Chapter 4

Sample Autocorrelation Function
and Sample Partial
Autocorrelation Function in the
presence of a Doublet Outlier and
a Patch of Additive Outliers

4.1 Introduction

In this chapter, we discuss the limiting behaviour of Sample Autocorrelation Function (SACF) and Sample Partial Autocorrelation Function (SPACF) of the time series in the presence of a Doublet Outlier (DO), i.e., as δ → ∞ and further look into the consequences of it.

Chan(1995) [11] discusses the effect of Additive Outlier (AO), Innovational Outlier (IO), Level Shift (LS) and Temporary Change (TC) on the SACF.

Suppose in the given time series \{X_t\}, there is an incidence of the above said outliers
with magnitude $\delta$ at time point $t = T$. Then, as $\delta \to \infty$, the $k^{th}$ order SACF has been studied by Chan(1995) [11]. This limiting behaviour of the SACF under different outlier models is summarised in the following table:

Table 4.1: Limiting behaviour of the SACF under different outlier models

<table>
<thead>
<tr>
<th>Model</th>
<th>$\lim_{\delta \to \infty} \rho_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AO</td>
<td>0</td>
</tr>
<tr>
<td>IO</td>
<td>$\rho(k)$</td>
</tr>
<tr>
<td>LC</td>
<td>$1 - \frac{k}{T-1}$</td>
</tr>
<tr>
<td>TC</td>
<td>$\omega^k$</td>
</tr>
</tbody>
</table>

Further, Maronna et al.,(2006) [30] have discussed the effect of a Patch of AO’s on the first order SACF. Suppose in the given time series $\{X_t\}$, there are $m$ AO’s of magnitude $\delta$ at time points, $t = T, T+1, \ldots, T+m-1$. Then, as $\delta \to \infty$, first order SACF is given by,

$$\lim_{\delta \to \infty} \hat{\rho}_1 = \frac{m-1}{m}.$$  

Along the lines of Chan(1995) [11], the effect of a DO on the SACF and SPACF has been worked out and further the consequences of this limiting behaviour on the method of moments estimates of the parameters in the models have been studied. Further, keeping Maronna et al.,(2006) [30] in the background, the limiting behaviour of the SACF under a patch of say $m$ Additive Outliers for higher orders have been studied. We have also extended this study on the SPACF as well in this chapter.

### 4.2 Sample Autocorrelation Function(SACF) of a time series in the presence of a Doublet Outlier

Let $\{X_t; 1, 2, \ldots, n\}$ be any (uniformly bounded) time series data. Let $\{Y_t; 1, 2, \ldots, n\}$ be contaminated data obtained by superimposing a DO of magnitude of $\delta$ at $t = T$
on \( \{X_t\} \). The following theorem provides the limiting behaviour of the SACF of \( \{Y_t; 1, 2, \ldots, n\} \) as \( \delta \) tends to infinity.

**Theorem 1.** Let us denote the SACF by \( \hat{\rho}_k \), \( k \geq 1 \). Then for a fixed \( n \)

\[
\lim_{\delta \to \infty} \hat{\rho}_k = \begin{cases} 
-\frac{1}{2}, & k = 1, \\
0, & k \geq 2.
\end{cases}
\]

**Proof:** By definition, when \( k = 1 \),

\[
\hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} Y_t Y_{t+1}}{\sum_{t=1}^{n} Y_t^2} \quad (4.1)
\]

Substituting \( \{Y_t; 1, 2, \ldots, n\} \) in terms of \( \{X_t; 1, 2, \ldots, n\} \), we get

\[
\hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} X_t X_{t+1}I(t \notin \{T - 1, T, T + 1\})}{\sum_{t=1}^{n} X_t^2I(t \notin \{T, T + 1\}) + (X_T + \delta)^2 + (X_{T+1} - \delta)^2} + \frac{\sum_{t=T}^{T+1} X_t X_{t+1} + \delta(X_{T-1} - X_T + X_{T+1} - X_{T+2})}{\sum_{t=1}^{n} X_t^2I(t \notin \{T, T + 1\}) + (X_T + \delta)^2 + (X_{T+1} - \delta)^2} \quad (4.2)
\]

On simplifying the above equation, we get

\[
\hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} X_t X_{t+1} + \delta(X_{T-1} - X_T + X_{T+1} - X_{T+2}) - \delta^2}{\sum_{t=1}^{n} X_t^2 + 2\delta^2 + 2\delta(X_T - X_{T+1})} \quad (4.3)
\]

Now, dividing both numerator and denominator of equation (4.3) by \( \delta^2 \), we get

\[
\hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} \frac{X_t X_{t+1}}{\delta^2} + \frac{\delta(X_{T-1} - X_T + X_{T+1} - X_{T+2})}{\delta^2} - \frac{\delta^2}{\delta^2}}{\sum_{t=1}^{n} \frac{X_t^2}{\delta^2} + \frac{2\delta^2}{\delta^2} + \frac{2\delta(X_T - X_{T+1})}{\delta^2}} \quad (4.4)
\]

100
On simplification, we get

\[
\hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} X_t X_{t+1}}{\delta^2} + \frac{(X_{T-1} - X_T + X_{T+1} - X_{T+2})}{\delta} \left( \frac{\sum_{t=1}^{n} X_t^2}{\delta^2} + 2 + \frac{2(X_T - X_{T+1})}{\delta} \right) - 1 \quad (4.5)
\]

Taking \(\lim_{\delta \to \infty}\) for equation (4.5), for a fixed \(n\), we get

\[
\lim_{\delta \to \infty} \hat{\rho}_1 = \lim_{\delta \to \infty} \frac{\sum_{t=1}^{n-1} X_t X_{t+1}}{\delta^2} + \frac{(X_{T-1} - X_T + X_{T+1} - X_{T+2})}{\delta} \left( \frac{\sum_{t=1}^{n} X_t^2}{\delta^2} + 2 + \frac{2(X_T - X_{T+1})}{\delta} \right) - 1
\]

\[
= \lim_{\delta \to \infty} \frac{\sum_{t=1}^{n-1} X_t X_{t+1}}{\delta^2} + \lim_{\delta \to \infty} \frac{(X_{T-1} - X_T + X_{T+1} - X_{T+2})}{\delta} \left( \frac{\sum_{t=1}^{n} X_t^2}{\delta^2} + 2 + \frac{2(X_T - X_{T+1})}{\delta} \right) - 1
\]

\[
= 0 + 0 - 1
\]

\[
= 0 + 2 + 0
\]

Hence,

\[
\lim_{\delta \to \infty} \hat{\rho}_1 = -\frac{1}{2} \quad (4.6)
\]

Similarly by definition, when \(k \geq 2\),

\[
\hat{\rho}_k = \frac{\sum_{t=1}^{n-k} Y_t Y_{t+k}}{\sum_{t=1}^{n} Y_t^2} \quad (4.7)
\]

Substituting \(\{Y_t; 1, 2, \ldots, n\}\) in terms of \(\{X_t; 1, 2, \ldots, n\}\), it is easy to check that for \(k \geq 2\),

\[
\hat{\rho}_k = \frac{\sum_{t=1}^{n-k} X_t X_{t+k} + \delta(X_{T-k} - X_{T-(k-1)} + X_{T+k} - X_{T+(k+1)})}{\sum_{t=1}^{n} X_t^2 + 2\delta^2 + 2\delta(X_T - X_{T+1})}, \quad k \geq 2 \quad (4.8)
\]
As before, dividing both numerator and denominator of equation (4.8) by \(\delta^2\), and taking \(\lim_{\delta \to \infty}\), we get

\[
\lim_{\delta \to \infty} \hat{\rho}_k = \lim_{\delta \to \infty} \frac{\sum_{t=1}^{n-k} X_t X_{t+k}}{\delta^2} + \frac{(X_{T-k} - X_{T-(k-1)} + X_{T+k} - X_{T+(k+1)})}{\delta^2} + 2 \frac{(X_T - X_{T+1})}{\delta} \frac{\sum_{t=1}^{n} X_t^2}{\delta^2} + 2 \frac{(X_T - X_{T+1})}{\delta} \frac{\sum_{t=1}^{n} X_t^2}{\delta^2} + 2 \lim_{\delta \to \infty} \frac{2(X_T - X_{T+1})}{\delta}
\]

\[
= \frac{0 + 0}{0 + 2 + 0}, \quad k \geq 2
\]

Therefore,

\[
\lim_{\delta \to \infty} \hat{\rho}_k = 0, \quad k \geq 2
\] (4.9)

The equations (4.6) and (4.9) prove the theorem. \(\Box\)

**Remark:** While taking limit as \(\delta \to \infty\) in Theorem-1, it is necessary to assume that \(\{X_t\}\) are uniformly bounded in \(t\). Hence the theorem may not hold for explosive time series for large \(n\) (ARMA\((p,q)\) models with \(\phi(z) = 0\) having roots within the unit circle).

### 4.3 Sample Partial Autocorrelation Function (SPACF)

of a time series in the presence of a Doublet Outlier

The following theorem provides the limiting behaviour of the SPACF \(\hat{\phi}_{kk}\) of \(\{Y_t; 1, 2, \ldots, n\}\) as \(\delta\) tends to infinity.
Theorem 2. For any $k \geq 1$,

$$\lim_{\delta \to \infty} \hat{\phi}_{kk} = -\frac{1}{k + 1}$$

Proof: By definition, for any $k \geq 1$, the Sample Partial Autocorrelation coefficients of a time series is given by a ratio of two determinants.

$$\hat{\phi}_{kk} = \frac{\begin{vmatrix} 1 & \hat{\rho}_1 & \hat{\rho}_2 & \hat{\rho}_3 & \cdots & \hat{\rho}_{k-2} & \hat{\rho}_{k-1} \\ \hat{\rho}_1 & 1 & \hat{\rho}_1 & \hat{\rho}_2 & \cdots & \hat{\rho}_{k-3} & \hat{\rho}_2 \\ \hat{\rho}_2 & \hat{\rho}_1 & 1 & \hat{\rho}_1 & \cdots & \hat{\rho}_{k-4} & \hat{\rho}_3 \\ \hat{\rho}_3 & \hat{\rho}_2 & \hat{\rho}_1 & 1 & \cdots & \hat{\rho}_{k-5} & \hat{\rho}_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{\rho}_{k-2} & \hat{\rho}_{k-3} & \hat{\rho}_{k-4} & \hat{\rho}_{k-5} & \cdots & 1 & \hat{\rho}_{k-1} \\ \hat{\rho}_{k-1} & \hat{\rho}_{k-2} & \hat{\rho}_{k-3} & \hat{\rho}_{k-4} & \cdots & \hat{\rho}_1 & \hat{\rho}_k \end{vmatrix}}{\begin{vmatrix} 1 & \hat{\rho}_1 & \hat{\rho}_2 & \hat{\rho}_3 & \cdots & \hat{\rho}_{k-2} & \hat{\rho}_{k-1} \\ \hat{\rho}_1 & 1 & \hat{\rho}_1 & \hat{\rho}_2 & \cdots & \hat{\rho}_{k-3} & \hat{\rho}_2 \\ \hat{\rho}_2 & \hat{\rho}_1 & 1 & \hat{\rho}_1 & \cdots & \hat{\rho}_{k-4} & \hat{\rho}_3 \\ \hat{\rho}_3 & \hat{\rho}_2 & \hat{\rho}_1 & 1 & \cdots & \hat{\rho}_{k-5} & \hat{\rho}_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{\rho}_{k-2} & \hat{\rho}_{k-3} & \hat{\rho}_{k-4} & \hat{\rho}_{k-5} & \cdots & 1 & \hat{\rho}_{k-1} \\ \hat{\rho}_{k-1} & \hat{\rho}_{k-2} & \hat{\rho}_{k-3} & \hat{\rho}_{k-4} & \cdots & \hat{\rho}_1 & \hat{\rho}_k \end{vmatrix}}$$

(4.10)
Keeping \( n \) fixed and taking \( \lim_{\delta \to \infty} \) for equation (4.10), we have by virtue of Theorem 3

\[
\lim_{\delta \to \infty} \hat{\phi}_{kk} = \begin{vmatrix}
1 & -1/2 & 0 & 0 & \cdots & 0 & -1/2 \\
-1/2 & 1 & -1/2 & 0 & \cdots & 0 & 0 \\
0 & -1/2 & 1 & -1/2 & \cdots & 0 & 0 \\
0 & 0 & -1/2 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 1 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & -1/2 & 0
\end{vmatrix}
\]

(4.11)

Consider the Numerator \( (N_k) \) of (4.11), the solution for this determinant is

\[
N_k = (-1)^{k+1}(-1/2)(-1/2)^{k-1}
\]

\[
= (-1)^{k+1}(-1/2)^k
\]

\[
= (-1)^{k+1}(-1)^k(1/2)^k
\]

\[
= (-1)^{2k+1}(1/2)^k
\]

\[
N_k = -(1/2)^k
\]

(4.12)
Next consider the Denominator \(D_k\) of (4.11), the solution for this determinant is

\[
D_k = 1(D_{k-1}) - (-1/2)(-1/2)(D_{k-2}) \\
D_k = D_{k-1} + (-1/2)^2D_{k-2}
\]

Rewriting the above, we get the difference equation

\[
D_k - D_{k-1} + \frac{1}{4}D_{k-2} = 0
\]  \hspace{1cm} (4.13)

Replacing \(k\) by \(k + 2\), we get

\[
D_{k+2} - D_{k+1} + \frac{1}{4}D_k = 0
\]  \hspace{1cm} (4.14)

Solving the above, we get

\[
D_k = \frac{+1 \pm \sqrt{1 - 4(1/4)}}{2} \\
D_k = 1/2
\]

Note: If the difference equation \(X_{t+2} + aX_{t+1} + bX_t = 0\) has roots both equal to say \(\mu\), then the complete solution for \(X_t\) is given by

\[
X_t = (t + 1)\mu^t \hspace{1cm} \text{forall} \hspace{0.5cm} t \geq 1
\]  \hspace{1cm} (4.15)

Hence

\[
D_k = (k + 1)(1/2)^k
\]  \hspace{1cm} (4.16)

Plugging (4.12) and (4.16) in (4.11), the proof is completed on simplification. \(\square\)

**Remark:** The Remark, made under Theorem-1 seems to be relevant for Theorem-2 also.
4.4 Simulation study to demonstrate the limiting behaviour of the SACF and SPACF in the presence of a DO

Since Theorems-1 and 2 hold for any uniformly bounded time series data, a simulation study was carried out to examine and demonstrate the limiting behaviour of the SACF and SPACF in the presence of a DO in the series with a high magnitude, for ARMA(1,1), AR(1) and MA(1) models which are known to be stationary. The plots of SACF and SPACF are as given below in figures:(4.1 - 4.4) for ARMA(1,1) model, figures:(4.5 - 4.6) for AR(1) model and figures:(4.7 - 4.8) for MA(1) model.

4.4.1 Plots of SACF and SPACF of ARMA(1,1) model

From the figures:(4.1 - 4.4), it is clear that whatever might be the values of the parameters $\phi$ and $\theta$, the occurrence of a DO in the series with a high magnitude, will disturb the Box-Jenkins model identification methodology. The limiting behaviour of the SACF and SPACF in relation to the different values of the parameters $\phi$ and $\theta$ becomes constant towing in line with Theorems-1 and 2, In fact for large $\delta$ and for moderately large $n$

1. The SACF is finite and Cuts off after lag 1 as against Infinite (damped exponentials and/or damped sine waves after first q-p lags), Tails off.

2. The SPACF is decaying exponentially as against the Infinite (dominated by damped exponentials and/or sine waves after first p-q lags), Tails off.

Consequently, in the presence of a DO with high magnitude, if we rely simply on the SACF and SPACF and try to identify the model, we will be fitting a MA(1) model instead of an ARMA(1,1) model, thus disturbing the identification step in Box-Jenkins methodology.
Figure 4.1: $\hat{\rho}_k$ and $\hat{\phi}_{kk}$ plots of $ARMA(1,1)$ in the Presence of a DO-A
Figure 4.2: $\hat{\rho}_k$ and $\hat{\phi}_{kk}$ plots of ARMA(1, 1) in the Presence of a DO-B
Figure 4.3: $\hat{\rho}_k$ and $\hat{\phi}_{kk}$ plots of ARMA(1, 1) in the Presence of a DO-C
Figure 4.4: $\hat{\rho}_k$ and $\hat{\phi}_{kk}$ plots of $ARMA(1, 1)$ in the Presence of a DO-D
4.4.2 Plots of SACF and SPACF of AR(1) model

From figures: 4.5 and 4.6, it’s clear that whenever there a DO with high magnitude, irrespective of the value of the autoregressive parameter \( \phi \) the patterns of SACF and SPACF across \( \phi \) are the same.

The SACF is finite and cuts off after lag 1 as against Infinite (damped exponentials and/or damped sine waves) and should tail off.

The SPACF is decaying exponentially as against finite and should cut off at lag 1.

Hence, if we rely on the SACF and SPACF for identification of the model, we will be fitting an MA(1) model instead of the actual AR(1) model.
Figure 4.5: $\hat{\rho}_k$ and $\hat{\phi}_{kk}$ plots of AR(1) in the Presence of a DO-A
Figure 4.6: $\hat{\rho}_k$ and $\hat{\phi}_{kk}$ plots of AR(1) in the Presence of a DO-B
4.4.3 Plots of SACF and SPACF of MA(1) model

From figures 4.7 and 4.8, it's clear that whenever there a DO with high magnitude, irrespective of the value of the moving average parameter $\theta$ the patterns of SACF and SPACF across $\theta$ are the same.

The SACF is finite and cuts off after lag 1 as against finite and should cut off at lag 1.

The SPACF is decaying exponentially as against Infinite (damped exponentials and/or damped sine waves) and should tail off.

Of course, the identification of the model in this case is not a problem, but it is the value of the parameter $\theta$ that is going to be in a serious problem. This aspect has been explored in the next section.
Figure 4.7: $\hat{\rho}_k$ and $\hat{\phi}_{kk}$ plots of MA(1) in the Presence of a DO-A
Figure 4.8: $\hat{\rho}_k$ and $\hat{\phi}_{kk}$ plots of AR(1) in the Presence of a DO-B
4.5 SACF in relation with sample size $n$, model parameters and magnitude of DO

The limiting behaviour of SACF, as reflected by Theorem-1 and the simulation thereon indicate that the limiting value as $\delta \to \infty$ depends on the model parameters and also on the fixed value of sample size $n$. As an academic curiosity a simulation study is carried out here to determine the least value of $\delta$ that reflects the limiting value of SACF across the coefficients of ARMA(1,1) and its particular cases. Tables:(4.2-4.4) and figures:(4.9-4.13) summarize the findings across the various values of sample size $n$ and the parameters of the underlying model.

4.5.1 Simulation study results of ARMA(1,1) model

Table 4.2: Minimum $\delta$ to reflect Theorem-1 as a function of $n$, $\phi$ and $\theta$

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$n \downarrow \theta \rightarrow$</th>
<th>-0.90</th>
<th>-0.60</th>
<th>-0.30</th>
<th>0.00</th>
<th>0.30</th>
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From table-4.2, it is clear that the least value of $\delta$ for attaining the limiting values of SACF given in Theorem-1 increases as sample size and/or $|\phi|, |\theta|$ increases.

### 4.5.2 Simulation study results of AR(1) model

Table 4.3: Minimum $\delta$ to reflect Theorem-1 as a function of $n$ and $\phi$

<table>
<thead>
<tr>
<th>$n \downarrow \phi \rightarrow$</th>
<th>-0.90</th>
<th>-0.60</th>
<th>-0.30</th>
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</tbody>
</table>

From table-4.3, it is clear that the least value of $\delta$ for attaining the limiting values of SACF given in Theorem-1 increases as sample size and/or $|\phi|$ increases. Further, to reflect Theorem-1, for positive values of $\phi$ the magnitude of $\delta$ required is relatively greater than that it is required when $\phi$ is negative.

### 4.5.3 Simulation study results of MA(1) model

Table 4.4: Minimum $\delta$ to reflect Theorem-1 as a function of $n$ and $\theta$

<table>
<thead>
<tr>
<th>$n \downarrow \theta \rightarrow$</th>
<th>-0.90</th>
<th>-0.60</th>
<th>-0.30</th>
<th>0.00</th>
<th>0.30</th>
<th>0.60</th>
<th>0.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>60</td>
<td>50</td>
<td>80</td>
<td>90</td>
<td>100</td>
<td>120</td>
<td>150</td>
</tr>
<tr>
<td>500</td>
<td>90</td>
<td>70</td>
<td>100</td>
<td>120</td>
<td>250</td>
<td>300</td>
<td>350</td>
</tr>
<tr>
<td>750</td>
<td>150</td>
<td>100</td>
<td>150</td>
<td>200</td>
<td>300</td>
<td>380</td>
<td>450</td>
</tr>
<tr>
<td>1000</td>
<td>200</td>
<td>130</td>
<td>200</td>
<td>250</td>
<td>350</td>
<td>400</td>
<td>500</td>
</tr>
</tbody>
</table>

From table-4.4, it is clear that the least value of $\delta$ for attaining the limiting values of SACF given in Theorem-1 increases as sample size and/or $|\theta|$ increases. Further, to reflect Theorem-1, for negative values of $\theta$ the magnitude of $\delta$ required is relatively lesser than that it is required when $\theta$ is positive.
Figure 4.9: $\phi_1$ across $\delta$ in ARMA(1,1)-A

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Figure 4.10: $\hat{\rho}_1$ across $\delta$ in ARMA(1,1)-B
Figure 4.11: $\hat{\rho}_1$ across $\delta$ in ARMA(1,1)-C
Figure 4.12: $\hat{\rho}_1$ across $\delta$ in $AR(1)$
Figure 4.13: $\hat{\rho}_1$ across $\delta$ in $MA(1)$
4.5.4 Conclusions drawn from the tables and graphs

The following conclusions can be drawn from the tables: (4.2 - 4.4) and figures: (4.9 - 4.13):

- For ARMA(1,1) case: For the case of ARMA(1,1) model, as $|\phi|$ and $|\theta|$ are equal and nearer to one, small $\delta$ is sufficient. When $\phi$ and $\theta$ are equal but have opposite sign, large $\delta$ is required than in the previous case. Further when $|\phi|$ and $|\theta|$ are equal and nearer to zero, very small $\delta$ is sufficient.

- For AR(1) case: As $n$ increases, the $\delta$ also increases for a fixed parameter value $\phi$ in the AR(1) model (refer to figure-4.12). When $\phi$ and $n$ are large, $\delta$ also increases. For a given sample size $n$, as $|\phi|$ moves away from zero, the minimum value of $\delta$ required for the SACF to attain its limiting value will increase and it further increases as $n$ increases.

- For MA(1) case: The conclusions are similar to that of AR(1) model (refer to figure-4.13). For a fixed sample size $n$, the minimum $\delta$ required for SACF to attain its reflect the limiting value over the range of $\theta$ is much less than that required for an AR(1) model over the range $\phi$.

Hence, from the above observation of the simulation study, there is interaction between the coefficients of the respective models, sample size $n$ and the magnitude of delta required for the limiting behaviour of the SACF to exist. When the parameter values are close to 1(0), we need a $\delta$ with large/small magnitude for a fixed sample size $n$. As $n$ increases, $\delta$ will also increase. We see that the minimum value of $\delta$ required for $\hat{\rho}_k$ to reach its limiting value (as $\delta \to \infty$) is influenced by the model and in turn the parameters $\phi$ and/or $\theta$ and the sample size $n$. But this influence is not explicitly seen in the expression of $\hat{\rho}_k$. Hence, this simulation study is of importance in revealing such interactions.
Note: Very similar behaviour was observed in a simulation study carried out for the case of a single AO.

4.6 Consequences of the limiting behaviour of the SACF

In this section, the consequences of the limiting behaviour of the SACF in the presence of a Doublet Outlier on the Moment Estimates of the parameters in AR(1), MA(1) and ARMA(1, 1) models have been explored.

4.6.1 Consequences of the limiting behaviour of the SACF in the AR(1) model

Consider the simple AR(1) Model

\[ X_t = \phi X_{t-1} + a_t. \]

Suppose that we intend to use a moment estimator for \( \phi \), then

\[ \hat{\rho}_1 = \phi \]

and

\[ \hat{\rho}_1 = \hat{\phi}. \]

Thus,

\[ \hat{\phi} = \hat{\rho}_1. \] (4.17)
Suppose we have a doublet outlier with magnitude $\delta$, then as $\lim_{\delta \to \infty}$,

$$
\lim_{\delta \to \infty} \hat{\phi} = \lim_{\delta \to \infty} r_1 = \lim_{\delta \to \infty} \hat{\rho}_1
$$

$$
\lim_{\delta \to \infty} \hat{\phi} = -1/2, \quad \text{from equation (4.6).} \quad (4.18)
$$

Hence, whatever might be the value of $\phi$, if there exists a doublet outlier and $\delta$ is very large, the moment estimate for $\phi$ will be equal to $-1/2$.

**Simulation study results of Method of Moments Estimates - AR(1) Case**

In order to demonstrate the limiting behaviour of the moment estimate of $\phi$ derived above for AR(1) model, a simulation study was carried out for $n = 200$ across various values of $\phi$ and $\delta$ and the results are presented in table-4.5.

<table>
<thead>
<tr>
<th>$\phi \downarrow \delta \rightarrow$</th>
<th>0</th>
<th>25</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.99</td>
<td>-0.97</td>
<td>-0.82</td>
<td>-0.62</td>
<td>-0.52</td>
<td>-0.49</td>
<td>-0.49</td>
<td>-0.50</td>
</tr>
<tr>
<td>-0.70</td>
<td>-0.70</td>
<td>-0.57</td>
<td>-0.53</td>
<td>-0.51</td>
<td>-0.51</td>
<td>-0.50</td>
<td>-0.50</td>
</tr>
<tr>
<td>-0.35</td>
<td>-0.38</td>
<td>-0.51</td>
<td>-0.51</td>
<td>-0.51</td>
<td>-0.50</td>
<td>-0.50</td>
<td>-0.50</td>
</tr>
<tr>
<td>0.35</td>
<td>0.31</td>
<td>-0.41</td>
<td>-0.48</td>
<td>-0.50</td>
<td>-0.50</td>
<td>-0.50</td>
<td>-0.50</td>
</tr>
<tr>
<td>0.70</td>
<td>0.69</td>
<td>-0.26</td>
<td>-0.43</td>
<td>-0.48</td>
<td>-0.50</td>
<td>-0.50</td>
<td>-0.50</td>
</tr>
<tr>
<td>0.99</td>
<td>0.97</td>
<td>0.53</td>
<td>-0.05</td>
<td>-0.31</td>
<td>-0.44</td>
<td>-0.49</td>
<td>-0.50</td>
</tr>
</tbody>
</table>

It is evident from the above table that, what ever be $\phi$, the moment estimate attains its limiting value as $\delta$ increases. Interestingly, the magnitude of $\delta$ for reflecting the limiting behaviour increases as $\phi$ increases to 1. However for negative values of $\phi$ requires relatively lesser magnitude of $\delta$ for the moment of estimate to attain its limiting value.
4.6.2 Consequences of the limiting behaviour of the SACF in MA(1) Model

Consider a simple MA(1) model

\[ X_t = a_t - \theta a_{t-1}. \numberedtag{4.19} \]

Suppose we intend to use a moment estimator for \( \theta \), then

\[ \rho_1 = -\frac{\theta}{1 + \theta^2} \]

and

\[ \hat{\rho}_1 = -\frac{\hat{\theta}}{1 + \hat{\theta}^2}. \]

Thus,

\[ \hat{\theta} = \frac{-1 \pm \sqrt{1 - 4 \hat{\rho}_1^2}}{2 \hat{\rho}_1}. \]

Substituting the limiting value of \( \hat{\rho}_1 \) in the presence of a DO with magnitude \( \delta \)

\[ \lim_{\delta \to \infty} \hat{\theta} = \frac{-1 \pm \sqrt{1 - 4(-1/2)^2}}{2(-1/2)}. \]

Hence,

\[ \lim_{\delta \to \infty} \hat{\theta} = 1. \numberedtag{4.20} \]

Thus, for any MA(1) model of the type (4.19), in the presence of a DO with a large \( \delta \), the moment estimate of \( \theta \) will be approximately 1.

**Simulation study results of Method of Moments Estimates - MA(1) Case**

In order to demonstrate the limiting behaviour of the moment estimate of \( \theta \) derived above for MA(1) model (4.19), a simulation study was carried out for \( n = 200 \) across various values of \( \theta \) and \( \delta \) and the results are presented in table-4.6.
Table 4.6: MA(1)-Method of Moments Parameter Estimates (n=200)

<table>
<thead>
<tr>
<th>$\theta \downarrow \delta \rightarrow$</th>
<th>0</th>
<th>25</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.99</td>
<td>-0.70</td>
<td>0.33</td>
<td>0.57</td>
<td>0.70</td>
<td>0.82</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>-0.70</td>
<td>-0.60</td>
<td>0.41</td>
<td>0.60</td>
<td>0.70</td>
<td>0.82</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>-0.35</td>
<td>-0.33</td>
<td>0.50</td>
<td>0.66</td>
<td>0.75</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.35</td>
<td>0.28</td>
<td>0.70</td>
<td>0.75</td>
<td>0.82</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.70</td>
<td>0.52</td>
<td>0.75</td>
<td>0.82</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.99</td>
<td>0.60</td>
<td>0.82</td>
<td>0.82</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

It is evident from the table that, whatever be $\theta$, the moment estimate attains its limiting value as $\delta$ increases. Interestingly, the magnitude of $\delta$ for reflecting the limiting behaviour decreases when $\theta$ increases to 1. However, for negative values of $\theta$, it requires relatively greater magnitude of $\delta$ for the moment of estimate to attain its limiting value.

4.6.3 Consequences of the limiting behaviour of the SACF in ARMA(1,1) model

Consider a simple ARMA(1,1) model

$$X_t - \phi X_{t-1} = a_t - \theta a_{t-1}. \quad (4.21)$$

Suppose we intend to use moment estimators for $\phi$ and $\theta$, then we first estimate $\phi$ where

$$\hat{\phi} = \frac{\hat{\rho}_2}{\hat{\rho}_1} \quad (4.22)$$

Having done so, we can then use the below to solve for $\hat{\theta}$.

$$\hat{\rho}_1 = \frac{(1 - \hat{\theta}\hat{\phi})(\hat{\phi} - \hat{\theta})}{(1 - 2\hat{\theta}\hat{\phi} + \hat{\theta}^2)}$$

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Suppose there exists a doublet outlier with magnitude $\delta$, then as $\lim_{\delta \to \infty}$,

$$
\lim_{\delta \to \infty} \hat{\phi} = \lim_{\delta \to \infty} \frac{\hat{\rho}_2}{\hat{\rho}_1} = 0\frac{(-1/2)}{(-1/2)} = 0.
$$

Therefore, if $\theta^* = \lim_{\delta \to \infty} \hat{\theta}$, using (4.23) we have

$$
\lim_{\delta \to \infty} \hat{\theta}_1 = \lim_{\delta \to \infty} \frac{(1 - \theta^* \hat{\phi})(\hat{\phi} - \theta^*)}{(1 - 2\theta^* \hat{\phi} + (\theta^*)^2)} = \frac{(1 - \theta^*(0))(0 - \theta^*)}{(1 - 2\theta^*(0) + (\theta^*)^2)}.
$$

Solving (4.24) for $\theta^*$, we get

$$
\lim_{\delta \to \infty} \hat{\theta} = 1.
$$

From (4.23) and (4.25), we can conclude that, whatever be the value of $\phi$ and $\theta$, in the presence of a DO with large $\delta$, the moment estimates of $\phi$ and $\theta$ will be equal to 0 and 1 respectively.

**Simulation study results of Method of Moments Estimates for ARMA(1,1)**

In order to demonstrate the limiting behaviour of the moment estimates of $\phi$ and $\theta$ derived above for ARMA(1,1) model (4.21), a simulation study was carried out for $n = 200$ across various values of $(\phi, \theta)$ and $\delta$ and the results are presented in table-4.7.

It is evident from the table that, whatever be the values of $(\phi$ and $\theta)$, their moment estimates attain limiting values as $\delta$ increases. The table also reveals an interaction between $\phi$ and $\theta$ for the attainment of the limiting value of the moments estimates. For a better understanding of this interaction one may require an intensive simulation study across all values of $\phi$ and $\theta$ and for different values of $n$. Interestingly the limiting behaviour of $\phi$ alone requires relatively a lesser value of $\delta$.
### Table 4.7: ARMA(1,1)-Method of Moments Parameter Estimates (n=200)

<table>
<thead>
<tr>
<th>$(\phi, \theta) \downarrow \delta \rightarrow$</th>
<th>Method of Moments Estimates of $\phi$ and $\theta$</th>
<th>0</th>
<th>200</th>
<th>700</th>
<th>2000</th>
<th>3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-0.20, -0.30)</td>
<td>-0.33, -0.48</td>
<td>-0.01, 0.86</td>
<td>0.00, 0.98</td>
<td>0.00, 0.99</td>
<td>0.00, 1.00</td>
<td></td>
</tr>
<tr>
<td>(-0.20, -0.90)</td>
<td>-0.11, -0.77</td>
<td>-0.01, 0.91</td>
<td>0.00, 0.94</td>
<td>0.00, 0.96</td>
<td>0.00, 0.98</td>
<td></td>
</tr>
<tr>
<td>(-0.20, 0.40)</td>
<td>-0.18, -0.36</td>
<td>-0.02, 0.91</td>
<td>0.00, 0.95</td>
<td>0.00, 0.97</td>
<td>0.00, 0.98</td>
<td></td>
</tr>
<tr>
<td>(-0.20, 0.80)</td>
<td>-0.17, 0.84</td>
<td>0.01, 0.93</td>
<td>0.00, 0.94</td>
<td>0.00, 0.97</td>
<td>0.00, 0.97</td>
<td></td>
</tr>
<tr>
<td>(-0.90, -0.33)</td>
<td>-0.91, -0.31</td>
<td>-0.02, 0.88</td>
<td>-0.01, 0.91</td>
<td>0.00, 0.97</td>
<td>0.00, 0.98</td>
<td></td>
</tr>
<tr>
<td>(-0.90, -0.98)</td>
<td>-0.89, -0.88</td>
<td>-0.02, 0.85</td>
<td>0.00, 0.93</td>
<td>0.00, 0.97</td>
<td>0.00, 0.99</td>
<td></td>
</tr>
<tr>
<td>(0.40, -0.30)</td>
<td>0.32, -0.34</td>
<td>0.00, 0.83</td>
<td>0.00, 0.95</td>
<td>0.00, 0.97</td>
<td>0.00, 0.98</td>
<td></td>
</tr>
<tr>
<td>(0.40, 0.89)</td>
<td>0.35, 0.77</td>
<td>0.01, 0.97</td>
<td>0.00, 0.98</td>
<td>0.00, 0.98</td>
<td>0.00, 0.99</td>
<td></td>
</tr>
<tr>
<td>(0.80, -0.30)</td>
<td>0.78, -0.21</td>
<td>0.00, 0.77</td>
<td>0.00, 0.93</td>
<td>0.00, 0.98</td>
<td>0.00, 0.99</td>
<td></td>
</tr>
<tr>
<td>(0.80, -0.90)</td>
<td>0.80, -0.81</td>
<td>-0.05, 0.53</td>
<td>0.00, 0.87</td>
<td>0.00, 0.96</td>
<td>0.00, 0.98</td>
<td></td>
</tr>
<tr>
<td>(0.80, 0.95)</td>
<td>0.68, 0.86</td>
<td>-0.01, 0.88</td>
<td>0.00, 0.91</td>
<td>0.00, 0.96</td>
<td>0.00, 0.98</td>
<td></td>
</tr>
</tbody>
</table>

### 4.7 SACF and SPACF in the presence of a Patch of Additive Outliers

Maronna et al., (2006) [30] have discussed about the limiting behaviour of $\hat{\rho}_1$ in the presence of a Patch (say $m$) of Additive Outliers. Here, we extend the result to higher orders SACF and further explore the limiting behaviour of SPACF $\hat{\phi}_{kk}$.

#### 4.7.1 SACF in the presence of a Patch of Additive Outliers

It is assumed that an observed time series $X_t; t = 1, 2, \ldots, n$ is superimposed with a patch of $m$ additive outliers at time $t = T, T + 1, \ldots, T + m - 1$ of size $\delta$ and the contaminated series is given by

$$Y_t = X_t + \delta I_{[T, T+1, \ldots, T+m-1]}(t)$$

$t = 1, 2, \ldots, n$  \hspace{1cm} (4.26)
where

\[ I_t^{(T, T+1, \ldots, T+m-1)} = \begin{cases} 1, & t \in (T, T + 1, \ldots, T + m - 1), \\ 0, & \text{otherwise.} \end{cases} \quad (4.27) \]

This modified model (4.26) is known as Patch of Additive Outlier Model. Using (2.2) equation (4.26) can be written as

\[ Y_t = \frac{\theta(B)}{\phi(B)} a_t + \delta I_t^{(T, T+1, \ldots, T+m-1)}. \quad (4.28) \]

The following theorem provides the limiting behaviour of the SACF of \( \{Y_t; 1, 2, \ldots, n\} \) as \( \delta \) tends to infinity.

**Theorem 3.** Let us denote the SACF by \( \hat{\rho}_k, \, k \geq 1 \). Then for a fixed \( n \)

\[ \lim_{\delta \to \infty} \hat{\rho}_k = \begin{cases} \frac{m - k}{m} & ; k = 1, 2, 3, \ldots, (m - 1), \\ 0 & ; k \geq m. \end{cases} \]

**Proof:** By definition, in lieu of equation 4.26 we have in terms of \( \{X_t; 1, 2, \ldots, n\} \)

\[ \hat{\rho}_1 = \frac{\sum_{t=1}^{n-1} X_tX_{t+1} + \delta \left( \sum_{t=T-1}^{T+m-2} X_t + \sum_{t=T+1}^{T+m} X_t \right) + (m - 1)\delta^2}{\sum_{t=1}^{n} X_t^2 + m\delta^2 + 2\delta \left( \sum_{t=T}^{T+m-1} X_t \right)} \quad (4.29) \]

\[ \hat{\rho}_2 = \frac{\sum_{t=1}^{n-2} X_tX_{t+2} + \delta \left( \sum_{t=T-2}^{T+m-3} X_t + \sum_{t=T+2}^{T-m+1} X_t \right) + (m - 2)\delta^2}{\sum_{t=1}^{n} X_t^2 + m\delta^2 + 2\delta \left( \sum_{t=T}^{T+m-1} X_t \right)} \quad (4.30) \]
Generalising the above for $k = m - 1$, we get

$$
\hat{\rho}_{m-1} = \frac{\sum_{t=1}^{n-(m-1)} X_t X_{t+(m-1)} + \delta \left( \sum_{t=T-(m-1)}^{T} X_t + \sum_{t=T+(m-1)}^{T+2m-2} X_t \right) + \delta^2}{\sum_{t=1}^{n} X_t^2 + m\delta^2 + 2\delta \left( \sum_{t=T}^{T+m-1} X_t \right)}
$$

(4.31)

Similarly

$$
\hat{\rho}_{m} = \frac{\sum_{t=1}^{n-m} X_t X_{t+m} + \delta \left( \sum_{t=T-m}^{T-1} X_t + \sum_{t=T+m}^{T-2m-1} X_t \right) \sum_{t=1}^{n} X_t^2 + m\delta^2 + 2\delta \left( \sum_{t=T}^{T+m-1} X_t \right)}
$$

(4.32)

$$
\vdots
$$

$$
\hat{\rho}_{m+j} = \frac{\sum_{t=1}^{n-(m+j)} X_t X_{t+(m+j)} + \delta \left( \sum_{t=T-(m+j)}^{T-j-1} X_t + \sum_{t=T+(m+j)}^{T+2m+j-1} X_t \right) \sum_{t=1}^{n} X_t^2 + m\delta^2 + 2\delta \left( \sum_{t=T}^{T+m-1} X_t \right)}
$$

(4.33)

The above equations can be compactly written as

$$
\hat{\rho}_k = \begin{cases}
\frac{\sum_{t=1}^{n-k} X_t X_{t+k} + \delta \left( \sum_{t=T-k}^{T-k+(m-1)} X_t + \sum_{t=T+k}^{T+k+(m-1)} X_t \right) + (m-k)\delta^2}{\sum_{t=1}^{n} X_t^2 + m\delta^2 + 2\delta \left( \sum_{t=T}^{T+m-1} X_t \right)} , & k \leq m - 1 \\
\sum_{t=1}^{n-k} X_t X_{t+k} + \delta \left( \sum_{t=T-k}^{T-k+(m-1)} X_t + \sum_{t=T+k}^{T+k+(m-1)} X_t \right)} \sum_{t=1}^{n} X_t^2 + m\delta^2 + 2\delta \left( \sum_{t=T}^{T+m-1} X_t \right) , & k \geq m.
\end{cases}
$$

(4.34)
From equation (4.34), consider \( \hat{\rho}_k \) for \( k = 1, 2, 3, \ldots, (m - 1) \)

\[
\hat{\rho}_k = \frac{\sum_{t=1}^{n-k} X_t X_{t+k} + \delta \left( \sum_{t=T-k}^{T-k+(m-1)} X_t + \sum_{t=T+k}^{T+k+(m-1)} X_t \right) + (m - k)\delta^2}{\sum_{t=1}^{n} X_t^2 + m\delta^2 + 2\delta \left( \sum_{t=T}^{T+m-1} X_t \right)}
\]  

(4.35)

Dividing the numerator and denominator of (4.35) by \( \delta^2 \), we get

\[
\hat{\rho}_k = \frac{\sum_{t=1}^{n-k} X_t X_{t+k}/\delta^2 + 1/\delta \left( \sum_{t=T-k}^{T-k+(m-1)} X_t + \sum_{t=T+k}^{T+k+(m-1)} X_t \right) + (m - k)}{\sum_{t=1}^{n} X_t^2/\delta^2 + m + 2/\delta \left( \sum_{t=T}^{T+m-1} X_t \right)}
\]  

(4.36)

For a fixed \( n \), consider \( \lim_{\delta \to \infty} \hat{\rho}_k \) in (4.36), we get

\[
\lim_{\delta \to \infty} \hat{\rho}_k = \frac{m - k}{m}, \quad k = 1, 2, 3, \ldots, (m - 1)
\]  

(4.37)

Similarly dividing the numerator and denominator of \( \hat{\rho}_k \) for \( k \geq m \) of the equation (4.34) and taking \( \lim_{\delta \to \infty} \), we get

\[
\lim_{\delta \to \infty} \hat{\rho}_k = 0, \quad k \geq m
\]  

(4.38)

Hence the theorem.

**Remark:** The remark, made under Theorem-1 seems to be relevant for Theorem-3 also.

### 4.7.2 SPACF in the presence of a Patch of Additive Outliers

By definition, the Sample Partial Autocorrelation coefficients for the time series (4.26) is calculated by using the ratio of two determinants given in (4.39) for
\[ k = 1, 2, 3, \ldots, \text{successively (Refer Box et al.,(1994) [20]).} \]

\[ \hat{\phi}_{kk} = \begin{vmatrix} 1 & \hat{\rho}_1 & \hat{\rho}_2 & \hat{\rho}_3 & \cdots & \hat{\rho}_{k-2} & \hat{\rho}_{k-1} \\
\hat{\rho}_1 & 1 & \hat{\rho}_1 & \hat{\rho}_2 & \cdots & \hat{\rho}_{k-3} & \hat{\rho}_{k-2} \\
\hat{\rho}_2 & \hat{\rho}_1 & 1 & \hat{\rho}_1 & \cdots & \hat{\rho}_{k-4} & \hat{\rho}_{k-3} \\
\hat{\rho}_3 & \hat{\rho}_2 & \hat{\rho}_1 & 1 & \cdots & \hat{\rho}_{k-5} & \hat{\rho}_{k-4} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\hat{\rho}_{k-2} & \hat{\rho}_{k-3} & \hat{\rho}_{k-4} & \hat{\rho}_{k-5} & \cdots & 1 & \hat{\rho}_{k-1} \\
\hat{\rho}_{k-1} & \hat{\rho}_{k-2} & \hat{\rho}_{k-3} & \hat{\rho}_{k-4} & \cdots & \hat{\rho}_1 & \hat{\rho}_k \end{vmatrix} \tag{4.39} \]

In order to study the limiting behaviour of \( \hat{\phi}_{kk} \) as \( \delta \to \infty \), we invoke Theorem-3 in (4.39). This results in evaluation of the above determinants involving the values of \( m \) and \( k \). Analytical evaluation of these determinants for general values of \( m \) and \( k \) is quite messy and we were not successful in establishing a general formula for SPACF for any values of \( m \) and \( k \). However, in order to extract the general pattern of SPACF for specific values of \( m \) and \( k \), we have numerically computed SPACF using (4.39) for \( m = 3, 4, \) and 5 and for different values of \( k \) and the results are presented in the following lemmas-(1-3). Theorem-4 is consolidated for any value of \( m \) and \( k \) based on the pattern that has been exhibited in these lemmas.
Lemma 1. For the time series given in (4.26) with \( m = 3 \), and as \( \delta \to \infty \),

\[
\begin{align*}
\hat{\phi}_{11} &= 2/3 & \hat{\phi}_{22} &= -1/5 & \hat{\phi}_{33} &= -1/4 \\
\hat{\phi}_{44} &= 1/3 & \hat{\phi}_{55} &= -1/8 & \hat{\phi}_{66} &= -1/7 \\
\hat{\phi}_{77} &= 2/9 & \hat{\phi}_{88} &= -1/11 & \hat{\phi}_{99} &= -1/10 \\
\hat{\phi}_{1010} &= 1/6
\end{align*}
\]

Proof: The definition of SPACF together with (4.39) and Theorem-3, we have, when \( m = 3 \)

\[
\lim_{\delta \to \infty} \hat{\phi}_{kk} = \begin{pmatrix}
1 & 2/3 & 1/3 & 0 & \cdots & 0 & 2/3 \\
2/3 & 1 & 2/3 & 1/3 & \cdots & 0 & 1/3 \\
1/3 & 2/3 & 1 & 2/3 & \cdots & 0 & 0 \\
0 & 1/3 & 2/3 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 2/3 & 0
\end{pmatrix}
\]

It is to be noted that the order of the two determinants in (4.40) will be \( k \times k \) for any given value of \( k \).

Numerically evaluating (4.40) for \( k = 1 \) to 10, we get the expressions in the lemma.

Hence the proof.
Lemma 2. For the time series given in (4.26) with \( m = 4 \), and as \( \delta \to \infty \),

\[
\begin{align*}
\hat{\phi}_{11} &= 3/4 & \hat{\phi}_{22} &= -1/7 & \hat{\phi}_{33} &= -1/6 & \hat{\phi}_{44} &= -1/5 \\
\hat{\phi}_{55} &= 3/8 & \hat{\phi}_{66} &= -1/11 & \hat{\phi}_{77} &= -1/10 & \hat{\phi}_{88} &= -1/9 \\
\hat{\phi}_{99} &= 1/4 & \hat{\phi}_{1010} &= -1/15 & \hat{\phi}_{1111} &= -1/14 & \hat{\phi}_{1212} &= -1/13 \\
\hat{\phi}_{1313} &= 3/16
\end{align*}
\]

Proof: The definition of SPACF together with (4.39) and Theorem-3, we have, when \( m = 4 \)

\[
\begin{vmatrix}
1 & 3/4 & 2/4 & 1/4 & \cdots & 0 & 3/4 \\
3/4 & 1 & 3/4 & 2/4 & \cdots & 0 & 2/4 \\
2/4 & 3/4 & 1 & 3/4 & \cdots & 0 & 1/4 \\
1/4 & 2/4 & 3/4 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 3/4 & 0
\end{vmatrix}
\]

(4.41)

\[
\lim_{\delta \to \infty} \hat{\phi}_{kk} =
\begin{vmatrix}
1 & 3/4 & 2/4 & 1/4 & \cdots & 0 & 0 \\
3/4 & 1 & 3/4 & 2/4 & \cdots & 0 & 0 \\
2/4 & 3/4 & 1 & 3/4 & \cdots & 0 & 0 \\
1/4 & 2/4 & 3/4 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 3/4 \\
0 & 0 & 0 & 0 & \cdots & 3/4 & 1
\end{vmatrix}
\]

It is to be noted that the order of the two determinants in (4.41) will be \( k \times k \) for any given value of \( k \).

Numerically evaluating (4.41) for \( k = 1 \) to 13, we get the expressions in the lemma.

Hence the proof. \( \square \)
Lemma 3. For the time series given in (4.26) with \( m = 5 \), and as \( \delta \to \infty \),

\[
\hat{\phi}_{11} = \frac{4}{5} \quad \hat{\phi}_{22} = -\frac{1}{9} \quad \hat{\phi}_{33} = -\frac{1}{8} \quad \hat{\phi}_{44} = -\frac{1}{7} \quad \hat{\phi}_{55} = -\frac{1}{6} \\
\hat{\phi}_{66} = \frac{2}{5} \quad \hat{\phi}_{77} = -\frac{1}{14} \quad \hat{\phi}_{88} = -\frac{1}{13} \quad \hat{\phi}_{99} = -\frac{1}{12} \quad \hat{\phi}_{1010} = -\frac{1}{11} \\
\hat{\phi}_{1111} = \frac{4}{15} \quad \hat{\phi}_{1212} = -\frac{1}{19} \quad \hat{\phi}_{1313} = -\frac{1}{18} \quad \hat{\phi}_{1414} = -\frac{1}{17} \quad \hat{\phi}_{1515} = -\frac{1}{16} \\
\hat{\phi}_{1616} = \frac{1}{5}
\]

Proof: The definition of SPACF together with (4.39) and Theorem-3, we have, when \( m = 5 \)

\[
\lim_{\delta \to \infty} \hat{\phi}_{kk} =
\begin{bmatrix}
1 & 4/5 & 3/5 & 2/5 & 1/5 & \cdots & 0 & 4/5 \\
4/5 & 1 & 4/5 & 3/5 & 2/5 & \cdots & 0 & 3/5 \\
3/5 & 4/5 & 1 & 4/5 & 3/5 & \cdots & 0 & 2/5 \\
2/5 & 3/5 & 4/5 & 1 & 4/5 & \cdots & 0 & 1/5 \\
1/5 & 2/5 & 3/5 & 4/5 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 4/5 & 0
\end{bmatrix}
\tag{4.42}
\]

It is to be noted that the order of the two determinants in (4.42) will be \( k \times k \) for any given value of \( k \).
Numerically evaluating (4.42) for $k = 1$ to 16, we get the expressions in the lemma. Hence the proof.

**Remark**: We believe that, the Lemmas 1 to 3 do provide sufficient information to obtain the following Theorem for any general value of $m$ and $k$. This theorem has been verified for other values of $m$ and $k$.

**Theorem 4.** For the time series given in (4.26) with a patch of $m$ additive outliers, for $k \geq 1$, let $(k - 1) = u_k m + v_k$, where $u_k$, $(u_k = [(k - 1)/m]; u_k \geq 0)$ and $v_k$, $(v_k = (k - 1) - u_k m; v_k = 0, 1, 2, \ldots, m - 1)$ are respectively the quotient and the remainder when $(k - 1)$ is divided by $m$. Then

$$
\lim_{\delta \to \infty} \hat{\phi}_{kk} = \begin{cases} 
\frac{m - 1}{(u_k + 1)m} &; v_k = 0 \\
\frac{1}{(u_k + 2)m - v_k} &; v_k = 1, 2, \ldots, m - 1.
\end{cases}
$$

**Proof**: A combined reading of Lemmas 1, 2 and 3 together with a careful search to extract the hidden patterns in the SPACF establishes the Theorem.

**Remark 1**: It is pertinent to note that the SPACF will be positive at lags $1, m + 1, 2m + 1, \ldots$, and negative at rest of the $m - 1$ intermediate lags. Interestingly the positive values of SPACF exhibit the decaying pattern as lag increases.

**Remark 2**: The remark, made under Theorem-1 seems to be relevant for Theorem-4 also.

### 4.7.3 Simulation Study: SACF, SPACF and Patch of Additive Outliers

In order to demonstrate Theorem-4 for a given values of $m$ and $k$, we have also obtained the plots of SPACF based on simulation work. The following figure-4.14 provides the SACF and SPACF plots for the generated time series data from an AR(1) model with $n = 200$, for different of values of $m$ and for $\delta = 0$ and 500.
Figure 4.14: $\hat{\rho}_k$ and $\hat{\phi}_{kk}$ plots of AR(1) in the Presence of a Patch of AO's
The simulation work confirm with the results stated in Theorem-4 especially with respect to Remark-1 that follows the theorem.

4.8 Discussion

The limiting behaviour of SACF has been discussed in the literature for AO, IO, LC and TC only. But the case of DO and patch of AO’s which have not been discussed so far, is a new effort in this chapter.

The Theorems -1 to 4 serve only a mathematical interest. Some exercises have been carried out subsequently in this chapter to provide a practical flavour, in terms of identifying the minimal $\delta$ required for a time series data of a given sample size $n$ to reflect the limiting behaviour. The consequences of DO on the estimation of the model parameters are also highlighted in this chapter.

The role of SACF and SPACF plots in the context of identification of the model in Box-Jenkins methodology is crucial for time series analysis. In the case of a DO, Theorems-1 and 2 provide a caution when data is contaminated with a DO of high magnitude. Theoretically in the presence of a DO of a large magnitude, masks the true model for time series, thereby seriously jeopardizing the function of SACF and SPACF as model identification tools. A combined reading of these two theorems suggest that a large DO would suggest an ARMA(0,0,1) model for any DO contaminated data.

Theorems-1 to 2 clearly indicate that the limiting behaviour of the SACF and the SPACF for DO case, when demonstrated on ARMA(1,1) models, depends on both the length of the data and the model coefficients. We have tried to identify the least
value of \( \delta \) that is necessary to attain the limiting value and have thereby exposed the role of the coefficients of the model and sample size \( n \). Further we have also demonstrated the impact of DO in the estimation of the coefficients in ARMA(1,1), AR(1) and MA(1) models. The results in these directions appear to be relevant from a practitioner’s point of view.

The presence of outliers in a time series produces serious effects on the sample autocorrelation and hence on partial autocorrelation coefficients. The existence of outliers can introduce substantial biases in SACF and SPACF and hence gives rise to misspecification in the Box-Jenkins approach to univariate time series analysis. The implications of this theoretical result are extensive for univariate time series analysis.

To summarize, we have demonstrated that relying purely on SACF and SPACF plots for model identification may not serve the purpose, since, for any data

- a DO with large \( \delta \) appears to suggest an MA(1) model and
- a patch of \( m \) AO’s with large \( \delta \) appears to suggest an MA(m-1) model.