1. SCHAUDER DECOMPOSITIONS IN A LOCALLY $K$-CONVEX SPACE

Throughout we consider $(X, \mathcal{J})$ to be a locally $K$-convex space whose topology is generated by a class $\mathcal{D}$ of non-archimedean semi-norms.

**Definition 1.1**

Let $X$ be a locally $K$-convex space. A sequence $(M_n)$ of subspaces of $X$ is said to be a Schauder decomposition of $X$ if for each $x \in X$ there exists a unique sequence $(x_n) \subset X$ with $x_n \in M_n$, $n \geq 1$ such that

$$x = \sum_{n=1}^{\infty} x_n$$

the convergence of the series being in the given topology of $X$. The unique expansion of $x$ in the above form gives raise to a sequence of projections $(P_n)$ on $X$ defined by $P_n(x) = x_n$, where $x = \sum_{n=1}^{\infty} x_n$. These projections are called the associated projections associated with the decomposition $(M_n)$.

If these associated sequence of projections are continuous then we say that $(M_n)$ is a Schauder decomposition of $X$. By definition of $P_n$, $n \geq 1$ it follows that

$$M_n = \text{Range } P_n = \mathbb{R}(P_n).$$

Associated with these projections $(P_n)$
one can define a sequence of projections \((Q_n)\) on \(X\) defined by

\[
Q_n(x) = \sum_{i=1}^{n} P_i(x) \quad n \geq 1
\]

If these sequence of projections \((Q_n)\) is equi-continuous then the decomposition \((M_n)\) is said to be on equi-Schauder decomposition of \(X\).

**Definition 1.2**

A sequence \((P_i)\) of projections on a locally \(K\)-convex space is said to be orthogonal if \(P_i P_j = 0\) for all \(i, j\) with \(i \neq j\). By the definition of decomposition we have that the projections associated with a decomposition are orthogonal.

We now prove

**Theorem 1.3**

Let \(X\) be a \(\sigma\)-quasi-barreled locally \(K\)-convex space. Let \((P_n)\) be a sequence of orthogonal projections on \(X\). Then \((M_n = R(P_n), P_n)\) is a Schauder decomposition of \(SP \cup M_n\) if and only if \((Q_n)\) is point-wise bounded sequence, where \(Q_n = \sum_{i=1}^{n} P_i\)
Proof:

Let \((M_n, P_n)\) be a Schauder decomposition of 

\[ x = \lim_{n \to \infty} Q_n(x) \]

Consider any \(p \in D\). Then we have 

\[ p \left( Q_n(x) \right) = p \left( \prod_{\nu} P_{n, \nu}(x) \right) \]

\[ \implies p(x) = \lim p \left( Q_n(x) \right) \]

i.e. \(p \left( Q_n(x) \right) \leq I\), for some \(I\),

\[ \implies Q_n\text{'s are pointwise bounded} \]

Conversely assume that \(\{Q_n\}\) is pointwise bounded,

\[ \implies \{Q_n\}\text{'s are equi-continuous sequence of projections.} \]

\[ \implies \text{For each } p \in D \text{ there exists a } q \in D \text{ such that } p \left( Q_n(x) \right) \leq q(x) \]

Now let 

\[ Y = \{ x \in X : Q_n(x) \to x \} \]

Now \(Q_n\text{'s are equi-continuous} \implies Y\text{ is a closed}\]

subspace of \(X\). By definition we have each 

\[ M_n \subset Y \implies U M_n \subset Y \implies Sp \cup (M_n) \subset Y \]

Again \(x \in Y \implies x = \lim Q_n(x) \implies x \in Sp \cup M_n\).
Hence $Y = \overline{\text{Sp} \cup M_n}$. Hence $(M_n)$ is a decomposition of $X$.

Again we have that $Q_n$'s are equicontinuous $\Rightarrow P_1$'s are continuous and hence $(M_n, P_n)$ is a Schauder decomposition of $X$.

2. ORTHOGONAL DECOMPOSITIONS

**Definition 2.1**

Let $X$ be a locally $K$-convex space whose topology is generated by a class $D$ of seminorms. A decomposition $(M_n, P_n)$ of $X$ is said to be an orthogonal decomposition of $X$ if for each $p \in D$,

$$p(x) = \max p(x_i), \text{ where } x = \sum_{i=1}^{\infty} x_i \in X.$$  

This concept of orthogonal decompositions is a natural generalization of the concept of orthogonal basis as considered by Kimpe(§).

**Definition 2.2**

Let $X$ be a locally $K$-convex space whose topology is generated by a class $D$ of seminorms. Let

$$Y = \left\{ (x_i) \mid \sum_{i=1}^{\infty} x_i \text{ converges in } X, x_i \in M_i, i \geq 1 \right\}$$

We equip $Y$ with the locally $K$-convex topology $\mathcal{F}_i$ generated by the class $D^* = \left\{ p^* \mid p \in D \right\}$, where

$$p^*(x_i) = \sup p \left( \sum_{i=1}^{n} x_i \right),$$

where $(x_i) \in Y$.
We now prove

**Proposition: 2.3**

Let $X$ be a locally $K$-convex space with an decomposition $(M_n)$. If $(M_n)$ is an orthogonal decomposition of $X$ then $(N_n)$ is an orthogonal decomposition of $(Y, J)$, where for each $i$, $N_i = \{ \delta_i^x \mid x \in X \}$ and $\delta_i^x = (y_j)$, $y_j = 0$, $i \neq j$ and $y_i = x$.

**Proof:**

We have $N_i = \{ \delta_i^x \mid x \in X \}$

For any $\bar{x} = (x_i) \in Y$ one has $\bar{x}^{(n)} = \sum_{i=1}^{n} \delta_i^x$

Now consider any $p \in D$. By the convergence of $\sum \delta_i x_i$ it follows that

$$p^* (\bar{x}^{(n)} - \bar{x}) = p^* (0, 0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots)$$

$$= \sup_{m} p (\sum_{n+1}^{m} x_i)$$

$$\Rightarrow 0 \text{ as } n \to \infty$$

$$\Rightarrow \bar{x}^{(n)} \to \bar{x} \Rightarrow \bar{x} = \sum \delta_i x_i$$

Again $(M_n)$ is a decomposition of $X \Rightarrow$ the expansion of $\bar{x}$ in the above form is unique. Hence $(N_i)$ is a decomposition of $Y.$
Now for each \( p* \in D* \), we have

\[
p*(x) = \sum_{i=1}^{n} \delta_i
\]

\[
= \text{Sup } p(\sum_{i=1}^{n} x_i)
\]

\[
= \text{Max } p(x_i) \quad \text{(since } (M_n) \text{ is orthogonal)}
\]

\[
= \text{Max } p*(\delta_i)
\]

Hence \((N_i)\) is an orthogonal decomposition of \((Y, \tau_Y)\).

We will now prove a result which gives a relationship between orthogonal decompositions and best approximation. We start with

**Definition: 2.4**

Let \( X \) be a n.a Banach space and \((M_n, P_n)\) be a Schauder decomposition for \( X \). Then for each \( x \in X \) there exists a unique sequence \((x_i) \in \wedge(x)\) with \( x_i \in M_i, \ i \geq 1 \) such that

\[
x = \sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} P_i(x)
\]

For each \( n \geq 1 \), we define,

\[
Q_n(x) = \sum_{i=1}^{n} P_i(x), \quad Q^{(n)}(x) = \sum_{i=n+1}^{\infty} P_i(x)
\]

\[
I_n = \left( \bigcup_{i=1}^{n} M_i \right), \quad I(n) = \left( \bigcup_{i=n+1}^{\infty} M_i \right)
\]
Definition 2.5

If $Z$ is a linear subspace of $E$ then $z_0 \in Z$ is said to be a best approximation of $x \in E$ from $Z$ if

$$\| x - z_0 \| = \inf \left\{ \| x - z \| : z \in Z \right\}.$$  We shall denote by $B_x(x)$, the elements of best approximation of $x$ from $Z$.

We denote $B_z(x) = B_{L_1}(x)$, $B(z) = B_{L_1}(x)$ etc.

We now prove

Proposition 2.6

If $(M_n, P_n)$ is an orthogonal decomposition of $X$ then $R_n(z) \in B_n(z)$ for all $z$ and for all $x$.

Proof:

Let $(M_n, P_n)$ be an orthogonal decomposition of $X$.

For each $n \geq 1$, $\| \sum_{i=1}^{n} x_i \| = \max_{1 \leq i \leq n} \| x_i \|$. Consider any $z \in X$ $\Rightarrow$ $\lim Q_n(z)$.

We have

$$\| z - R_n(z) \| = \| Q_n(z) \| \leq \max_{1 \leq i \leq n} \| P_i(z) \| \leq \max \left( \| P_i(z) \|, \| \sum_{n+1}^{\infty} (y_{n+1} - P_i(z)) \| \right) = \| z - y \| \text{ for any } y = \sum_{n+1}^{\infty} y_i \in L(n)$$
\[ \Rightarrow \| z - R_n(z) \| \leq \| z - y \| \text{ for all } y \in L(n) \]

\[ \Rightarrow \| z - R_n(z) \| \leq \inf_{y \in L(n)} \| z - y \| \]

\[ \Rightarrow R_n(z) \in B(n)(z) \]

3. **ASSOCIATED VECTOR VALUED SEQUENCE SPACE**

**Definition 3.1**

Let \( X \) be a locally convex space and \((M_i, P_i)\) be a Schauder decomposition of \( X \). Now \((M_i, P_i)\) is a Schauder decomposition of \( X \) implies \( x = \sum_{i=1}^{\infty} P_i(x) \). for each \( x \in X \)

\[ \Rightarrow f(x) = \sum_{i=1}^{\infty} f(P_i(x)) \text{ for each } f \in X^* \]

\[ = \sum_{i=1}^{\infty} f_i(P_i^*(f))(x) \]

\[ \Rightarrow \text{Range } (P_i^*P_i^*) \text{ is a } \sigma(X^*, X) \text{ decomposition of } X \]

Again \( P_i^* \)'s are continuous \( \Rightarrow P_i^* \) is continuous in \( \sigma(X^*, X) \). Hence we have that \((\text{Range } P_i^*, P_i^*)\) is a \( \sigma(X^*, X) \) Schauder decomposition of \( X^* \). Hence associated with a Schauder decomposition \((M_i, P_i)\) one can define the VWSS \( \wedge(X) \) and \( \triangle(X^*) \) defined by \( \alpha_j \).
\( \bigwedge(X) = \left\{ (x_i) \mid x_i \in X, i \geq 1 \text{ and } \sum x_i \text{ converges in } X \right\} \)

\[ = \left\{ (p_i(x)) \mid x \in X \right\} \]

\( \triangle(X^*) = \left\{ (f_i) \mid f_i \in X^*, i \geq 1 \text{ and } \sum f_i \text{ converges in } (X^*, \sigma(X^*, X)) \right\} \)

From the definition of \( \bigwedge(X) \) we have

**Proposition 3.2**

The spaces \( \bigwedge(X) \) and \( \triangle(X^*) \) are algebraically isomorphic to \( X \) and \( (X^*, \sigma(X^*, X)) \) respectively.

Now we prove

**Proposition 3.3**

If \( \bigwedge(X) \) and \( \triangle(X^*) \) are TVSS as defined above then

\( (i) = \triangle(X^*) \subset \bigwedge(X^*) \) and

\( (ii) \quad \text{If } (X^*, \sigma(X^*, X)) \text{ is sequentially complete} \)

then \( \triangle(X^*) = \bigwedge(X^*) \).

**Proof:**

(1) Let \( (f_i) \in \triangle(X^*) \implies f_i = p_i^*(f) \)

where \( f = \sum_{i=1}^{\infty} f_i \)

Again for any \( (x_i) \in \bigwedge(X) \) we have that

\[ x = \sum_{i=1}^{\infty} x_i \in X \text{ and } \implies x_i = p_i(x), i \geq 1 \]
Now consider \( f_i(x_1) = (P_i^* f) (x_1) \)
\[
= f(P_i(x))
\]
\[
\rightarrow 0 \text{ as } i \rightarrow \infty
\]

since \( x = \Sigma P_i(x) \implies f(x) = \Sigma f(P_i(x)) < \infty \).

Hence we have that \((f_1) \in \bigwedge^X (X^*)\)

i.e. \( \triangle(X^*) \subset \bigwedge^X (X^*) \).

(ii) Assume that \( X^* \) is \( \sigma(X^*, X) \) sequentially complete.

Consider any \((f_i) \in \bigwedge^X (X^*)\).

\[
\lim_{i \to \infty} f_i(x_i) = 0 \text{ for all } (x_i) \in \bigwedge(x)
\]

Now consider \( \Sigma f_i \) we have for any \( x \in X, x = \Sigma x_i, x_i \in \mathcal{M}_i \Rightarrow (x_i) \in \bigwedge(X) \).

\[
\lim_{i \to \infty} f_i(x_i) = 0. \text{ Again } x = \Sigma x_i \Rightarrow x = \Sigma P_i(x).
\]

We have

\[
P_i^* (f_i(x)) = f_i (P_i(x)) = f_i(x_1) \rightarrow 0 \text{ as } i \rightarrow \infty (3.4)
\]

Again \((R(P_i^*), P_i)\) is weak Schauder decomposition of \( X^* \Rightarrow P_i^* (f_i(x_j)) = 0 \text{ for all } x_j \in \mathcal{M}_j, j \neq i. \)

\[
\Rightarrow P_i f_i = f_i, \ i \geq 1. \text{ Again by (3.4) it follows that}
\]
the sequence \( \{ s_n(x) \} \) of partial sums of \( \Sigma f_n \) forms a
\( \sigma^-(X^*, X) \) Cauchy sequence in \( X^* \) and hence converges to
a function \( f \) in \( \sigma^-(X^*, X) \). Hence \( (f_i) \in \Delta(X^*) \).

i.e. \( \wedge^X (X^*) = \Delta(X^*) \). Finally we prove

**Proposition 3.**

If \((X, \sigma^-(X, X^*))\) and \((X^*, \sigma^-(X^*, X))\) are
sequentially complete then \( \wedge(X) \) is a perfect sequence space.

**Proof:**

Let \( X \) and \( X^* \) be weakly sequentially complete.
Now \( X^* \) is weakly sequentially complete \( \implies \wedge^X (X^*) = \Delta(X^*) \)
Now \( X \) is weakly sequentially complete.

\[ \implies \wedge(X) \text{ is complete with respect to } \sigma^-(\wedge(X), \wedge^X (X^*)) \]

\[ = \sigma^-(\wedge(X), \Delta(X^*)) \]. Consider \( \tilde{y} = (y_i) \in \wedge^X (X) \)

\[ \implies \tilde{y}^{(n)} = \sum_{i=1}^{n} y_i \in \sigma^- (\wedge^X (X), \wedge^X (X^*)) \]

\[ \implies \tilde{y}^{(n)} \rightarrow \tilde{y} \text{ in } \sigma^- (\wedge^X (X), \wedge^X (X^*)) \]

Again for all \( n \), we have \( \tilde{y}^{(n)} \in \wedge(X) \implies \{ \tilde{y}^{(n)} \} \) is a
Cauchy sequence in \( (\wedge(X), \sigma^-(\wedge(X), \wedge^X (X^*)) \)

\[ \implies \tilde{y}^{(n)} \rightarrow \tilde{y} \in \wedge(X) \implies \wedge^X (X) \subset \wedge(X) \]

\[ \implies \wedge(X) = \wedge^X (X) \], as the other inclusion is always true.