measure on \( IFS(X) \), if \( S \) satisfies the following axiomatic requirements:

\[(S1) \text{ If } \tilde{A} \text{ is a crisp set, then } S(\tilde{A}, \tilde{A}^c) = 0;\]
\[(S2) S(\tilde{A}, \tilde{B}) = 1 \iff \tilde{A} = \tilde{B}, \text{ i.e., } \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x) \& \nu_{\tilde{A}}(x) = \nu_{\tilde{B}}(x);\]
\[(S3) S(\tilde{A}, \tilde{B}) = S(\tilde{B}, \tilde{A});\]
\[(S4) \text{ If } \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}, \text{ then } S(\tilde{A}, \tilde{C}) \leq S(\tilde{A}, \tilde{B}) \text{ and } S(\tilde{A}, \tilde{C}) \leq S(\tilde{B}, \tilde{C}).\]

**Similarity Measure on IVIFSs:**
Xu and Chen (2008) proposed that a real-valued function \( S : IVIFS(X) \times IVIFS(X) \to [0, 1] \), is called the similarity measure on \( IVIFS(X) \), if \( S \) satisfies the following axiomatic requirements:

\[(S1) 0 \leq S(\tilde{A}_*, \tilde{B}_*) \leq 1;\]
\[(S2) S(\tilde{A}_*, \tilde{B}_*) = 1 \iff \tilde{A}_* = \tilde{B}_*;\]
\[(S3) S(\tilde{A}_*, \tilde{B}_*) = S(\tilde{B}_*, \tilde{A}_*);\]
\[(S4) \text{ If } \tilde{A}_* \subseteq \tilde{B}_* \subseteq \tilde{C}_*, \text{ then } S(\tilde{A}_*, \tilde{C}_*) \leq S(\tilde{A}_*, \tilde{B}_*) \text{ and } S(\tilde{A}_*, \tilde{C}_*) \leq S(\tilde{B}_*, \tilde{C}_*).\]
Apart from similarity measures for IFSs, we have the entropies (information measures) for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. These entropies play an important role in many fields of research such as pattern recognition, approximate reasoning, decision making etc.

**Entropy Measure on IFSs:**
Szmidt and Kacprzyk (2001) proposed that a real-valued function $E : \mathcal{IFS}(X) \rightarrow [0, 1]$ is called the entropy measure on $\mathcal{IFS}(X)$, if $E$ satisfies the following properties:

(E1) $E(\tilde{A}) = 0 \iff \tilde{A}$ is crisp set;

(E2) $E(\tilde{A}) = 1 \iff \mu_{\tilde{A}}(x) = \nu_{\tilde{A}}(x), \forall x \in X$;

(E3) $E(\tilde{A}) \leq E(\tilde{B})$ if $\tilde{A}$ is less fuzzy than $\tilde{B}$, i.e., $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) \geq \nu_{\tilde{B}}(x)$ for $\mu_{\tilde{B}}(x) \leq \nu_{\tilde{B}}(x)$ or $\mu_{\tilde{A}}(x) \geq \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) \leq \nu_{\tilde{B}}(x)$ for $\mu_{\tilde{B}}(x) \geq \nu_{\tilde{B}}(x), \forall x \in X$;

(E4) $E(\tilde{A}) = E(\tilde{A}^c)$, where $\tilde{A}^c$ is the complement of $\tilde{A}$.

**Entropy Measure on IVIFSs:**
Liu et al. (2005) proposed that a real-valued function $E : \mathcal{IVIFS}(X) \rightarrow [0, 1]$ is called the entropy measure on $\mathcal{IVIFS}(X)$, if $E$ satisfies the following properties:

(E1) $E(\tilde{A}_*) = 0 \iff \tilde{A}_*$ is crisp set;

(E2) $E(\tilde{A}_*) = 1 \iff \mu^L_{\tilde{A}_*}(x) = \mu^U_{\tilde{A}_*}(x)$ and $\nu^L_{\tilde{A}_*}(x) = \nu^U_{\tilde{A}_*}(x), \forall x \in X$;

(E3) $E(\tilde{A}_*) \leq E(\tilde{B}_*)$ if $\tilde{A}_*$ is less fuzzy than $\tilde{B}_*$, i.e., $\tilde{A}_* \subseteq \tilde{B}_*$, for $\mu^L_{\tilde{B}_*}(x) \leq \nu^L_{\tilde{B}_*}(x)$ and $\mu^U_{\tilde{B}_*}(x) \leq \nu^U_{\tilde{B}_*}(x)$, or $\tilde{B}_* \subseteq \tilde{A}_*$ for $\mu^L_{\tilde{B}_*}(x) \geq \nu^L_{\tilde{B}_*}(x)$ and $\mu^U_{\tilde{B}_*}(x) \geq \nu^U_{\tilde{B}_*}(x), \forall x \in X$;

(E4) $E(\tilde{A}_*) = E(\tilde{A}_*^c)$, where $\tilde{A}_*^c$ is the complement of $\tilde{A}_*$.
4.3 ‘NTV’ Based Similarity Measures for IFSs and IVIFSs

In this section, we propose similarity measures for IFSs and IVIFSs along with their weighted form based on the ‘NTV’ metric defined by Neito et al. (2003) on \( n \)-dimensional unit hypercube \( I^n \).

Neito et al. (2003) defined ‘NTV’ metric, \( d_{NTV}(p, q) \), on \( I^n \) as follows:

\[
d_{NTV}(p, q) = \frac{\sum_{i=1}^{n} |p_i - q_i|}{\sum_{i=1}^{n} \max \{p_i, q_i\}},
\]

where \( p = (p_1, p_2, \ldots, p_n) \) and \( q = (q_1, q_2, \ldots, q_n) \) are \( n \)-dimensional vectors in \( I^n \).

Let \( \tilde{A} = \{\{x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\}\} \) and \( \tilde{B} = \{\{x, \mu_{\tilde{B}}(x), \nu_{\tilde{B}}(x)\}\} \) are two single-element IFSs. Based on the ‘NTV’ metric, we propose a new similarity measure between \( \tilde{A} \) and \( \tilde{B} \) as follows:

\[
S^1_{NTV}(\tilde{A}, \tilde{B}) = 1 - \frac{|\mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)| + |\nu_{\tilde{A}}(x) - \nu_{\tilde{B}}(x)| + |\pi_{\tilde{A}}(x) - \pi_{\tilde{B}}(x)|}{\max \{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + \max \{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\} + \max \{\pi_{\tilde{A}}(x), \pi_{\tilde{B}}(x)\}}. \tag{4.3.1}
\]

Also, we know that

\[
|\mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)| = \max \{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} - \min \{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},
\]

\[
|\nu_{\tilde{A}}(x) - \nu_{\tilde{B}}(x)| = \max \{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\} - \min \{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\},
\]

\[
|\pi_{\tilde{A}}(x) - \pi_{\tilde{B}}(x)| = \max \{\pi_{\tilde{A}}(x), \pi_{\tilde{B}}(x)\} - \min \{\pi_{\tilde{A}}(x), \pi_{\tilde{B}}(x)\}.
\]

Hence, the similarity measure (4.3.1) reduces to

\[
S^1_{NTV}(\tilde{A}, \tilde{B}) = \frac{\min \{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + \min \{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\} + \min \{\pi_{\tilde{A}}(x), \pi_{\tilde{B}}(x)\}}{\max \{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + \max \{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\} + \max \{\pi_{\tilde{A}}(x), \pi_{\tilde{B}}(x)\}}. \tag{4.3.2}
\]

The similarity measure (4.3.2) is defined for single-element IFS. Further, we define similarity measure of two IFSs \( \tilde{A} \) and \( \tilde{B} \) under the universe of discourse \( X = \{x_1, x_2, \ldots, x_n\} \).
Let \( \tilde{A} = \{ (x_i, \mu_\tilde{A}(x_i), \nu_\tilde{A}(x_i)) | x_i \in X \} \) and \( \tilde{B} = \{ (x_i, \mu_\tilde{B}(x_i), \nu_\tilde{B}(x_i)) | x_i \in X \} \) are two IFSs, then similarity measure between \( \tilde{A} \) and \( \tilde{B} \) is defined as

\[
S_{NTV}(\tilde{A}, \tilde{B}) = \frac{1}{n} \sum_{i=1}^{n} \left( \min \{ \mu_\tilde{A}(x_i), \mu_\tilde{B}(x_i) \} + \min \{ \nu_\tilde{A}(x_i), \nu_\tilde{B}(x_i) \} + \min \{ \pi_\tilde{A}(x_i), \pi_\tilde{B}(x_i) \} \right).
\]

**Theorem 4.3.1:** \( S_{NTV}(\tilde{A}, \tilde{B}) \) is a valid similarity measure.

**Proof.** In order to prove that similarity measure (4.3.3) is a valid similarity measure, we prove the four properties (S1) to (S4) as listed by Hung and Yang (2006):

(S1) By the definition of equality of two IFSs, it is easy to show that \( S_{NTV}(\tilde{A}, \tilde{B}) = 1 \) if and only if \( \tilde{A} = \tilde{B} \).

(S2) If \( \tilde{A} \) is a crisp set, then either \( \mu_\tilde{A}(x_i) = 1, \nu_\tilde{A}(x_i) = 0, \pi_\tilde{A}(x_i) = 0 \) or \( \mu_\tilde{A}(x_i) = 0, \nu_\tilde{A}(x_i) = 1, \pi_\tilde{A}(x_i) = 0 \) for all \( x_i \in X \).

Moreover, for \( \tilde{A}^c \), either \( \mu_\tilde{A}^c(x_i) = 0, \nu_\tilde{A}^c(x_i) = 1, \pi_\tilde{A}^c(x_i) = 0 \) or \( \mu_\tilde{A}^c(x_i) = 1, \nu_\tilde{A}^c(x_i) = 0, \pi_\tilde{A}^c(x_i) = 0 \) for all \( x_i \in X \);

\( \Rightarrow S_{NTV}(\tilde{A}, \tilde{A}^c) = 0 \).

(S3) In view of the proposed similarity measure, it is easy to verify that \( S_{NTV}(\tilde{A}, \tilde{B}) = S_{NTV}(\tilde{B}, \tilde{A}) \).

(S4) Let \( \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \), then, we have

\[
\mu_\tilde{A}(x_i) \leq \mu_\tilde{B}(x_i) \leq \mu_\tilde{C}(x_i), \nu_\tilde{A}(x_i) \geq \nu_\tilde{B}(x_i) \geq \nu_\tilde{C}(x_i) \text{ and } \pi_\tilde{A}(x_i) \leq \pi_\tilde{B}(x_i) \leq \pi_\tilde{C}(x_i), \forall x_i \in X \text{ which implies}
\]

\[
\min \{ \mu_\tilde{A}(x_i), \mu_\tilde{B}(x_i) \} = \min \{ \mu_\tilde{A}(x_i), \mu_\tilde{C}(x_i) \};
\]

\[
\max \{ \mu_\tilde{A}(x_i), \mu_\tilde{B}(x_i) \} \leq \max \{ \mu_\tilde{A}(x_i), \mu_\tilde{C}(x_i) \};
\]

\[
\min \{ \nu_\tilde{A}(x_i), \nu_\tilde{B}(x_i) \} \geq \min \{ \nu_\tilde{A}(x_i), \nu_\tilde{C}(x_i) \};
\]

\[
\max \{ \nu_\tilde{A}(x_i), \nu_\tilde{B}(x_i) \} \leq \max \{ \nu_\tilde{A}(x_i), \nu_\tilde{C}(x_i) \};
\]

\[
\min \{ \pi_\tilde{A}(x_i), \pi_\tilde{B}(x_i) \} = \min \{ \pi_\tilde{A}(x_i), \pi_\tilde{C}(x_i) \};
\]

\[
\max \{ \pi_\tilde{A}(x_i), \pi_\tilde{B}(x_i) \} \leq \max \{ \pi_\tilde{A}(x_i), \pi_\tilde{C}(x_i) \},
\]

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which further implies that

\[
\begin{align*}
\min \{ \mu_A(x_i), \mu_B(x_i) \} & \geq \min \{ \mu_A(x_i), \mu_C(x_i) \}, \\
\max \{ \mu_A(x_i), \mu_B(x_i) \} & = \max \{ \mu_A(x_i), \mu_C(x_i) \}; \\
\min \{ \nu_A(x_i), \nu_B(x_i) \} & \geq \min \{ \nu_A(x_i), \nu_C(x_i) \}, \\
\max \{ \nu_A(x_i), \nu_B(x_i) \} & \geq \max \{ \nu_A(x_i), \nu_C(x_i) \}; \\
\min \{ \pi_A(x_i), \pi_B(x_i) \} & \geq \min \{ \pi_A(x_i), \pi_C(x_i) \}, \\
\max \{ \pi_A(x_i), \pi_B(x_i) \} & \geq \max \{ \pi_A(x_i), \pi_C(x_i) \}.
\end{align*}
\]

Hence, we have

\[
\begin{align*}
\min \{ \mu_A(x_i), \mu_B(x_i) \} & + \min \{ \nu_A(x_i), \nu_B(x_i) \} + \min \{ \pi_A(x_i), \pi_B(x_i) \} \\
\max \{ \mu_A(x_i), \mu_B(x_i) \} & + \max \{ \nu_A(x_i), \nu_B(x_i) \} + \max \{ \pi_A(x_i), \pi_B(x_i) \} \\
\geq & \frac{\min \{ \mu_A(x_i), \mu_C(x_i) \} + \min \{ \nu_A(x_i), \nu_C(x_i) \} + \min \{ \pi_A(x_i), \pi_C(x_i) \}}{\max \{ \mu_A(x_i), \mu_C(x_i) \} + \max \{ \nu_A(x_i), \nu_C(x_i) \} + \max \{ \pi_A(x_i), \pi_C(x_i) \}}.
\end{align*}
\]

(4.3.4)

Similarly, we have

\[
\begin{align*}
\min \{ \mu_B(x_i), \mu_C(x_i) \} & \geq \min \{ \mu_A(x_i), \mu_C(x_i) \}, \\
\max \{ \mu_B(x_i), \mu_C(x_i) \} & = \max \{ \mu_A(x_i), \mu_C(x_i) \}; \\
\min \{ \nu_B(x_i), \nu_C(x_i) \} & \geq \min \{ \nu_A(x_i), \nu_C(x_i) \}, \\
\max \{ \nu_B(x_i), \nu_C(x_i) \} & \leq \max \{ \nu_A(x_i), \nu_C(x_i) \}; \\
\min \{ \pi_B(x_i), \pi_C(x_i) \} & \geq \min \{ \pi_A(x_i), \pi_C(x_i) \}, \\
\max \{ \pi_B(x_i), \pi_C(x_i) \} & = \max \{ \pi_A(x_i), \pi_C(x_i) \},
\end{align*}
\]

which implies that

\[
\begin{align*}
\min \{ \mu_A(x_i), \mu_B(x_i) \} & = \min \{ \mu_A(x_i), \mu_C(x_i) \}, \\
\max \{ \mu_A(x_i), \mu_B(x_i) \} & = \max \{ \mu_A(x_i), \mu_C(x_i) \}; \\
\min \{ \nu_A(x_i), \nu_B(x_i) \} & \geq \min \{ \nu_A(x_i), \nu_C(x_i) \}, \\
\max \{ \nu_A(x_i), \nu_B(x_i) \} & \geq \max \{ \nu_A(x_i), \nu_C(x_i) \}; \\
\min \{ \pi_A(x_i), \pi_B(x_i) \} & \geq \min \{ \pi_A(x_i), \pi_C(x_i) \}, \\
\max \{ \pi_A(x_i), \pi_B(x_i) \} & \geq \max \{ \pi_A(x_i), \pi_C(x_i) \}.
\end{align*}
\]

Hence, we have

\[
\begin{align*}
\min \{ \mu_A(x_i), \mu_B(x_i) \} & + \min \{ \nu_A(x_i), \nu_B(x_i) \} + \min \{ \pi_A(x_i), \pi_B(x_i) \} \\
\max \{ \mu_A(x_i), \mu_B(x_i) \} & + \max \{ \nu_A(x_i), \nu_B(x_i) \} + \max \{ \pi_A(x_i), \pi_B(x_i) \} \\
\geq & \frac{\min \{ \mu_A(x_i), \mu_C(x_i) \} + \min \{ \nu_A(x_i), \nu_C(x_i) \} + \min \{ \pi_A(x_i), \pi_C(x_i) \}}{\max \{ \mu_A(x_i), \mu_C(x_i) \} + \max \{ \nu_A(x_i), \nu_C(x_i) \} + \max \{ \pi_A(x_i), \pi_C(x_i) \}}.
\end{align*}
\]

(4.3.5)
From equation (4.3.4) and (4.3.5), we have $S_{NTV} (\tilde{A}, \tilde{B}) \geq S_{NTV} (\tilde{A}, \tilde{C})$
and $S_{NTV} (\tilde{B}, \tilde{C}) \geq S_{NTV} (\tilde{A}, \tilde{C})$.

Therefore, $S_{NTV} (\tilde{A}, \tilde{B})$ is a valid similarity measure between IFSs $\tilde{A}$ and $\tilde{B}$. □

Further, we associate some weights depending upon importance of the elements of the universal set to define the weighted form of the similarity measure (4.3.3).

Let $w = (w_1, w_2, \ldots, w_n)$ be the weight vector of the elements $x_i, i = 1, 2, \ldots, n$.

We propose the following weighted similarity measure:

$$S'_{NTV} (\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} w_i \left( \frac{\min \{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\} + \min \{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\} + \min \{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}}{\max \{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\} + \max \{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\} + \max \{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}} \right),$$

(4.3.6)

where $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

**Remark:** If $w = (1/n, 1/n, \ldots, 1/n)$, then the weighted similarity measure (4.3.6) reduces to the similarity measure (4.3.3).

Next, we consider two IVIFSs as

$$\tilde{A}_* = \left\{ \langle x, [\mu_{\tilde{A}}^{L}(x), \mu_{\tilde{A}}^{R}(x)], [\nu_{\tilde{A}}^{L}(x), \nu_{\tilde{A}}^{R}(x)] \rangle \big| x \in X \right\}$$

and

$$\tilde{B}_* = \left\{ \langle x, [\mu_{\tilde{B}}^{L}(x), \mu_{\tilde{B}}^{R}(x)], [\nu_{\tilde{B}}^{L}(x), \nu_{\tilde{B}}^{R}(x)] \rangle \big| x \in X \right\}.$$ 

Analogous to the ‘NTV’ similarity measure for IFS in (4.3.3), we propose the following similarity measure for IVIFSs:

$$S_{NTV} (\tilde{A}_*, \tilde{B}_*) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{M_L(\mu, \nu) + M_U(\mu, \nu)}{N_L(\mu, \nu) + N_U(\mu, \nu)} \right),$$

(4.3.7)

and the weighted form of the similarity measure (4.3.7) is given by

$$S'_{NTV} (\tilde{A}_*, \tilde{B}_*) = \sum_{i=1}^{n} w_i \left( \frac{M_L(\mu, \nu) + M_U(\mu, \nu)}{N_L(\mu, \nu) + N_U(\mu, \nu)} \right),$$

(4.3.8)

where

$$M_L(\mu, \nu) = \min \left\{ \mu_{\tilde{A}}^{L}(x_i), \mu_{\tilde{B}}^{L}(x_i) \right\} + \min \left\{ \nu_{\tilde{A}}^{L}(x_i), \nu_{\tilde{B}}^{L}(x_i) \right\} + \min \left\{ \pi_{\tilde{A}}^{L}(x_i), \pi_{\tilde{B}}^{L}(x_i) \right\},$$

$$N_L(\mu, \nu) = \max \left\{ \mu_{\tilde{A}}^{L}(x_i), \mu_{\tilde{B}}^{L}(x_i) \right\} + \max \left\{ \nu_{\tilde{A}}^{L}(x_i), \nu_{\tilde{B}}^{L}(x_i) \right\} + \max \left\{ \pi_{\tilde{A}}^{L}(x_i), \pi_{\tilde{B}}^{L}(x_i) \right\},$$

$$M_U(\mu, \nu) = \min \left\{ \mu_{\tilde{A}}^{U}(x_i), \mu_{\tilde{B}}^{U}(x_i) \right\} + \min \left\{ \nu_{\tilde{A}}^{U}(x_i), \nu_{\tilde{B}}^{U}(x_i) \right\} + \min \left\{ \pi_{\tilde{A}}^{U}(x_i), \pi_{\tilde{B}}^{U}(x_i) \right\},$$

$$N_U(\mu, \nu) = \max \left\{ \mu_{\tilde{A}}^{U}(x_i), \mu_{\tilde{B}}^{U}(x_i) \right\} + \max \left\{ \nu_{\tilde{A}}^{U}(x_i), \nu_{\tilde{B}}^{U}(x_i) \right\} + \max \left\{ \pi_{\tilde{A}}^{U}(x_i), \pi_{\tilde{B}}^{U}(x_i) \right\}.$$
\[ N_U(\mu, \nu) = \max \left\{ \mu^{U}_{A^*}(x_i), \mu^{U}_{B^*}(x_i) \right\} + \max \left\{ \nu^{U}_{A^*}(x_i), \nu^{U}_{B^*}(x_i) \right\} + \max \left\{ \pi^{U}_{A^*}(x_i), \pi^{U}_{B^*}(x_i) \right\}. \]

**Theorem 4.3.2:** Similarity measure \( S_{NTV}(\tilde{A}, \tilde{B}) \) is a valid similarity measure.

**Proof.** The proof of the theorem follows on the similar lines as the proof of theorem 4.3.1. \qed

### 4.4 Entropy Measures based on Proposed Similarity Measures

In this section, we introduce entropy measures based on the proposed similarity measures for IFSs and IVIFSs, respectively. We first recall some entropy formulas for IFSs.

For an IFS \( \tilde{A} = \{ \langle x_i, \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i) \rangle | x_i \in X \} \), Szmidt and Kacprzyk (2001) defined two kind of cardinalities of \( \tilde{A} \). The least cardinality or min-sigma-count of \( \tilde{A} \) given by

\[ \min \sum \text{count}(\tilde{A}) = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i), \]

and the biggest cardinality or max-sigma-count of \( \tilde{A} \) given by

\[ \max \sum \text{count}(\tilde{A}) = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i) + \pi_{\tilde{A}}(x_i). \]

Using these two cardinalities, Szmidt and Kacprzyk (2001) proposed an entropy measure for \( \tilde{A} \) as

\[ E_{SK}(\tilde{A}) = \frac{1}{n} \sum_{i=1}^{n} \frac{\max \text{count}(\tilde{A}_i \cap \tilde{A}_i^c)}{\max \text{count}(\tilde{A}_i \cup \tilde{A}_i^c)}, \]

where for each \( i, \tilde{A}_i \) denote the single-element IFS corresponding to the element \( x_i \) in \( X \), and described as \( \tilde{A}_i = \{ \langle x_i, \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i) \rangle \} \). Also,

\[ \tilde{A}_i \cap \tilde{A}_i^c = \{ \langle x_i, \min \{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\}, \max \{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\} \rangle \}, \]

\[ \tilde{A}_i \cup \tilde{A}_i^c = \{ \langle x_i, \max \{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\}, \min \{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\} \rangle \}. \]
For an IFS \( \tilde{A} \), Wang et al. (1997) gave a different entropy formula

\[
E_{WL}(\tilde{A}) = \frac{1}{n} \sum_{i=1}^{n} \min \left\{ \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i) \right\} + \pi_{\tilde{A}}(x_i).
\]

(4.4.2)

Hung and Liu et al. (2005) introduced fuzzy entropy for a vague sets. Using the equivalence of two theories of vague sets and IFSs [Bustince and Burillo (1996b)], Wei et al. (2011) transform the Hung and Liu (2005) fuzzy entropy for a vague set to fuzzy entropy for an IFS \( \tilde{A} \) as

\[
E_{HL}(\tilde{A}) = \frac{1}{n} \sum_{i=1}^{n} \frac{1 - |\mu_{\tilde{A}}(x_i) - \nu_{\tilde{A}}(x_i)| + \pi_{\tilde{A}}(x_i)}{1 + |\mu_{\tilde{A}}(x_i) - \nu_{\tilde{A}}(x_i)| + \pi_{\tilde{A}}(x_i)}.
\]

(4.4.3)

Wei et al. (2011) also proved that all these entropies given by (4.4.1), (4.4.2) and (4.4.3) are equivalent. In fuzzy set theory, Kosko (1990) gave the idea to drive entropies from the distance and similarity measures. Xuecheng (1992) found various entropies from the similarity measures for the fuzzy sets by the relation \( E(\tilde{A}) = S(A, A^c) \).

Similarly, we derive entropies for IFSs and IVIFSs from the proposed similarity measures (4.3.3) and (4.3.7) as follows:

\[
E_T(\tilde{A}) = S_{NTV}(\tilde{A}, \tilde{A}^c)
= \frac{1}{n} \sum_{i=1}^{n} \left( \min \left\{ \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i) \right\} + 0.5\pi_{\tilde{A}}(x_i) \right)
\]

(4.4.4)

and

\[
E_T(\tilde{A}) = \frac{1}{n} \sum_{i=1}^{n} \left( \min \left\{ \mu_{\tilde{A}}^{L}(x_i), \nu_{\tilde{A}}^{L}(x_i) \right\} + \min \left\{ \mu_{\tilde{A}}^{U}(x_i), \nu_{\tilde{A}}^{U}(x_i) \right\} + 0.5\left( \pi_{\tilde{A}}^{L}(x_i) + \pi_{\tilde{A}}^{U}(x_i) \right) \right).
\]

(4.4.5)

**Theorem 4.4.1:** \( E_T(\tilde{A}) \) is a valid information measure for the intuitionistic fuzzy set.

**Proof.** In order to prove that the entropy (4.4.4) is a valid measure, we prove all the four properties \((E1)\) to \((E4)\) as listed by Szmidt and Kacprzyk (2001).

\((E1)\) If \( \tilde{A} \) is a crisp set, then either \( \mu_{\tilde{A}}(x_i) = 1, \nu_{\tilde{A}}(x_i) = 0, \pi_{\tilde{A}}(x_i) = 0 \) or
\[
\mu_{\tilde{A}}(x_i) = 0, \nu_{\tilde{A}}(x_i) = 1, \pi_{\tilde{A}}(x_i) = 0, \forall x_i \in X.
\]

From this we have \( S(\tilde{A}, \tilde{A}^c) = 0 \Rightarrow E_T(\tilde{A}) = 0 \).
Conversely, if $E_T(\tilde{A}) = 0$, then $\min \{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\} + 0.5 \times \pi_{\tilde{A}}(x_i) = 0$, $\forall x_i \in X$; which implies either $\mu_{\tilde{A}}(x_i) = 1$, $\nu_{\tilde{A}}(x_i) = 0$, $\pi_{\tilde{A}}(x_i) = 0$ or
$\mu_{\tilde{A}}(x_i) = 0$, $\nu_{\tilde{A}}(x_i) = 1$, $\pi_{\tilde{A}}(x_i) = 0$, $\forall x_i \in X$;
$\Rightarrow \tilde{A}$ is a crisp set.

(E2) Let $\mu_{\tilde{A}}(x_i) = \nu_{\tilde{A}}(x_i)$, $\forall x_i \in X$
$\Rightarrow \mu_{\tilde{A}^c}(x_i) = \nu_{\tilde{A}^c}(x_i) = \mu_{\tilde{A}}(x_i) = \nu_{\tilde{A}}(x_i)$,
$\Rightarrow \tilde{A}^c = \tilde{A} \Leftrightarrow S_{NTV}(\tilde{A}, \tilde{A}^c) = 1 \Leftrightarrow E_T(\tilde{A}) = 1$.

(E3) It is easy to verify that, $S_{NTV}(\tilde{A}, \tilde{A}^c) = S_{NTV}(\tilde{A}^c, \tilde{A}) \Leftrightarrow E_T(\tilde{A}) = E_T(\tilde{A}^c)$.

(E4) Suppose that $\mu_B(x_i) \leq \nu_B(x_i)$ for each $x_i \in X$, then $\tilde{A} \subseteq \tilde{B}$, i.e.,
$\mu_{\tilde{A}}(x_i) \leq \mu_B(x_i)$, $\nu_{\tilde{A}}(x_i) \geq \nu_B(x_i)$;
$\Rightarrow \mu_{\tilde{A}}(x_i) \leq \mu_B(x_i) \leq \nu_B(x_i) \leq \nu_{\tilde{A}}(x_i)$;
$\Rightarrow \tilde{A} \subseteq \tilde{B} \subseteq \tilde{B}^c \subseteq \tilde{A}^c$.
Therefore, we have $S_{NTV}(\tilde{A}, \tilde{A}^c) \leq S_{NTV}(\tilde{B}, \tilde{A}^c) \leq S_{NTV}(\tilde{B}, \tilde{B}^c)$.
Similarly, if $\mu_{\tilde{A}}(x_i) \geq \mu_B(x_i)$, $\nu_{\tilde{A}}(x_i) \leq \nu_B(x_i)$, for $\mu_B(x_i) \geq \nu_B(x_i)$,
then we have $\nu_{\tilde{A}}(x_i) \leq \nu_B(x_i) \leq \mu_B(x_i) \leq \mu_{\tilde{A}}(x_i)$,
$\Rightarrow \tilde{A}^c \subseteq \tilde{B}^c \subseteq \tilde{B} \subseteq \tilde{A}$,
$\Rightarrow S_{NTV}(\tilde{A}^c, \tilde{A}) \leq S_{NTV}(\tilde{B}^c, \tilde{A}) \leq S_{NTV}(\tilde{B}^c, \tilde{B})$,
$\Rightarrow S_{NTV}(\tilde{A}, \tilde{A}^c) \leq S_{NTV}(\tilde{A}, \tilde{B}^c) \leq S_{NTV}(\tilde{B}, \tilde{B}^c)$,
$\Rightarrow E_T(\tilde{A}) = S_{NTV}(\tilde{A}, \tilde{A}^c) \leq S_{NTV}(\tilde{B}, \tilde{B}^c) = E_T(\tilde{B})$,
$\Rightarrow E_T(\tilde{A}) \leq E_T(\tilde{B})$.
Since $E_T(\tilde{A})$ satisfies all the four properties of an entropy measure, therefore, it is a valid entropy for the IFSs. □

**Theorem 4.3.2**: $E_T(\tilde{A}_*)$ is a valid information measure for the interval-valued intuitionistic fuzzy set.

**Proof.** In order to prove that the entropy (4.4.5) is a valid measure, we prove all the four properties (E1) to (E4) as listed by Liu et al. (2005).

(E1) Let $\tilde{A}_*$ be a crisp set. Then either we have

$[\mu_{\tilde{A}_*}(x_i), \mu_{\tilde{A}_*}(x_i)] = [1, 1], [\nu_{\tilde{A}_*}(x_i), \nu_{\tilde{A}_*}(x_i)] = [0, 0]$ & $[\pi_{\tilde{A}_*}(x_i), \pi_{\tilde{A}_*}(x_i)] = [0, 0]
or

\[ \left[ \mu_{A_i}^L(x_i), \mu_{A_i}^U(x_i) \right] = [0, 0], \left[ \nu_{A_i}^L(x_i), \nu_{A_i}^U(x_i) \right] = [1, 1] \& \left[ \pi_{A_i}^L(x_i), \pi_{A_i}^U(x_i) \right] = [0, 0] \]

for each \( x_i \in X \).

Hence, we have \( S(\tilde{A}_s, \tilde{A}_s^c) = 0 \Rightarrow E_T(\tilde{A}_s) = 0 \).

Conversely, suppose that \( E_T(\tilde{A}_s) = 0 \), then we have

\[
\min \left\{ \mu_{A_i}^L(x_i), \nu_{A_i}^L(x_i) \right\} + \min \left\{ \mu_{A_i}^U(x_i), \nu_{A_i}^U(x_i) \right\} + 0.5 \left( \pi_{A_i}^L(x_i) + \pi_{A_i}^U(x_i) \right) = 0;
\]

Since each term in the above equation is non-negative, therefore,

\[
\min \left\{ \mu_{A_i}^L(x_i), \nu_{A_i}^L(x_i) \right\} = 0, \min \left\{ \mu_{A_i}^U(x_i), \nu_{A_i}^U(x_i) \right\} = 0
\]

and \( \pi_{A_i}^L(x_i) + \pi_{A_i}^U(x_i) = 0 \) for each \( x_i \in X \);

which further implies that \( \tilde{A}_s \) is a crisp set.

(E2) If \( \left[ \mu_{A_i}^L(x), \mu_{A_i}^U(x) \right] = \left[ \nu_{A_i}^L(x), \nu_{A_i}^U(x) \right] \) for each \( x_i \in X \), then from equation (4.4.5) we obtain \( E_T(\tilde{A}_s) = 1 \).

Conversely, if we suppose that \( E_T(\tilde{A}_s) = 1 \), then we get

\[
\min \left\{ \mu_{A_i}^L(x_i), \nu_{A_i}^L(x_i) \right\} + \min \left\{ \mu_{A_i}^U(x_i), \nu_{A_i}^U(x_i) \right\}
\]

\[
= \max \left\{ \mu_{A_i}^L(x_i), \nu_{A_i}^L(x_i) \right\} + \max \left\{ \mu_{A_i}^U(x_i), \nu_{A_i}^U(x_i) \right\} ;
\]

which implies that \( \left[ \mu_{A_i}^L(x), \mu_{A_i}^U(x) \right] = \left[ \nu_{A_i}^L(x), \nu_{A_i}^U(x) \right], \forall x_i \in X \).

(E3) It is easy to verify that \( S_{NTV}(\tilde{A}_s, \tilde{A}_s^c) = S_{NTV}(\tilde{A}_c^c, A_s) \Rightarrow E_T(\tilde{A}_s) = E_T(\tilde{A}_c^c) \).

(E4) Let \( \tilde{A}_s \) is less fuzzy than \( \tilde{B}_s \), i.e., \( \tilde{A}_s \subseteq \tilde{B}_s \).

\[
\Rightarrow \mu_{A_i}^L(x_i) \leq \mu_{B_i}^L(x_i), \mu_{A_i}^U(x_i) \leq \mu_{B_i}^U(x_i) \& \nu_{A_i}^L(x_i) \geq \nu_{B_i}^L(x_i), \nu_{A_i}^U(x_i) \geq \nu_{B_i}^U(x_i)
\]

for \( \mu_{B_i}^L(x_i) \leq \nu_{B_i}^L(x_i) \) and \( \mu_{B_i}^U(x_i) \leq \nu_{B_i}^U(x_i) \), \( \forall x_i \in X \).

Then it follows that \( \mu_{A_i}^L(x_i) \leq \mu_{B_i}^L(x_i) \leq \nu_{B_i}^L(x_i) \leq \nu_{A_i}^L(x_i) \)

and \( \mu_{A_i}^U(x_i) \leq \mu_{B_i}^U(x_i) \leq \nu_{B_i}^U(x_i) \leq \nu_{A_i}^U(x_i), \forall x_i \in X \);

\[
\Rightarrow A_s \subseteq B_s \subseteq \tilde{B}_s \subseteq \tilde{A}_s^c
\]

Therefore, we have \( S_{NTV}(\tilde{A}_s, \tilde{A}_s^c) \leq S_{NTV}(\tilde{B}_s, \tilde{A}_s^c) \leq S_{NTV}(\tilde{B}_s, \tilde{B}_s^c) \).

Similarly, if \( \mu_{A_i}^L(x_i) \geq \mu_{B_i}^L(x_i), \mu_{A_i}^U(x_i) \geq \mu_{B_i}^U(x_i) \) and \( \nu_{A_i}^L(x_i) \leq \nu_{B_i}^L(x_i), \nu_{A_i}^U(x_i) \leq \nu_{B_i}^U(x_i) \),

\[
\nu_{A_i}^L(x_i) \leq \nu_{B_i}^L(x_i) \text{ for } \mu_{B_i}^L(x_i) \geq \nu_{B_i}^L(x_i) \text{ and } \mu_{B_i}^U(x_i) \geq \nu_{B_i}^U(x_i), \forall x_i \in X ;
\]

which follows that \( \nu_{A_i}^L(x_i) \leq \nu_{B_i}^L(x_i) \leq \mu_{B_i}^L(x_i) \leq \mu_{A_i}^L(x_i) \)

and \( \nu_{A_i}^U(x_i) \leq \nu_{B_i}^U(x_i) \leq \mu_{B_i}^U(x_i) \leq \mu_{A_i}^U(x_i), \forall x_i \in X ;
\]

\[
\Rightarrow \tilde{A}_s^c \subseteq \tilde{B}_s \subseteq \tilde{B}_s \subseteq \tilde{A}_s
\]

\[
\Rightarrow S_{NTV}(\tilde{A}_s, \tilde{B}_s) \leq S_{NTV}(\tilde{B}_s, \tilde{A}_s) \leq S_{NTV}(\tilde{B}_s, \tilde{B}_s) ;
\]

\[
\Rightarrow S_{NTV}(\tilde{A}_s, \tilde{B}_s) \leq S_{NTV}(\tilde{A}_s, \tilde{B}_s) \leq S_{NTV}(\tilde{B}_s, \tilde{B}_s) ;
\]

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\[ E_T(\tilde{A}_*) = S_{NTV}(\tilde{A}_*, \tilde{A}_c^*) \leq S_{NTV}(\tilde{B}_*, \tilde{B}_c^*) = E_T(\tilde{B}_*); \]
\[ E_T(\tilde{A}_*) \leq E_T(\tilde{B}_*). \]
Since \( E_T(\tilde{A}_*) \) satisfies all the four properties of an entropy measure, therefore, it is a valid entropy for the IVIFS.

4.5 Multiple-Criteria Decision Making with IFS and IVIFS

In this section, we present a new method which is based on the proposed weighted similarity measures, where the objective weights are calculated using the proposed entropies to deal with the Multiple-Criteria Decision Making (MCDM) problems under the intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. Ratings of the alternatives, importance/weights of criteria and importance of decision makers in a group decision committee are the three most significant factors which can affect on the results of decision making problems.

Let \( A = \{ A_1, A_2, \ldots, A_m \} \) be the set of possible alternatives, \( D = \{ D_1, D_2, \ldots, D_l \} \) be the set of decision makers and \( C = \{ C_1, C_2, \ldots, C_n \} \) be the set of criteria with which the performance of alternatives are measured. Assume that the weight information of the criteria and the decision makers are completely unknown. Let \([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]\) be the interval-valued intuitionistic fuzzy number, where \([a_{ij}, b_{ij}]\) indicates the degree that alternative \( A_i \) satisfies the criterion \( C_j \), \([c_{ij}, d_{ij}]\) indicates the degree that alternative \( A_i \) does not satisfies the criterion \( C_j \) and \([a_{ij}, b_{ij}] \subset [0, 1], [c_{ij}, d_{ij}] \subset [0, 1]\) such that \( b_{ij} + d_{ij} \leq 1, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \)

Now, we propose the following algorithm to solve the above multiple-criteria decision making problem:

**Step 1:** Determine the weights of decision makers in the decision group.

Assume that decision group contains \( l \) decision makers. The importance/weights of the decision makers in the selection committee may not be equal. The importance/weights of decision makers are considered as linguistic variables
expressed by interval-valued intuitionistic fuzzy numbers (IVIFNs).

Let $D_k = (a_k, b_k, c_k, d_k)$ be an interval-valued intuitionistic fuzzy number for rating of $k$th decision maker. Then the subjective weight of $k$th decision maker can be defined as:

$$\lambda_k = \frac{(a_k + b_k + (1 - b_k - d_k)\left(\frac{a_k}{a_k+c_k}\right) + (1 - a_k - c_k)\left(\frac{b_k}{b_k+d_k}\right))}{\sum_{k=1}^{l}(a_k + b_k + (1 - b_k - d_k)\left(\frac{a_k}{a_k+c_k}\right) + (1 - a_k - c_k)\left(\frac{b_k}{b_k+d_k}\right))}$$

(4.5.1)

and $\sum_{k=1}^{l} \lambda_k = 1$. The linguistic variables for the importance of the decision makers are provided in the Table 4.1. If the importance of all the decision makers is same namely extremely importance, the rating of the $k$th decision maker can be expressed as $([1, 1],[0, 0][0, 0])$. Then the weight of each decision maker will be $1/l$.

**Step 2:** Construct the aggregated interval-valued intuitionistic fuzzy decision matrix by pulling the individual decision opinions into a group opinions.

Let $D^k = \left(\begin{array}{cccc} r_{ij}^{(1)} & & & \\ \vdots & \ddots & \ddots & \vdots \\ r_{ij}^{(l)} & \cdots & \cdots & r_{ij} \end{array}\right)_{m \times n}$ is an interval-valued intuitionistic fuzzy decision matrix for $k$th ($k = 1, 2, \ldots, l$) decision maker and $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_l$ is the weight vector for decision makers, $\sum_{k=1}^{l} \lambda_k = 1, \lambda_k \in [0, 1]$. In group decision-making process, all the individual decision opinions need to be fused into group opinions to construct aggregated interval-valued intuitionistic fuzzy decision matrix. In order to do, we utilize interval-valued intuitionistic fuzzy weighted average (IIFWA) operator due to Xu and Chen (2007) as follows:

$$r_{ij} = IIFWA_{\lambda} \left(\begin{array}{cccc} r_{ij}^{(1)} & & & \\ \vdots & \ddots & \ddots & \vdots \\ r_{ij}^{(l)} & \cdots & \cdots & r_{ij} \end{array}\right)$$

$$= \left(1 - \prod_{k=1}^{l}(1 - a_{ij}^{(k)})^{\lambda_k}, 1 - \prod_{k=1}^{l}(1 - b_{ij}^{(k)})^{\lambda_k}, 1 - \prod_{k=1}^{l}(1 - c_{ij}^{(k)})^{\lambda_k}, 1 - \prod_{k=1}^{l}(1 - d_{ij}^{(k)})^{\lambda_k}\right).$$

The aggregated interval-valued intuitionistic fuzzy decision matrix can be defined as:

$$D = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix}$$

**Step 3:** Determine the aggregated interval-valued intuitionistic fuzzy weights of the criteria using IIFWA operator.
All criteria may not be assumed to be of equal importance. Let $W$ represents a set of grades of importance for given criteria’s. In order to obtain $W$, all the individual decision maker opinions for the importance of each of criterion need to be combined. Let $w^{(k)}_j = \left( [a_{ij}^{(k)}, a_{ij}^{(k)}], [c_{ij}^{(k)}, d_{ij}^{(k)}] \right)$ be an IVIFN assigned to criterion $C_j$ by the $k$th decision maker. Then the aggregated weights of the criteria are calculated using the IIFWA operator due to Xu and Chen (2007) as follows:

$$w_j = IIFWA_{\lambda} \left( w^{(1)}_j, w^{(2)}_j, \ldots, w^{(l)}_j \right)$$

$$= \left( \left[ 1 - \prod_{k=1}^l (1 - a_{ij}^{(k)})^{\lambda_k}, 1 - \prod_{k=1}^l (1 - b_{ij}^{(k)})^{\lambda_k} \right], \left[ \prod_{k=1}^l (c_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^l (d_{ij}^{(k)})^{\lambda_k} \right] \right).$$

(4.5.2)

The aggregated weights of the criteria can be defined as:

$W = [w_1, w_2, \ldots, w_n]^T$, here $w_j = ([a_j, b_j], [a_j, b_j])$, $j = 1, 2, \ldots, n$.

**Step 4:** Construct the aggregated weighted interval-valued intuitionistic fuzzy decision matrix.

After the aggregated weights of criteria and the aggregated interval valued intuitionistic fuzzy decision matrix are determined, the aggregated weighted interval-valued intuitionistic fuzzy decision matrix can be defined as follows:

$$D' = D \otimes W = (r'_{ij})_{m \times n},$$

(4.5.3)

where $r'_{ij} = \left( \left[ a'_{ij}, a'_{ij} \right], \left[ c'_{ij}, d'_{ij} \right] \right)$ is an element of the aggregated weighted interval-valued intuitionistic fuzzy decision matrix.

**Step 5:** Determine the objective weights of criteria using the proposed interval-valued intuitionistic fuzzy entropy measure (4.4.5).

Hwang and Yoon (1981) introduced a method based on information entropy to determine the weights of attributes. Rao (2007), Rao and Singh (2012) methods also suggested the calculation of objective weights using entropy. Xu (2004), Xu and Hui (2009) assigns a small weight to an attribute with similar attribute values across alternatives because such attribute does not help in differentiating alternatives. Furthermore, the method requires all elements in a decision matrix to be normalized to the range $[0, 1]$ so that each column of the decision matrix sums to one.
The entropy of the $j^{th}$ criterion $C_j$, $j = 1, 2, \ldots, n$ for the $m$ available alternatives can be obtained from entropy measure (4.4.5) as follows:

$$E_j = \frac{1}{m} \sum_{i=1}^{m} \left( \frac{\min \{a_{ij}, c_{ij}\} + \min \{b_{ij}, d_{ij}\} + (1 - (a_{ij} + b_{ij} + c_{ij} + d_{ij})/2)}{\max \{a_{ij}, c_{ij}\} + \max \{b_{ij}, d_{ij}\} + (1 - (a_{ij} + b_{ij} + c_{ij} + d_{ij})/2)\right)$$

and the attribute weight $w_j$ for each criterion $C_j$ based on entropy value can be defined as

$$w_j = \frac{1 - E_j}{n - \sum_{j=1}^{n} E_j}, \ j = 1, 2, \ldots, n.$$ 

**Step 6:** Obtain the interval-valued intuitionistic fuzzy positive-ideal solution (IVIFPIS) and the interval-valued intuitionistic fuzzy negative-ideal solution (IVIFNIS).

Let $J_1$ and $J_2$ be benefit criteria and cost criteria, respectively. The interval-valued intuitionistic fuzzy positive-ideal solution, denoted as $A^+$, and the interval-valued intuitionistic fuzzy negative-ideal solution, denoted as $A^-$, are defined as follows:

$$A^+ = \left(\left([a_1^+, b_1^+], [c_1^+, d_1^+]\right), \left([a_2^+, b_2^+], [c_2^+, d_2^+]\right), \ldots, \left([a_n^+, b_n^+], [c_n^+, d_n^+]\right)\right),$$

$$A^- = \left(\left([a_1^-, b_1^-], [c_1^-, d_1^-]\right), \left([a_2^-, b_2^-], [c_2^-, d_2^-]\right), \ldots, \left([a_n^-, b_n^-], [c_n^-, d_n^-]\right)\right)\right),$$

where for each $j = 1, 2, \ldots, n$,

$$\left([a_j^+, b_j^+], [c_j^+, d_j^+]\right) = (\langle[\max a_{ij}, \max b_{ij}], [\min a_{ij}, \min b_{ij}] | j \in J_1\rangle, \langle[\min a_{ij}, \max b_{ij}], [\max a_{ij}, \max b_{ij}] | j \in J_2\rangle\rangle),$$

$$\left([a_j^-, b_j^-], [c_j^-, d_j^-]\right) = (\langle[\min a_{ij}, \min b_{ij}], [\max a_{ij}, \max b_{ij}] | j \in J_1\rangle, \langle[\max a_{ij}, \max b_{ij}], [\min a_{ij}, \min b_{ij}] | j \in J_2\rangle\rangle).$$

**Step 7:** Calculate the similarity of alternatives with the IVIFPIS and IVIFNIS based on proposed weighted similarity measure (4.3.8), respectively as follows:

The similarity between alternatives can be found based on the proposed weighted similarity measure (4.3.8) as follows:

$$S(A_i, A^+) = \sum_{j=1}^{n} w_j \left(\frac{p + q}{s + t}\right),$$

where $p$, $q$, $s$, and $t$ are the weights assigned to the criteria.
and
\[ S(A_i, A^-) = \sum_{j=1}^{n} w_j \left( \frac{p' + q'}{s' + t'} \right), \]

where

\[ p = \min\left\{ a_{ij}, a_{ij}^+ \right\} + \min\left\{ c_{ij}, c_{ij}^+ \right\} + \min\left\{ 1 - b_{ij} - d_{ij}, 1 - b_{ij}^+ - d_{ij}^+ \right\}, \]

\[ q = \min\left\{ b_{ij}, b_{ij}^+ \right\} + \min\left\{ d_{ij}, d_{ij}^+ \right\} + \min\left\{ 1 - a_{ij} - c_{ij}, 1 - a_{ij}^+ - c_{ij}^+ \right\}, \]

\[ s = \max\left\{ a_{ij}, a_{ij}^+ \right\} + \max\left\{ c_{ij}, c_{ij}^+ \right\} + \max\left\{ 1 - b_{ij} - d_{ij}, 1 - b_{ij}^+ - d_{ij}^+ \right\}, \]

\[ t = \max\left\{ b_{ij}, b_{ij}^+ \right\} + \max\left\{ d_{ij}, d_{ij}^+ \right\} + \max\left\{ 1 - a_{ij} - c_{ij}, 1 - a_{ij}^+ - c_{ij}^+ \right\}, \]

\[ p' = \min\left\{ a_{ij}, a_{ij}^- \right\} + \min\left\{ c_{ij}, c_{ij}^- \right\} + \min\left\{ 1 - b_{ij} - d_{ij}, 1 - b_{ij}^- - d_{ij}^- \right\}, \]

\[ q' = \min\left\{ b_{ij}, b_{ij}^- \right\} + \min\left\{ d_{ij}, d_{ij}^- \right\} + \min\left\{ 1 - a_{ij} - c_{ij}, 1 - a_{ij}^- - c_{ij}^- \right\}, \]

\[ s' = \max\left\{ a_{ij}, a_{ij}^- \right\} + \max\left\{ c_{ij}, c_{ij}^- \right\} + \max\left\{ 1 - b_{ij} - d_{ij}, 1 - b_{ij}^- - d_{ij}^- \right\}, \]

\[ t' = \max\left\{ b_{ij}, b_{ij}^- \right\} + \max\left\{ d_{ij}, d_{ij}^- \right\} + \max\left\{ 1 - a_{ij} - c_{ij}, 1 - a_{ij}^- - c_{ij}^- \right\}, \]

**Step 8:** Calculate the relative closeness coefficient to the interval-valued intuitionistic fuzzy ideal solution.

The relative closeness coefficient of an alternative \( A_i \) with respect \( A^+ \) and \( A^- \) is defined as follows:

\[ C_i^* = \frac{S(A_i, A^+)}{S(A_i, A^+ + S(A_i, A^-))}, \quad i = 1, 2, \ldots, m. \]  \hspace{1cm} (4.5.4)

**Step 9:** Rank all the alternatives.

After the relative closeness coefficient of each alternative is determined, alternatives are ranked according to descending order of \( C_i^* \)'s and select one that has largest rank, denoted by \( C_k^* \) among the values \( C_i^* \), \( i = 1, 2, \ldots, m \). Hence, \( C_i^* \) is the best choice.

**Remark 4.5.2:** Since the intuitionistic fuzzy set is a particular case of interval-valued intuitionistic fuzzy set, therefore above proposed algorithm for IVIFs may similarly be outline for IFSs. For this, we will have to make the following changes:
In step 1, the subjective weight given by the equation (4.5.1) will be replaced by
the weight as suggested in Boran et al. (2009).

In step 2 and 3, the interval-valued intuitionistic fuzzy weighted average (IIFWA)
operator due to Xu and Chen (2007) will be replaced by the Xu (2007b) intu-
itionistic fuzzy weighted average (IFWA) operator.

In step 5, the entropy measure given by the equation (4.4.5) will be replaced by
the entropy measure given by the equation (4.4.4).

In step 5, the weighted similarity measure given by the equation (4.3.8) will be
replaced by the weighted similarity measure given by the equation (4.3.6).

Table 4.1: Importance of Decision Makers with their Weights.

<table>
<thead>
<tr>
<th>Linguistic terms</th>
<th>$DM_1$</th>
<th>$DM_2$</th>
<th>$DM_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>Very Important</td>
<td>Medium</td>
<td>Important</td>
</tr>
<tr>
<td></td>
<td>0.393</td>
<td>0.236</td>
<td>0.372</td>
</tr>
</tbody>
</table>

Table 4.2: Linguistic Terms for Rating the Criteria by Decision Makers

<table>
<thead>
<tr>
<th>Linguistic terms</th>
<th>IFNs</th>
<th>IVIFNs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very Important (VI)</td>
<td>(0.90, 0.10)</td>
<td>([0.90, 0.95], [0.00, 0.05])</td>
</tr>
<tr>
<td>Important (I)</td>
<td>(0.85, 0.10)</td>
<td>([0.85, 0.90], [0.05, 0.10])</td>
</tr>
<tr>
<td>Medium (M)</td>
<td>(0.50, 0.40)</td>
<td>([0.50, 0.55], [0.35, 0.40])</td>
</tr>
<tr>
<td>Unimportant (U)</td>
<td>(0.20, 0.70)</td>
<td>([0.20, 0.25], [0.65, 0.70])</td>
</tr>
<tr>
<td>Very Unimportant (VU)</td>
<td>(0.05, 0.90)</td>
<td>([0.05, 0.10], [0.85, 0.90])</td>
</tr>
</tbody>
</table>

4.6 Numerical Examples

Example 4.6.1: An automobile company desires to select the most appropriate sup-
plier for one of the key elements in its manufacturing process. After pre-evaluation,
five suppliers ($A_1$, $A_2$, $A_3$, $A_4$, $A_5$) have remained as alternatives for further evalua-
tion. In order to evaluate alternative suppliers, a committee of three decision makers
$DM_1$, $DM_2$ and $DM_3$ has been formed. Four criteria are considered as:
### Table 4.3: Linguistic Terms for Rating the Alternatives

<table>
<thead>
<tr>
<th>Linguistic terms</th>
<th>IFNs</th>
<th>IVIFNs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extremely Good (EG)/Extremely High (EH)</td>
<td>(0.95, 0.05)</td>
<td>([0.90, 0.95], [0.00, 0.05])</td>
</tr>
<tr>
<td>Very Very Good (VVG)/Very Very High (VVH)</td>
<td>(0.85, 0.10)</td>
<td>([0.85, 0.90], [0.05, 0.10])</td>
</tr>
<tr>
<td>Very good (VG)/Very High (VH)</td>
<td>(0.80, 0.15)</td>
<td>([0.80, 0.85], [0.10, 0.15])</td>
</tr>
<tr>
<td>Good (G)/High (H)</td>
<td>(0.75, 0.20)</td>
<td>([0.75, 0.80], [0.15, 0.20])</td>
</tr>
<tr>
<td>Medium Good (MG)/Medium High (MH)</td>
<td>(0.60, 0.25)</td>
<td>([0.60, 0.65], [0.20, 0.25])</td>
</tr>
<tr>
<td>Fair (F)/Medium (M)</td>
<td>(0.50, 0.35)</td>
<td>([0.50, 0.55], [0.30, 0.35])</td>
</tr>
<tr>
<td>Medium Poor (MP)/Medium Low (ML)</td>
<td>(0.40, 0.55)</td>
<td>([0.40, 0.45], [0.30, 0.55])</td>
</tr>
<tr>
<td>Poor (P)/Low (L)</td>
<td>(0.30, 0.65)</td>
<td>([0.30, 0.35], [0.60, 0.65])</td>
</tr>
<tr>
<td>Very Poor (VP)/Very Low (VL)</td>
<td>(0.20, 0.75)</td>
<td>([0.20, 0.25], [0.70, 0.75])</td>
</tr>
<tr>
<td>Very Very Poor (VVP)/Very Very Low (VVL)</td>
<td>(0.10, 0.85)</td>
<td>([0.10, 0.15], [0.80, 0.85])</td>
</tr>
</tbody>
</table>

- \( X_1 \): Product quality.
- \( X_2 \): Relationship closeness.
- \( X_3 \): Delivery performance.
- \( X_4 \): Price.

The proposed method is currently applied to solve this problem and the computational procedure is as follows:

Importance degree of the decision makers on group decision is shown in Table 4.1. Linguistic terms used for the ratings of the decision makers and criteria are given in Table 4.2. In order to obtain the weights of the decision makers, equation (4.5.1) is utilized:

\[
\lambda_{DM1} = 0.393, \lambda_{DM2} = 0.372, \lambda_{DM3} = 0.236.
\]

Now the aggregated interval-valued intuitionistic fuzzy decision matrix based on the opinions of decision makers is constructed using IIFWA operator. The linguistic terms shown in Table 4.3 are used to rate each alternative supplier with respect to each criterion by three decision makers. The ratings given by the decision makers to five alternatives is shown in Table 4.4.
Table 4.4: Rating of the Alternatives

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Suppliers</th>
<th>Decisions makers</th>
<th>Criteria</th>
<th>Suppliers</th>
<th>Decisions makers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>DM₁</td>
<td>DM₂</td>
<td>DM₃</td>
<td></td>
</tr>
<tr>
<td>X₁</td>
<td>A₁</td>
<td>G</td>
<td>G</td>
<td>G</td>
<td>X₃</td>
</tr>
<tr>
<td></td>
<td>A₂</td>
<td>MG</td>
<td>G</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A₃</td>
<td>VVG</td>
<td>VG</td>
<td>VG</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A₄</td>
<td>MG</td>
<td>G</td>
<td>G</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A₅</td>
<td>F</td>
<td>MG</td>
<td>MG</td>
<td></td>
</tr>
<tr>
<td>X₂</td>
<td>A₁</td>
<td>MG</td>
<td>G</td>
<td>MG</td>
<td>X₄</td>
</tr>
<tr>
<td></td>
<td>A₂</td>
<td>F</td>
<td>MG</td>
<td>G</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A₃</td>
<td>VG</td>
<td>G</td>
<td>VG</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A₄</td>
<td>F</td>
<td>F</td>
<td>MG</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A₅</td>
<td>MP</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.5: Importance Weight of the Criteria

<table>
<thead>
<tr>
<th>Criteria</th>
<th>DM₁</th>
<th>DM₂</th>
<th>DM₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>X₁</td>
<td>VI</td>
<td>VI</td>
<td>I</td>
</tr>
<tr>
<td>X₂</td>
<td>I</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>X₃</td>
<td>I</td>
<td>I</td>
<td>M</td>
</tr>
<tr>
<td>X₄</td>
<td>M</td>
<td>I</td>
<td>M</td>
</tr>
</tbody>
</table>

The aggregated interval-valued intuitionistic fuzzy decision matrix based on aggregation of decision makers’ opinions is constructed as follows:

\[
D = \begin{bmatrix}
(0.750, 0.800), (0.150, 0.200) & (0.642, 0.694), (0.187, 0.237) & (0.790, 0.840), (0.110, 0.160) & (0.750, 0.800), (0.150, 0.200) \\
(0.611, 0.664), (0.217, 0.268) & (0.634, 0.687), (0.210, 0.262) & (0.668, 0.719), (0.178, 0.229) & (0.579, 0.629), (0.220, 0.270) \\
(0.822, 0.872), (0.076, 0.128) & (0.790, 0.840), (0.110, 0.160) & (0.783, 0.833), (0.116, 0.167) & (0.783, 0.833), (0.116, 0.167) \\
(0.790, 0.751), (0.168, 0.218) & (0.540, 0.591), (0.258, 0.369) & (0.771, 0.822), (0.128, 0.178) & (0.668, 0.719), (0.178, 0.229) \\
(0.564, 0.614), (0.234, 0.285) & (0.463, 0.514), (0.366, 0.418) & (0.705, 0.754), (0.167, 0.217) & (0.526, 0.576), (0.272, 0.323)
\end{bmatrix}
\]

The importance weights of the criteria provided by decision makers can be linguistic terms. These linguistic terms are represented as interval-valued intuitionistic fuzzy numbers in Table 4.5 and opinions of decision makers on criteria are aggregated using equation (4.5.2) to determine the aggregated weights of criteria. The interval-valued intuitionistic fuzzy weights of criteria after aggregation of opinions of decision makers
After the weights of the criteria and the rating of the alternatives has been determined, the aggregated weighted interval-valued intuitionistic fuzzy decision matrix is constructed utilizing equation (4.5.3) as follows:

\[
W = \begin{bmatrix}
(0.884, 0.936), [0.000, 0.065]) \\
(0.850, 0.900), [0.050, 0.100]) \\
(0.766, 0.825), [0.103, 0.167]) \\
(0.624, 0.685), [0.221, 0.288])
\end{bmatrix}
\]

The entropy of the \(j^{th}\) criterion \(X_j, j = 1, 2, \ldots, 4\) for the available alternatives can be obtained from entropy measure (4.4.5). The objectives weights of criteria based on entropy are \(w_1 = 0.359, w_2 = 0.230, w_3 = 0.303, w_4 = 0.108\).

Product quality, relationship closeness and delivery performance are benefit criteria \(J_1 = \{X_1, X_2, X_3\}\) and price is cost criteria \(J_2 = \{X_4\}\). Then interval-valued intuitionistic fuzzy positive-ideal solution and interval-valued intuitionistic fuzzy negative ideal solution are

\[
A^+ = \{(0.726, 0.816), [0.076, 0.184]), ([0.671, 0.756], [0.154, 0.244]), \}
\]

\[
(0.605, 0.693), [0.201, 0.301]), ([0.328, 0.395, [0.433, 0.518])\}
\]

and

\[
A^- = \{(0.498, 0.575), [0.234, 0.331]), ([0.394, 0.462], [0.398, 0.476]), \}
\]

\[
(0.511, 0.594), [0.263, 0.358]), ([0.489, 0.571, [0.311, 0.407])\}. \]

Similarity of each alternative with the IVIFPIS and IVIFNIS based on proposed weighted similarity measure (4.3.8) is calculated in Table 4.6.

Finally, using equation (4.5.4), the value of relative closeness of each alternative for the final ranking is shown in Table 4.7.

Thus, the preference order of alternatives is \(A_1, A_2, A_3, A_4\) and \(A_5\) according to decreasing order of \(C_i\) is

\[A_3 > A_1 > A_4 > A_2 > A_5.\]
Table 4.6: *Similarities with the IVIFPIS and IVIFNIS*

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>$S^+$</th>
<th>$S^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.873</td>
<td>0.772</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.769</td>
<td>0.883</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.966</td>
<td>0.711</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.818</td>
<td>0.818</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.722</td>
<td>0.953</td>
</tr>
</tbody>
</table>

Table 4.7: *Relative Closeness Coefficients*

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>$C_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.531</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.465</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.576</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.500</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.431</td>
</tr>
</tbody>
</table>

4.7 Conclusions

The proposed new similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on ‘NTV’ metric along with their weighted form are valid similarity measures. The new intuitionistic fuzzy entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets analogously obtained through the proposed similarity measures are also valid information measures. Further, a new algorithm for MCDM using the proposed weighted similarity measures in which the weights have been calculated using the proposed entropies, has been illustrated through a numerical example.