CHAPTER VII
7. MAGNETIZED KERR-DE SITTER GEOMETRY**

In this paper, we have derived Kerr-de Sitter metric in a magnetic field using null tetrad formalism. It is observed that the dragging effect on a particle moving in the field of a magnetized Kerr object in de Sitter universe is enhanced by the introduction of the magnetic field in the instant case. A magnetized Kerr-de Sitter black hole has two event horizons. We have also derived the Sagnac effect in magnetized Kerr-de Sitter metric and have found that Sagnac effect is affected by magnetic field.

7.1 INTRODUCTION

Melvin’s magnetic universe (Melvin 1964) is a collection of parallel magnetic lines of force in equilibrium under their mutual gravitational attraction. The magnetic universe is described by electrovac solution to the Einstein-Maxwell equations. This space-time is invariant under rotation about and translation along an axis of symmetry. Wheeler (1964) demonstrated that a magnetic universe could also be obtained in Newton’s theory of gravitation and showed that it is unstable according to elementary Newtonian analysis. Further Melvin (1965) showed his universe to be stable against small radial perturbations, and Thorne (1965) proved the stability of the magnetic universe against arbitrary large perturbations. W. A. Hiscock (1981) studied the

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magnetized black hole solutions discovered by Ernst (1976). The Schwarzschild-Melvin black hole solution is the unique static, axisymmetric black hole solution of the source less Einstein-Maxwell equations which asymptotically resembles Melvin’s magnetic universe (MMU).

The MMU is a solution of the Einstein’s equation that describes a matter-free universe, endowed only with magnetic field (Melvin 1964, 1965; Bonnor 1953, 1954). The Schwarzschild-Melvin black hole solution is not asymptotically flat. This solution can provide insights in understanding more realistic situations and this has just been the motivation of several current studies about the Schwarzschild-Melvin solution as well as other Melvin-like black hole solutions. Recently Castello-Branco et al. (2012) investigated the distribution of gravitational energy in the space-time of a Schwarzschild black hole immersed in a cosmic magnetic field.

The metric describing the Schwarzschild black hole in an external magnetic field (or simply the magnetized Schwarzschild black hole) for stationary, axially symmetric magnetic solutions is given by

\[
ds^2 = \Lambda_M'^2 \left[ -\left( 1 - \frac{2m}{r} \right) dt^2 + \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2 d\theta^2 \right] + \Lambda_M'^{-2} r^2 \sin^2 \theta \, d\phi^2
\]  

(1)

where \( \Lambda_M = 1 + \frac{1}{4} B_0^2 r^2 \sin^2 \theta \), \( B_0 \) is a constant that corresponds to the value of the magnetic field everywhere on the polar axis and represents the strength of the
cosmological magnetic field. For \( m = 0 \), equation (1) yields the metric of MMU in spherical-like coordinates and for \( B_0 = 0 \), it reduces to the Schwarzschild metric.

The purpose of the present paper is to investigate the magnetic universe for a rotating object embedded in de Sitter universe. We have derived the magnetized Kerr-deSitter metric using null tetrad formalism (K. D. Krorg 2010). We have examined three types of geometries of the metric. Firstly, it is observed that the dragging effect (K. D. Krori 2010) on a particle moving in the field of a magnetized Kerr-de Sitter object is affected by magnetic field. We have found that a Coriolis-like dragging is experienced by a particle moving in the field of a magnetized Kerr-de Sitter object in de Sitter space-time. Secondly, magnetized Kerr de Sitter black hole is discussed. It is found that there are two distinct infinite red shift surfaces for this space-time.

The subject of derivation of the Sagnac effect has attracted much attention. G.Rizzi and M.L. Ruggiero (2003) and later M.L. Ruggiero alone (2005) derived the effect in analogy with Aharonov-Bohm effect. Krori et al. (2008) derived the Sagnac effect by direct, simple method in different types of metrics viz, flat, Schwarzschild, Kerr, etc. Lastly, we have derived the Sagnac effect by a direct, simple method following Krori et.al (2008) in magnetized Kerr metric and it is observed that the Sagnac effect is also changed by the magnetic field.

The paper is organized as follows. In section 2, we have derived the magnetized Kerr-de Sitter metric using null tetrad formalism; in section 3, we have investigated the dragging effect on a particle moving in the field of a magnetized Kerr-de Sitter object; in section 4, we have found that there are two event horizon for the magnetized Kerr de
Sitter black hole; in section 5, we give the derivation of Sagnac effect in magnetized Kerr-de Sitter metric. The paper deals with a brief conclusion in section 6.

7.2 DERIVATION OF MAGNETIZED KERR-DE SITTER METRIC

We use the null tetrad formalism to derive the metric for magnetized Kerr-de Sitter solution. To find the metric for a rotating object we introduce a rotation parameter $a$ and construct covariant and contravariant forms of the null tetrad satisfying the properties $l_{\alpha}l^{\alpha} = 0$ etc. such that for $a \to 0$, we recover the metric describing the Schwarzschild black hole in an external magnetic field or simply magnetized Schwarzschild black hole.

This calls for an ingenuous search leading to the following forms of the null tetrad with

$$\Delta = \frac{\Delta_r}{1 + \frac{\Lambda}{3} a^2 \cos^2 \theta}, \quad \rho = \frac{r - ia \cos \theta}{1 - i \sqrt{\frac{\Lambda}{3} a \cos \theta}}$$

(2)

where $\Delta_r = -\frac{\Lambda}{3} r^2 (r^2 + a^2) + r^2 - 2mr + a^2$

$$\rho^2 = \rho \rho^* = \frac{r^2 + a^2 \cos^2 \theta}{1 + \frac{\Lambda}{3} a^2 \cos^2 \theta}$$

(3)
satisfying the properties

\[ l^a l_a = 0 = n^a n_a \]  \hspace{1cm} (4) \\
\[ i^a n_a = 1 \]  \hspace{1cm} (5) \\
\[ m^a m_a = m^{\alpha *} m_{\alpha *} = 0 \]  \hspace{1cm} (6) \\
\[ m^a m^* \alpha = -1 \]  \hspace{1cm} (7)

\( l^a, n^a, m^a, m^{\alpha *} \) are called null vectors because each of them has zero magnitude.

Here we take

\[ l_a = \frac{\Lambda_M}{\Delta} \left[ -\rho^2, 0, \frac{a \Delta \sin^2 \theta}{\left(1 + \frac{\Lambda}{3} a^2\right) \left(1 + \frac{\Lambda}{5} a^2\right) a^2} \right] \]  \hspace{1cm} (8) \\
\[ n_a = \frac{\Lambda_M}{2 \rho^2} \left[ \rho^2, 0, \frac{a \Delta \sin^2 \theta}{\left(1 + \frac{\Lambda}{3} a^2\right) \left(1 + \frac{\Lambda}{5} a^2\right) a^2} \right] \]  \hspace{1cm} (9) \\
\[ m_a = \frac{\Lambda_M}{\sqrt{2} \rho^2} \left[ 0, -\rho^2, -i(r^2 + a^2) \sin \theta, \frac{-ia \sin \theta}{\Lambda_M^2 \left(1 + \frac{\Lambda}{3} a^2\right) \left(1 + \frac{\Lambda}{5} a^2\right)} \right] \]  \hspace{1cm} (10) \\
\[ m^* \alpha = \frac{\Lambda_M}{\sqrt{2} \rho} \left[ 0, -\rho^2, i(r^2 + a^2) \sin \theta, \frac{ia \sin \theta}{\Lambda_M^2 \left(1 + \frac{\Lambda}{3} a^2\right) \left(1 + \frac{\Lambda}{5} a^2\right)} \right] \]  \hspace{1cm} (11)
and
\[ l^a = \frac{1}{\Delta \lambda_m} \left[ \Delta, 0, -\frac{a \left( 1 + \frac{\Lambda}{3} a^2 \right)}{1 + \frac{\Lambda}{3} a^2 \cos^2 \theta}, \frac{(r^2 + a^2) \left( 1 + \frac{\Lambda}{3} a^2 \right)}{1 + \frac{\Lambda}{3} a^2 \cos^2 \theta} \right] \] (12)

\[ n^a = \frac{1}{2 \rho^2 \Delta m} \left[ -\Delta, 0, -\frac{a \left( 1 + \frac{\Lambda}{3} a^2 \right)}{1 + \frac{\Lambda}{3} a^2 \cos^2 \theta}, \frac{(r^2 + a^2) \left( 1 + \frac{\Lambda}{3} a^2 \right)}{1 + \frac{\Lambda}{3} a^2 \cos^2 \theta} \right] \] (13)

\[ m^a = \frac{1}{\sqrt{2} \rho \Delta m} \left[ 0, 1, \frac{i \Lambda_m^2 \left( 1 + \frac{\Lambda}{3} a^2 \right)}{\sin \theta \left( 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \right)}, \frac{-ia \sin \theta \left( 1 + \frac{\Lambda}{3} a^2 \right)}{1 + \frac{\Lambda}{3} a^2 \cos^2 \theta} \right] \] (14)

\[ m^a = \frac{1}{\sqrt{2} \rho \Delta m} \left[ 0, 1, \frac{-i \Lambda_m^2 \left( 1 + \frac{\Lambda}{3} a^2 \right)}{\sin \theta \left( 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \right)}, \frac{ia \sin \theta \left( 1 + \frac{\Lambda}{3} a^2 \right)}{1 + \frac{\Lambda}{3} a^2 \cos^2 \theta} \right] \] (15)

Equations (8)-(15) satisfy the properties (4)-(7).
The metric coefficients are obtained with the help of equations (8)-(11) as

\[ g_{11} = -\Lambda_M^2 \left( \frac{r^2 + a^2 \cos^2 \theta}{\Lambda_r} \right), \quad g_{22} = -\Lambda_M^2 \left( \frac{r^2 + a^2 \cos^2 \theta}{1 + \frac{\Lambda}{3} a^2 \cos^2 \theta} \right) \]

\[ g_{33} = \Lambda_M^{-2} \left[ \frac{a^2 \Lambda_M^4 \sin^4 \theta \Lambda_r}{(r^2 + a^2 \cos^2 \theta)^2} \right] \left[ \frac{(r^2 + a^2)^2 \sin^2 \theta \left(1 + \frac{\Lambda}{3} a^2 \cos^2 \theta\right)}{(r^2 + a^2 \cos^2 \theta)^2 \left(1 + \frac{\Lambda}{3} a^2 \cos^2 \theta\right)^2} \right] \]

\[ g_{44} = \Lambda_M^2 \left[ \frac{\Lambda_r}{(r^2 + a^2 \cos^2 \theta)^2} \right] \left[ \frac{a^2 \sin^2 \theta \left(1 + \frac{\Lambda}{3} a^2 \cos^2 \theta\right)}{(r^2 + a^2 \cos^2 \theta)^2 \left(1 + \frac{\Lambda}{3} a^2 \cos^2 \theta\right)^2} \right] \]

\[ g_{34} = \Lambda_M^2 \left[ \frac{a \sin^2 \theta \Lambda_r}{(r^2 + a^2 \cos^2 \theta)^2} \right] \left[ \frac{a \sin^2 \theta \left(1 + \frac{\Lambda}{3} a^2 \cos^2 \theta\right)(r^2 + a^2) \Lambda_M^{-2}}{(r^2 + a^2 \cos^2 \theta)^2 \left(1 + \frac{\Lambda}{3} a^2 \cos^2 \theta\right)^2} \right] \]

(16)

The corresponding metric obtained making use of (16) is
\[ ds^2 = \Lambda_M^2 \left[ \frac{\Delta_r}{(r^2 + a^2 \cos^2 \theta)^{\frac{1}{3}}} \frac{a^2 \sin^2 \theta \left(1 + \frac{\Lambda}{3} \frac{a^2}{r^2} \cos^2 \theta\right)^2}{(r^2 + a^2 \cos^2 \theta)^{\frac{2}{3}}} \right] dt^2 \\
- \Lambda_M^2 \frac{(r^2 + a^2 \cos^2 \theta)d r}{\Lambda_r} - \Lambda_M^2 \frac{(r^2 + a^2 \cos^2 \theta)d \theta}{1 + \frac{\Lambda}{3} \frac{a^2}{r^2} \cos^2 \theta} \\
- \Lambda_M^{-2} \frac{(r^2 + a^2)^2 \sin^2 \theta \left(1 + \frac{\Lambda}{3} \frac{a^2}{r^2} \cos^2 \theta\right)^2}{(r^2 + a^2 \cos^2 \theta)^{\frac{2}{3}}} \frac{a^2 \Lambda M^4 \sin^4 \theta \Delta \theta}{(r^2 + a^2 \cos^2 \theta)^{\frac{2}{3}}} \\
- 2 \Lambda_M^2 \frac{(r^2 + a^2)^{\frac{1}{2}} \left(1 + \frac{\Lambda}{3} \frac{a^2}{r^2} \cos^2 \theta\right)^2}{(r^2 + a^2 \cos^2 \theta)^{\frac{2}{3}}} \Lambda_M^2 \frac{\Delta r}{(r^2 + a^2 \cos^2 \theta)^{\frac{2}{3}}} \\
as \sin^2 \theta \left( d \theta \right) \left( d \phi \right) \] 

(17)

where \( \Lambda_M = 1 + \frac{1}{4} B_0^2 r^2 \sin^2 \theta \)

The metric (17) is called magnetized Kerr-de Sitter metric. For \( B_0 = 0 \) (i.e. when there is no magnetic field) equation (17) reduces to Kerr-de Sitter metric and for \( a = 0 \) (i.e. rotation parameter = 0), equation (17) yields the magnetized Schwarzschild-de Sitter metric which is

\[ ds^2 = \Lambda_M^2 \left( 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} \right) dt^2 - \Lambda_M^2 \left( 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} \right)^{-1} dr^2 - r^2 \Lambda_M^2 d \theta^2 - r^2 \Lambda_M^{-2} \sin^2 \theta d \phi^2 \]
and when \( B_0 = 0, a = 0, \Lambda = 0 \) we get magnetized Schwarzschild metric (Castello-Branco, da Rocha-Neto 2012).

### 7.3 Dragging Effect of a Magnetized Kerr-de Sitter Object

In order to study the dragging effect on a particle moving in the magnetic field of a Kerr object embedded in de Sitter universe, we confine our discussion to

(i) the approximate metric given by equation (17)

(ii) considering the motion of a particle in the equatorial plane (for which \( \dot{\theta} = \frac{d\theta}{ds} = 0 \) and \( \theta = \) a constant = \( \frac{\pi}{2} \))

The equation (17) can readily be reduced to the approximate metric considering it to the first order in \( \frac{a}{r} \) as

\[
\begin{align*}
    ds^2 &= \Lambda_M^{-2} \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right) dt^2 - \Lambda_M^{-2} \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 - \Lambda_M^{-2} r^2 d\theta^2 - \Lambda_M^{-2} r^2 \sin^2 \theta d\phi^2 \\
    &\quad - 2 \left[1 - \Lambda_M^{-2} \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right)\right] a \sin^2 \theta d\phi dt
\end{align*}
\]

(18)

The Lagrangian \( L \) following Chandrasekhar (1983) appropriate to the motion of the particle is written from equation (18) as
\[ 2L = \Lambda_{M_0}^2 \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right) \dot{t}^2 - \Lambda_{M_0}^2 \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right) \dot{r}^2 - \Lambda_{M_0}^{-2} r^2 \dot{\phi}^2 \]
\[ - 2 \left[ 1 - \Lambda_{M_0}^2 \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right) \right] a \dot{\phi} \]

where \( i = \frac{dt}{ds}, \quad \dot{r} = \frac{dr}{ds}, \quad \dot{\phi} = \frac{d\phi}{ds} \) and \( \Lambda_{M_0} = 1 + \frac{1}{4} B_0^2 r^2 \) \( \left( \theta = \frac{\pi}{2} \right) \)

The generalized momenta are given by

\[ p_t = \frac{\partial L}{\partial \dot{t}} = \Lambda_{M_0}^2 \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right) \dot{t} - \left[ 1 - \Lambda_{M_0}^2 \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right) \right] a \dot{\phi} \]
\[ = E = \text{cons tant} \]

\[ p_\phi = -\frac{\partial L}{\partial \dot{\phi}} = \Lambda_{M_0}^{-2} r^2 \dot{\phi} + \left[ 1 - \Lambda_{M_0}^2 \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right) \right] a \dot{t} \]
\[ = L = \text{cons tant} \]

And, \( p_r = -\frac{\partial L}{\partial \dot{r}} = \Lambda_{M_0}^2 \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right)^{-1} \dot{r} \)

Considering the particle in instantaneous radial motion, \( \dot{\phi} = 0 \) equation (21) gives for \( \ddot{\phi} \)

\[ r \ddot{\phi} + \Lambda M_0^2 \left[ 1 - \Lambda M_0^2 \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right) \right] a \dot{t} \]
\[ - \Lambda M_0^2 \left[ \Lambda M_0 B_0^2 \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right) + 2 \Lambda M_0^2 \left( \frac{M}{r^3} - \frac{\Lambda}{3} \right) \right] a \dot{r} = 0 \]

\[ \text{(23)} \]
Also equation (20) yields

\[ i = \Lambda M_0^{-2} \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right)^{-1} E \]

(24)

\[ \ddot{i} = \left[ \Lambda M_0^{-2} \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right)^{-1} \right] \dot{a} \bar{\phi} - \frac{2}{\Lambda M_0} \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right)^{\frac{3}{2}} \frac{E \ddot{r}}{\left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right)^{\frac{3}{2}}} - \frac{EB_0^2}{\Lambda M_0} \frac{r}{\left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right)^{\frac{3}{2}}} \dot{r} \]

(25)

Using equations (24) and (25) in equation (23), we obtain, to the first order in \( \frac{a}{r} \), (with \( E = 1 \))

\[ r \ddot{\phi} = \left[ -\frac{B_0^2}{\Lambda M_0 \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right)^{\frac{3}{2}}} + \frac{2}{\Lambda M_0} \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right)^{\frac{3}{2}} \frac{a \ddot{r}}{\left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right)^{\frac{3}{2}}} \right] \dot{a} = 0 \]

(26)

For \( r \gg 2M \), equation (26) gives
\[ r \ddot{\phi} + \left[ \frac{2M}{r^3 \left( 1 - \frac{\Delta r}{3} \right)^2} - \frac{B_0^2}{\Lambda \Lambda_{M_0} \left( 1 - \frac{\Delta r}{3} \right)^2} + \frac{2\Lambda}{3 \left( 1 - \frac{\Delta r}{3} \right)^2} \right] a \dot{r} = 0 \] 

From equation (27) we observe that a Coriolis-like dragging is experienced by a particle moving in the magnetic field of a Kerr object in de Sitter universe. This dragging effect is increased due to the introduction of magnetic field in this case.

### 7.4 INFINITE RED-SHIFT AND EVENT HORIZON: MAGNETIZED KERR-DE SITTER BLACK HOLE

Let a light wave be emitted from a point on a star and received at a point on the earth. Then the proper-time interval \( d\tau_0 \) on the star is related to the coordinate interval \( dt \) by

\[ ds = d\tau_0 = \sqrt{g_{44}(x_0^\mu)} \, dt \quad c = 1 \] 

(28)

since the space coordinates are fixed for the point on the star.

From equation (28)

\[ d\tau_0 = \sqrt{g_{44}(x_0^\mu)} \, dt \] 

(29)

The proper-time interval \( d\tau \) is related to the coordinate-time interval at the point on the earth by

\[ d\tau = \sqrt{g_{44}(x^\mu)} \, dt \] 

(30)
If \( n \) waves of frequency \( \nu_0 \) are emitted from an atom at the point on the star, then

\[
n = \nu_0 d\tau_0
\]

These \( n \) waves are received on the earth during a different time interval \( d\tau \) with a different frequency, \( \nu \). Thus on the earth

\[
n = \nu d\tau
\]

From equation (29) and (30),

\[
\frac{d\tau_0}{d\tau} = \frac{\sqrt{g_{44}(x_0^\mu)}}{\sqrt{g_{44}(x^\mu)}},
\]

and from equation (31) and equation (32)

\[
\frac{d\tau_0}{d\tau} = \frac{\nu}{\nu_0}
\]

Using equation (33), equation (34) gives

\[
\frac{\nu}{\nu_0} = \frac{\sqrt{g_{44}(x_0^\mu)}}{\sqrt{g_{44}(x^\mu)}}
\]

Writing \( \nu = \frac{c}{\lambda} \) (\( \lambda \) being the wave length on the earth), equation (35) leads to

\[
\lambda_0 = \lambda \frac{\sqrt{g_{44}(x_0^\mu)}}{\sqrt{g_{44}(x^\mu)}}.
\]

For infinite red-shift surface (i.e. for \( \lambda \to \infty \)) \( g_{44}(x_0^\mu) = 0 \)

Corresponding to magnetized Kerr-de Sitter metric, equation (17), the infinite red-shift surface is given by
\[
g_{44}(x^\mu) = \Lambda_{\mathcal{M}_1}^2 \left[ \frac{\Delta_{\mathcal{R}}}{(r_0^2 + a^2 \cos^2 \theta) \left(1 + \frac{\Lambda a^2}{3}\right)^2} - \frac{a^2 \sin^2 \theta \left(1 + \frac{\Lambda}{3} a^2 \cos^2 \theta\right)}{(r_0^2 + a^2 \cos^2 \theta) \left(1 + \frac{\Lambda a^2}{3}\right)^2} \right] = 0 \quad (37)
\]

where \( \Lambda_{\mathcal{M}_1} = \left[1 + \frac{1}{4} B_6 \frac{r_0^2 \sin^2 \theta}{a^2} \right] \).

From equation (37), since \( \Lambda_{\mathcal{M}_1} \neq 0 \),

\[
\Delta_{\mathcal{R}} - a^2 \sin^2 \theta \left(1 + \frac{\Lambda}{3} a^2 \cos^2 \theta\right) = 0 \quad (38)
\]

That is

\[
r_0^2 - 2mr_0 + a^2 \cos^2 \theta + A = 0 \quad (39)
\]

We call the term containing \( \Lambda \) as \( A \), where \( A = -\frac{\Lambda}{3} \left(r_0^2 + r_0^2 a^2 + a^4 \sin^2 \theta \cos^2 \theta\right) \),

which is a small quantity. Equation (39) implies

\[
r_0 = m + \sqrt{m^2 - (a^2 + A)} \quad (40)
\]

From equation (17), the coordinate speed of light signal is obtained with \( ds^2 = 0 \). Hence
\[
\left( \frac{dr}{dt} \right)^2 = \frac{\Lambda_r}{\Lambda_M^2 \left( r^2 + a^2 \cos^2 \theta \right)} \left[ \Lambda_M^2 \left( \frac{\Delta_r - a^2 \sin^2 \theta \left( 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \right)}{r^2 + a^2 \cos^2 \theta \left( 1 + \frac{\Lambda a^2}{3} \right)} \right) \right]
\]

\[\left( r^2 + a^2 \right) \sin^2 \theta \left( 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \right) - \Lambda_M^2 \left( \Lambda_M a^2 \Delta_r \sin^2 \theta \right) \left( \frac{d\phi}{dt} \right)^2 \]

\[\left( r^2 + a^2 \right) \left( 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \right) - \Lambda_M^2 \Lambda_r \left( a \sin^2 \theta \right) \left( \frac{d\phi}{dt} \right)^2 \]

(41)

It is observed from equation (41) that

\[ \frac{dr}{dt} = 0 \quad \text{for} \quad r_\pm = m \pm \sqrt{m^2 - (a^2 + A_0)} \]

(42)

where \[ A_0 = \frac{\Lambda}{3} \left( \frac{1}{r_+^4} + \frac{1}{r_-^2} a^2 \right) \]

This means at \( r = r_+ \) and \( r = r_- \), a light signal has zero coordinate speed. Hence a magnetized Kerr-de Sitter black hole has two event horizons. It is observed that when \( \Lambda = 0 \), the magnetized Kerr-de Sitter black hole and Kerr black hole has two equivalent event horizons.
### 7.5 DERIVATION OF SAGNAC EFFECT IN MAGNETIZED KERR-DE SITTER METRIC


In this paper we derive the Sagnac effect in magnetized Kerr-de Sitter metric.

For \( r = R \) (constant) and \( \theta = \frac{\pi}{2} \) (constant), equation (17) becomes (with \( \phi = \bar{\phi} \))

\[
\begin{align*}
\frac{ds^2}{\Lambda M_R^2} &= \frac{\Delta_R^2}{R^2 \left( 1 + \frac{\Lambda a^2}{3} \right)^2} - \frac{a^2}{R^2 \left( 1 + \frac{\Lambda a^2}{3} \right)^2} dt^2 - \Lambda M_R^{-2} \frac{(R^2 + a^2)^2}{R^2 \left( 1 + \frac{\Lambda a^2}{3} \right)^2} - \frac{\Lambda^2 M_R^4 \Delta_R^2}{R^2 \left( 1 + \frac{\Lambda a^2}{3} \right)^2} d\bar{\phi}^2 \\
-2A M_R^2 &= \frac{(R^2 + a^2)^2}{R^2 \left( 1 + \frac{\Lambda a^2}{3} \right)^2} - \frac{\Delta R}{R^2 \left( 1 + \frac{\Lambda a^2}{3} \right)^2} d\bar{\phi} dt
\end{align*}
\]

(43)

Here we write \( \Lambda M_s = 1 + \frac{B_0^2}{4} R^2 \) and \( \Delta_R = -\frac{\Lambda}{3} R^2 (R^2 + a^2) + R^2 - 2mR + a^2 \)

We apply the transformation
\[ \bar{\phi} = \phi + \Omega t, \] (44)

where \( \Omega \) is an angular speed.

From equation (43) and equation (44),

\[
ds^2 = \Lambda M_s^{-2} \left\{ \frac{\Delta_R - a^2}{R^2 \left( 1 + \frac{\Delta a^2}{3} \right)^2} \right\} - \Lambda M_s^{-2} \left\{ \frac{(R^2 + a^2)^2 - a^2 \Lambda M_s^4 \Delta_R}{R^2 \left( 1 + \frac{\Delta a^2}{3} \right)^2} \right\} \Omega^2 \\
- 2a \Omega \left\{ \frac{(R^2 + a^2) - \Lambda M_s^2 \Delta_R}{R^2 \left( 1 + \frac{\Delta a^2}{3} \right)^2} \right\} dt^2 - \Lambda M_s^{-2} \left\{ \frac{(R^2 + a^2)^2 - a^2 \Lambda M_s^4 \Delta_R}{R^2 \left( 1 + \frac{\Delta a^2}{3} \right)^2} \right\} d\phi^2 \\
- 2 \Lambda M_s^{-2} \left\{ \frac{(R^2 + a^2)^2 - a^2 \Lambda M_s^4 \Delta_R}{R^2 \left( 1 + \frac{\Delta a^2}{3} \right)^2} \right\} \Omega \left\{ \frac{(R^2 + a^2) - \Lambda M_s^2 \Delta_R}{R^2 \left( 1 + \frac{\Delta a^2}{3} \right)^2} \right\} a d\phi dt
\]

\[= Ud\Omega^2 - Vd\phi^2 - Wd\phi dt\]
\[
U = \Lambda_{M_5}^{-2} \left[ \frac{\Delta_k - a^2}{R^2 \left( 1 + \frac{\Lambda}{3} a^2 \right)^2} \right] - \Lambda_{M_5}^{-2} \Omega \left[ \frac{\left( R^2 + a^2 \right)^2 - \Lambda_{M_5}^4 \Delta_k a^2}{R^2 \left( 1 + \frac{\Lambda}{3} a^2 \right)^2} \right] \\
- 2a \Omega \left[ \frac{\left( R^2 + a^2 \right) - \Lambda_{M_5}^2 \Delta_k}{R^2 \left( 1 + \frac{\Lambda}{3} a^2 \right)^2} \right] 
\] (46)

\[
V = \Lambda_{M_5}^{-2} \left[ \frac{(R^2 + a^2)^2 - a^2 \Lambda_{M_5}^4 \Delta_k}{R^2 \left( 1 + \frac{\Lambda a^2}{3} \right)^2} \right] 
\] (47)

\[
W = 2 \Lambda_{M_5}^{-2} \Omega \left[ \frac{(R^2 + a^2)^2 - a^2 \Lambda_{M_5}^4 \Delta_k}{R^2 \left( 1 + \frac{\Lambda a^2}{3} \right)^2} \right] + a \left[ \frac{(R^2 + a^2) - \Delta_k \Lambda_{M_5}^2}{R^2 \left( 1 + \frac{\Lambda a^2}{3} \right)^2} \right] 
\] (48)

From equation (45), the proper time interval is

\[
d\tau = U^{1/2} dt
\]

\[
= \frac{ds}{dt} \frac{ds}{d\phi} + \frac{V}{U^{1/2}} \frac{d\phi}{dt} + \frac{W}{U^{1/2}} \frac{d\phi}{dt} \] (49)
From equation (49), for a beam of particles co-propagating along a semicircular path of radius R, we have

\[
\tau_1 = \int_0^\pi d\tau
\]

\[
= \int_0^\pi \frac{ds}{dt} \frac{d\phi}{U^{1/2}} + \int_0^\pi \frac{V}{U^{1/2}} \frac{d\phi}{dt} + \int_0^\pi \frac{W}{U^{1/2}} d\phi
\]

\[
= A \text{ (1st integral)} + B \text{ (2nd integral)} + \frac{W\pi}{U^{1/2}} \tag{50}
\]

Again from equation (49), for a similar beam of particles counter-propagating in a semicircular path of radius R,

\[
\tau_2 = \int_0^\pi d\tau
\]

\[
= \int_0^\pi \frac{ds}{dt} \frac{d\phi}{U^{1/2}} + \int_0^\pi \frac{V}{U^{1/2}} \frac{(-d\phi)}{dt} + \int_0^\pi \frac{W}{U^{1/2}} d\phi
\]

\[
= A \text{ (1st integral)} + B \text{ (2nd integral)} - \frac{W\pi}{U^{1/2}} \tag{51}
\]

From equation (50) and equation (51), the proper time-difference between two identical oppositely circulating beams of particles is given by

\[
\Delta \tau = \tau_1 - \tau_2 = \frac{2W\pi}{U^{1/2}}
\]
This shows that the Sagnac effect is governed by the magnetic field. If the magnetic field, $B_0 = 0$ and $\Lambda = 0$ we obtain the result derived by Ruggiiero (2005) and Krori et. al (2008).

### 7.6 CONCLUSIONS

In this paper, we have derived the magnetized Kerr-de Sitter metric using null tetrad method. It is seen that a Coriolis-like dragging is experienced by a particle moving in the field of a magnetized Kerr-de Sitter object. This effect is increased due to
the introduction of the magnetic field. We have also found that the magnetized Kerr-de Sitter black hole surface and simply Kerr-de Sitter black hole surface are equivalent. Magnetic field does not affect black hole. Also Sagnac effect has been derived in magnetized Kerr-de Sitter metric and it is affected by the magnetic field. In the absence of the magnetic field and cosmological constant \((B_0 = 0\text{ and } \Lambda = 0)\) we obtain the various properties of Kerr geometry.

7.7 REFERENCES


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