CHAPTER II
2. ACCELERATING UNIVERSE IN HIGHER DIMENSIONS IN PRESENCE OF SCALAR FIELD

In this paper we present five-dimensional matter dominated cosmological models in presence of scalar field. Einstein’s cosmological constant \( \Lambda \) appears as integration constant. The dark energy density and the energy density of ordinary matter decrease during the expansion of the universe.

2.1 INTRODUCTION

The physical cosmology of present day required two important concepts. (i) the matter which does not interact with the electromagnetic force-dark matter and (ii) the hypothetical energy that tends to increase the rate of expansion of the universe-dark energy. The recent cosmological observations point towards an accelerated expansion of the universe. Instead of slowing down the expanding universe is speeding up. An intense search is going on to find out the true nature of this acceleration. The advent of inflationary theory [1,2,3] led to the belief that 96% of matter content of the universe is hidden mass constituted by 23% dark matter and 73% dark energy[4]. The present day

universe is four dimensional. But attempts are being made by a section of workers to recast the theory in a higher-dimensional space-time. One reason which inspires this motivation is the considerable success in solving long standing problems concerning consistency of general relativity and quantum mechanics by considering the space time having dimensions higher than four. Modern developments of superstring theory and Yang Mills super gravity in their field theory limit need higher-dimensional space time. For this reason in recent years there has been considerable interest in theories with higher dimensional space time, in which extra dimensions are contracted to a very small size, apparently beyond our ability for measurement. A model of higher dimensions are proposed by Kaluza and Klein[5,6] who tried to unify gravity with electromagnetic interaction by introducing an extra dimensions which is an extension of Einstein’s general relativity in five dimensions. Recent activities in extra dimensions stem from the space time Matter (STM) theory [7] proposed by Wesson and his collaborators. The universe is dominated by a form of matter with negative pressure which is referred to as dark energy. A kind of repulsive force which acts as anti-gravity is responsible for accelerating universe. At present $\Lambda$ with a dynamical character is preferred over a constant $\Lambda$ specially a time dependent which decreases slowly from its large value to reach its small value at present era [9]. A scalar field $\phi$ with a potential $V(\phi)$, which is known as quintessence and decreases slowly with time may be another candidate for dark energy. Quintessence exerts negative pressure and is dynamic in nature. The effect of the scalar field leads the universe to reach its critical energy and to accelerate its expansion. We have studied the dynamical behaviour of a higher dimensional cosmological model in presence non relativistic matter and scalar field. The exact
analytical solutions of the Einstein’s equation are derived for FRW universe by assuming a particular relation between time derivative of the scalar field and that of a Hubble function [10]. Einstein’s cosmological constant appears as integration constant. The pressure of dark energy behaves as Einstein’s cosmological constant. The dark energy density decreases as the universe expands.

In section 2 of the paper we present the field equations for five dimensional spatially flat, homogeneous and anisotropic cosmological models. In section 3 of this paper we obtain the solutions of the field equation by assuming a relation between the time derivative of scalar field and that of the Hubble function.

2.2 FIELD EQUATIONS

We consider the spatially flat homogeneous and anisotropic five dimensional space time model described by the line element

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)\right] + b^2(t)dy^2$$  \hspace{1cm} (1)

where \(a(t)\) and \(b(t)\) being function of time represent the scale factors of the four dimensional flat FRW cosmological space time and extra dimension respectively. We consider that the universe is filled with scalar field \(\phi\) having potential \(V(\phi)\) and normal matt. The Einstein’s field equations are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{(M)\mu\nu} + T^{(\phi)\mu\nu}$$  \hspace{1cm} (2)
where $8\pi G = c = 1$ and $R_{ij}$ are the Ricci tensor and $T^{(M)}_{ij}$, $T^{(\phi)}_{ij}$ are the energy momentum tensors of normal matter and quintessence respectively and

$$T^{(M)}_{\mu \nu} = \rho_m, \quad T^{(M)}_{11} = T^{(M)}_{22} = T^{(M)}_{33} = T^{(M)}_{44} = -p_m$$

$$T^{(\phi)}_{ij} = \phi_i \phi_j - g_{ij} \left[ \frac{1}{2} g^{kl} \phi_k \phi_l - V(\phi) \right]$$

where $\rho_m$, $p_m$ are respectively the energy density and pressure of the ordinary matter.

Einstein’s field equations for the metric (1) are

$$3 \frac{\ddot{a}^2}{a^2} + 3 \frac{\dot{a} \dot{b}}{ab} = \rho_m + \rho_\phi$$

$$2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + 2 \frac{\dot{a} \dot{b}}{ab} + \frac{\ddot{b}}{b} = -p_\phi$$

$$3 \frac{\ddot{a}^2}{a^2} + 3 \frac{\ddot{a}}{a} = -p_\phi$$

where dot denotes time derivatives and $\rho_\phi, p_\phi$ represent the energy density and pressure of the scalar field and $\rho_m$ denotes the energy density of the ordinary matter respectively.

$\rho_\phi$ and $p_\phi$ are given by

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)$$

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi)$$

The energy conservation equation for normal matter is
\[ \dot{\rho}_m + \left( 3 \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) (p_m + \rho_m) = 0 \]  \hspace{1cm} (10)

Also the evolution equation for the scalar field is

\[ \ddot{\phi} + \left( 3 \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) \dot{\phi} + V'(\phi) = 0 \]  \hspace{1cm} (11)

where dash denotes derivatives with respect to \( \phi \).

### 2.3 SOLUTIONS OF THE FIELD EQUATIONS

We assume the relation between the metric coefficients read as

\[ b = Qa^{-n} \]  \hspace{1cm} (12)

where \( Q \) and \( n \) are positive constant.

The field equations (5)-(7) take the form

\[ 3(1-n)H^2 = \rho_m + \rho_\phi \]  \hspace{1cm} (13)

\[ (2-n)H + \left( 3 - 2n + n^2 \right)H^2 = -p_\phi \]  \hspace{1cm} (14)

\[ 3\dot{H} + 6H^2 = -p_\phi \]  \hspace{1cm} (15)

where \( H = \frac{\dot{a}}{a} \) is the Hubble parameter.

The evolution equation for the scalar field is
\[
\dot{\phi} + \left( 3 \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) \phi + V'(\phi) = 0
\]  \hspace{1cm} (16)

In the case of dust the energy density of the ordinary matter \(\rho_m\) is related with the scale factors through

\[
\rho_m = \rho_{m0} \left( \frac{a}{a_0} \right)^3 \left( \frac{b}{b_0} \right)
\]  \hspace{1cm} (17)

where the lower index ‘0’ indicates the present day values of the corresponding quantities. The Hubble parameter \(H\) is expressed as a function of the scalar field \(\phi\) is

\[H = H(\phi)\.

From equation (14) and (8) we get,

\[
\dot{\phi} = -(2-n)H' \pm \Phi(\phi)
\]  \hspace{1cm} (18)

where

\[
\Phi(\phi) = \left[ (2-n)^2 H'^2 + 2V(\phi) - 2(3-2n+n^2) \right] H^2
\]  \hspace{1cm} (19)

Equation (19) may be written as

\[
\Phi^2(\phi) = \left[ (2-n)^2 H'^2 + 2V(\phi) - 2(3-2n+n^2) \right] H^2
\]

This implies

\[
V(\phi) = (3-2n+n^2) H^2 - \frac{(2-n)^2}{2} H'^2 - \frac{\Phi^2(\phi)}{2}
\]  \hspace{1cm} (20)
The equation (13) and (14) yield

$$ (2-n)\dot{H} + \rho_m + \dot{\phi}^2 = -n(n+1)H^2 $$

(21)

From equation (13) and (16)-(21) we have

$$ \left( (2-n)H^* - (3-n)H \right) \Phi + \left( (2-n)H^* + 2\Phi \right) \Phi' \pm (3-n)(2-n)HH^* + 2(3-2n+n^2)HH^* = 0 $$

(22)

Equation (22) contains two time dependent functions $H(\phi)$ and $\Phi(\phi)$ and their derivatives. So it is not possible to solve both of them. To solve it we assume the following relationship

$$ \Phi(\phi) = \pm \beta H' $$

(23)

Where $\beta$ is a positive constant.

Substituting (23) in (24) we obtain

$$ 2\left[ (2-n)\beta - \beta^2 \right] H^* - \left[ n^2 + (1-\beta)n + 3\beta \right] H = 0 $$

(24)

Equation (24) can be integrated and we get

$$ H(\phi) = Ce^{\frac{\left[ n^2 + (1-\beta)n + 3\beta \right]}{2(2-n)\beta - \beta^2}} + Be^{-\frac{\left[ n^2 + (1-\beta)n + 3\beta \right]}{2(2-n)\beta - \beta^2}} $$

(25)

where $C$ and $B$ are positive integration constant.

Using (23) in (18) we get

$$ \dot{\phi} = -(2-n-\beta)H' $$

(26)
For the quintessence field finding scalar potential is important which could explain current cosmological observations.

From equation (20), (23) and (25) we get

\[
V(\phi) = \left(\frac{n^2 - 2n + 3}{4}(2-n-\beta)\right) C e^{\psi^{2/1-\beta, n+3/5\beta}} - B e^{-\psi^{2/1-\beta, 2-n-3/5\beta}} + \Lambda \tag{27}
\]

where

\[
\Lambda = 4(3-2n+n^2)BC \tag{28}
\]

From equation (9), (20), (23) and (26) we get

\[
\rho_\phi = \left(3-2n+n^2\right)H^2 + \left[\frac{\{2-(n+\beta)\}^2 + \beta^2 - (2-n)^2}{2}\right]H^2 \tag{29}
\]

The energy density of ordinary matter is given by

\[
\rho_m = \left(-n(n+1)\right)H^2 + \left[\frac{\{2-(n+\beta)\}^2 + \beta^2 - (2-n)^2}{2}\right]H^2 \tag{30}
\]

Also from equations (25) and (30) we have

\[
\frac{3n^2 + 3n + \beta(3-n)}{2} = \rho_m + 2\left[n^2 + n + \beta(3-n)\right]BC \tag{31}
\]

For dust the energy density of ordinary matter \(\rho_m\) is given by

\[
\rho_m = \rho_m(1+z)^{-3-n} \text{ where } (1+z) = \frac{a}{a_0}
\]
The expression of the energy density of scalar field is

$$\rho_\phi = \frac{6 - 9n - 3n^2 - (3 - n)\beta}{3n^2 + 5n + (3 - n)\beta} \rho_\kappa (1 + z)^{\gamma - n} + \frac{12(1 - n)[n^2 + n + (3 - n)\beta]}{3n^2 + 5n + (3 - n)\beta} BC$$  \hspace{1cm} (32)

From equation (32), it is observed that the dark energy density decreases monotonously as $a^{-(3-n)}$ (where $n < 3$) as the universe is expanding. Also dark energy density and the energy density of ordinary matter decrease at same rates during the expansion of the universe. From equation (9), (20),(23),(25) and (26), the expression for the pressure of scalar field is given by

$$p_\phi = -\left[\frac{n^2(n + \beta - 1) + n\beta - 2n}{2\beta}\right] C^2e^{\frac{\pi^2(1-\beta)+3\beta}{\sqrt{2(2-n)\beta^{-n}}}} + B^2e^{\frac{-\pi^2(1-\beta)+3\beta}{\sqrt{2(2-n)\beta^{-n}}}}$$

$$-\left[\frac{n^2(3\beta+1-n)+n(2-9\beta)+12\beta}{\beta}\right] BC$$  \hspace{1cm} (33)

With suitable choice of $\beta$ and $n < 3$, it can be seen that the pressure of the scalar field may be found to be negative. The negative pressure of scalar field indicates that the universe is expanding.

2.4 CONCLUSIONS

In this paper we have discussed 5D homogeneous cosmological model. We know that the best candidates for unification of the forces of nature in a quantum gravitational environment seem to exist in finite form if there are many more dimensions of space.
This implies that true constant of nature are defined in higher dimensions. We have taken only one extra spatial dimension but we believe most of the findings may be extended if we take a larger number of extra dimensions. The exact solutions of Einstein equations are obtained by assuming a particular relation between the time derivative of the scalar field and that of the Hubble function. For $n > 0$, $a(t)$ increases while $b(t)$ decreases. Thus extra dimension becomes insignificant as the time proceeds after the creation and we are left with the real four-dimensional world. The expansion scale factors and the potential $V(\varphi)$ are determined. The dark energy density decreases during the expansion of the universe. Our solution reduces to that of Wang. W.F & Yang. S. Z [10] when $n = 0$, i.e. four dimensional model. This work is an extension of Wang. W. F & Yang. S. Z’s work in four dimensions.
2.5 REFERENCES


