Chapter 5

Comparison between Majorana Representation and Multiaxial Representation

5.1 Introduction

In this chapter, we study a comparison between MR and MAR for the $N$-qubit GHZ state to bring out the differences and similarities between the two representations. Depending on whether $N$ is odd or even we have different axes and classification for both representations in the case of $N$-qubit GHZ state. We show that pure state classification based on MR is not a special case of our classification scheme based on MAR.

5.2 MR of GHZ State

Consider symmetric $N$-qubit GHZ state

$$|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}} \left[ |\uparrow \cdots \uparrow_N \rangle + |\downarrow \cdots \downarrow_N \rangle \right] \equiv \frac{1}{\sqrt{2}} \left[ |jj\rangle + |j-j\rangle \right].$$

(5.1)

The MR polynomial equations (2.9) and (2.11), takes the form,

$$(-1)^{2j} Z^{2j} + 1 = 0$$

(5.2)

Depending on whether $N$ is odd or even we have the following solutions:

5.2.1 Odd $N$(Half odd integral $j$)

In this case $Z^{2j} = +1$ or

$$Z = e^{\frac{2\pi i r}{2j}}; \quad r = 0, 1, 2, \ldots, 2j - 1.$$
Thus the $2j$ distinct spinors characterizing $N$-qubit GHZ state are

$$\left(\frac{\pi}{2},0\right),\left(\frac{\pi}{2},\frac{2\pi}{2j}\right),\left(\frac{\pi}{2},\frac{4\pi}{2j}\right),...\left(\frac{\pi}{2},\frac{2(2j-1)\pi}{2j}\right). \tag{5.4}$$

### 5.2.2 Even $N$(integral $j$)

In this case $Z^{2j} = +1$ or

$$Z = e^{\frac{2\pi i}{2j}(r-\frac{1}{2})}, \quad r = 0, 1, 2, ..., 2j - 1. \tag{5.5}$$

Thus we have $2j$ distinct spinors namely

$$\left(\frac{\pi}{2},\frac{\pi}{2j}\right),\left(\frac{\pi}{2},\frac{3\pi}{2j}\right),\left(\frac{\pi}{2},\frac{5\pi}{2j}\right),...\left(\frac{\pi}{2},\frac{(4j-1)\pi}{2j}\right) \tag{5.6}$$

or equivalently $j$ distinct axes.

According to Bastin et al. (2009), $N$-qubit GHZ state belong to $D_{1,1,1...1}^{N}$ or equivalently $D_{2j}^{2j,1,1...1}$ for both odd and even $N$'s.

### 5.3 MAR of GHZ State

To find out the axes, consider the density matrix of $N$-qubit GHZ state in the $|jm\rangle$ basis; $m = +j...-j$

$$\rho_{GHZ} = \frac{1}{2} \begin{pmatrix} 1 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \ldots & 1 \end{pmatrix}. \tag{5.7}$$

$$t^{k}_{q} = \sum_{m=-j}^{+j} \rho_{mm'} [k] C(jk;jmqm'), \quad \text{with} \quad m' = m + q. \tag{5.8}$$

Since $\rho_{jj} = \rho_{j-j} = \rho_{-jj} = \rho_{-j-j} = \frac{1}{2}$ are the only non-zero matrix elements of $\rho_{GHZ}$, the $t^{k}_{q}$'s can be computed as,

$$t^{k}_{q} = 0, \quad \text{for all} \quad q \neq 0, 2j. \tag{5.9}$$
Further,

\[ t_k^0 = \rho_{jj}[^k]C(jkj; j0j) + \rho_{-jj}[^k]C(jkj; -j0 - j) = 0 \quad \text{for odd } k's. \] (5.10)

Here we have used the symmetry property of Clebsch-Gordan coefficients namely

\[ C(jkj; j0j) = (-1)^k C(jkj; -j0 - j). \]

Also

\[ t_k^0 = \frac{[k]}{2} (2j)! \left[ \frac{2j + 1}{(2j - k)! (2j + k + 1)!} \right]^{1/2}, \quad \text{for even } k's \] (5.11)

since

\[ C(abc; c0c) = (2c)! \left[ \frac{(2c + 1)}{(2c - b)! (2c + b + 1)!} \right]^{1/2} \]

(eq. (42) in page 252 of Varshalovich (1988)).

To write the polynomial equation for MAR, we compute \[ t_{2j}^{2j} \] as

\[ t_{2j}^{2j} = (-1)^{2j} t_{-2j}^{2j} = (-1)^{2j} \rho_{-jj} [2j] C(j2jj; j - 2j - j) \]

\[ = (-1)^{2j} \frac{[2j]}{2} \left[ \frac{(2j + 1)(4j)!}{(4j + 1)!} \right]^{1/2}. \] (5.12)

Here we have used the expression

\[ C(abc; a\beta\gamma) = \delta_{\gamma-\beta,a} \left[ \frac{(2c + 1)(2a)!(-a + b + c)!(b - \beta)!(c + \gamma)!}{(a + b + c + 1)!(a - b + c)!(a + b - c)!(b + \beta)!(c - \gamma)!} \right]^{1/2} \]

(eq. (36) in page 251 of Varshalovich (1988)).

As in the case of MR, here also we take up the case of odd \( N \) and even \( N \) separately.

### 5.3.1 Odd \( N \) (half odd integral \( j \))

Since \( t_k^0 = 0 \) for odd \( k's \) and \( t_k^0 \neq 0 \) for even \( k's \), there exist \( k \) axes collinear to \( z \)-axis as explained in sec. 3.2 for every even \( k \) \((k = 2, 4, 6 \ldots 2j - 1)\). Thus, the total number of axes collinear to \( z \)-axis, characterizing the odd \( N \)-qubit GHZ state is,

\[ 2 + 4 + 6 + \ldots + 2j - 1 = j^2 - \frac{1}{4}. \] (5.13)
Further, for the highest value of \( k \),

\[
P(Z) = \sqrt{\frac{4j}{C_{2j}}} t^{2j}_{2j} Z^0 + \sqrt{\frac{4j}{C_{0}}} t^{2j}_{-2j} Z^{4j} = 0,
\]

(5.14)

since \( t^{2j}_{2j} = -t^{2j}_{-2j} \), we have

\[
P(Z) = Z^{4j} - 1 = 0,
\]

(5.15)

\[
Z = e^{\frac{2\pi i r}{2j}}, \quad r = 0, 1 \ldots 4j - 1.
\]

(5.16)

There exist \( 4j \) solutions or \( 2j \) axes namely

\[
\left( \frac{\pi}{2}, 0 \right), \left( \frac{\pi}{2}, \frac{\pi}{2j} \right), \left( \frac{\pi}{2}, \frac{2\pi}{2j} \right) \ldots \left( \frac{\pi}{2}, \frac{4j - 1)\pi}{2j} \right).
\]

(5.17)

Therefore, the degeneracy configuration of the statistical tensor parameters are given by

\[
t^2 \in D^2_{2j}, \quad t^4 \in D^4_{4}, \ldots, t^{2j-1} \in D^{2j-1}_{2j-1}, \quad t^{2j} \in D^{2j}_{1,1,1\ldots 1_{2j}}
\]

(5.18)

Thus according to our classification, the degeneracy configuration of \( N \)-qubit GHZ state for odd \( N \) is

\[
\{D^2_{2j}, D^4_{4}, \ldots, D^{2j-1}_{2j-1}, D^{2j}_{1,1,1\ldots 1_{2j}}\}.
\]

5.3.2 Even \( N \) (integral \( j \))

Since \( t^k_0 \neq 0 \) for \( k = 2, 4, 6, \ldots, 2j - 2 \), there exist \( k \) axes collinear to \( z \)-axis. Thus, in this case the total number of axes collinear to the \( z \)-axis is,

\[
2 + 4 + 6 + \ldots + 2j - 2 = j(j - 1)
\]

(5.19)

The polynomial equation for the highest \( k \) is,

\[
P(Z) = \sqrt{\frac{4j}{C_{2j}}} t^0_0 Z^{2j} + \sqrt{\frac{4j}{C_{4j}}} t^{2j}_{2j} Z^0 + \sqrt{\frac{4j}{C_{0}}} t^{2j}_{-2j} Z^{4j} = 0.
\]

(5.20)
Since in this case $t_{2j}^2 = t_{-2j}^2$, we have

$$P(Z) = \sqrt{\frac{2j}{C_{2j}}} t_{0}^{2j} Z^{2j} + t_{2j}^{2j} (Z^{4j} + 1) = 0. \tag{5.21}$$

Substituting $t_{0}^{2j}$ and $t_{2j}^{2j}$ from (Eq. 5.11) and (Eq. 5.12) respectively,

$$P(Z) = [\frac{(4j)!}{(2j)!(2j)!}]^{1/2} 2 (2j)! [\frac{2j + 1}{(4j + 1)!}]^{1/2} Z^{2j} + \left[\frac{(4j)! (2j + 1)}{(4j + 1)!}\right]^{1/2} (Z^{4j} + 1) = 0 \tag{5.22}$$

which leads to

$$P(z) = Z^{4j} + 2Z^{2j} + 1 = 0. \tag{5.23}$$

Thus

$$(Z^{2j} + 1)^2 = 0,$$

$$Z = e^{\frac{2\pi i}{2j}(r - \frac{1}{2})}, \quad r = 0, 1, ..., 2j - 1 \tag{5.24}$$

There exist two identical sets of solutions or $j$ axes namely

$$\left(\frac{\pi}{2}, \frac{\pi}{2j}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2j}\right), \left(\frac{\pi}{2}, \frac{5\pi}{2j}\right)... \left(\frac{\pi}{2}, \frac{(4j - 1)\pi}{2j}\right). \tag{5.25}$$

Therefore, the degeneracy configuration of the statistical tensor parameters are given by

$$t^2 \in D^2_2, \quad t^4 \in D^4_4 \quad t^{2j-2} \in D^{2j-2}_{2j-2}, \quad t^{2j} \in D^{2j}_{2, 2, \ldots, 2} \tag{5.26}$$

Thus according to our classification the degeneracy configuration of $N$-qubit GHZ state for even $N$ is

$$\{D^2_2, D^4_4, D^6_6, ..., \underbrace{D^{2j-2}_{2j-2}}_y, \underbrace{D^{2j}_{2, 2, \ldots, 2}}_y\}$$

Let us now consider The MR and MAR of the 3-qubit and 4-qubit GHZ states.
5.4 Some examples

MR of 3-qubit GHZ state:

Consider

\[ |\psi_{\text{GHZ}}\rangle = \frac{|\frac{3}{2}, \frac{3}{2}\rangle + |\frac{3}{2}, -\frac{3}{2}\rangle}{\sqrt{2}} = \frac{|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}. \]

Since \( N \) is odd, according to (Eq. 5.2), the polynomial equation is given by \( Z^3 = 1 \) and the three distinct spinors are,

\[ \left( \frac{\pi}{2}, 0 \right), \left( \frac{\pi}{2}, \frac{2\pi}{3} \right), \left( \frac{\pi}{2}, \frac{4\pi}{3} \right). \]  \hspace{1cm} (5.27)

Thus,

\[ |\psi_{\text{GHZ}}\rangle \in D_{1,1,1}^3. \]

Spinors characterizing the MR of the 3-qubit GHZ state are shown in figure 5.1.
Figure 5.1: MR of the 3-qubit GHZ state.

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**MAR of 3-qubit GHZ state:**

Corresponding density matrix for 3-qubit GHZ state is

\[
\rho_{GHZ} = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
\end{pmatrix}.
\]  \hspace{1cm} (5.28)

The non-zero \(t^k_q\)'s from (Eq. 1.11) are:

\[
t_0^2 = \rho_{\pi/2} \sqrt{5} C(\frac{2}{2}; \frac{3}{2}, \frac{3}{2}; \frac{0}{2}, \frac{-3}{2}, \frac{-3}{2}) + \rho_{-\pi/2} \sqrt{5} C(\frac{2}{2}; \frac{-3}{2}, \frac{-3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}) = 1 \]  \hspace{1cm} (5.29)

\[
t_3^3 = \rho_{\pi/2} \sqrt{7} C(\frac{2}{3}; \frac{3}{2}, \frac{3}{2}, \frac{-3}{2}, \frac{-3}{2}, \frac{3}{2}) = -1 \]  \hspace{1cm} (5.30)

\[
t_{-3}^3 = \rho_{\pi/2} \sqrt{7} C(\frac{2}{3}; \frac{-3}{2}, \frac{-3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{-3}{2}) = 1. \]  \hspace{1cm} (5.31)

Solving the polynomial equation for \(t_q^3\); \(q = 3, -3\) (Eq. 5.14), we have

\[
Z = e^{2\pi i r}, \quad r = 0, 1, ..., 5
\]  \hspace{1cm} (5.32)

Thus, the three distinct axes are:

\[
\{(\frac{\pi}{2}, 0), (\frac{\pi}{2}, \pi)\}, \{(\frac{\pi}{2}, \frac{\pi}{3}), (\frac{\pi}{2}, \frac{4\pi}{3})\}, \{(\frac{\pi}{2}, \frac{2\pi}{3}), (\frac{\pi}{2}, \frac{5\pi}{3})\}.
\]  \hspace{1cm} (5.33)

Also, since \(t_0^2 = 1\), there exist two axes collinear to \(z\)-axis.

Therefore, \(t^2 \in \mathcal{D}_2\), \(t^3 \in \mathcal{D}_{1,1,1}\) and \(\rho \in \{\mathcal{D}_2, \mathcal{D}_{1,1,1}\}\).

Axes characterizing MAR of 3-qubit GHZ state are shown in figure 5.2
Figure 5.2: MAR of $t^2$ and $t^3$ characterizing the 3-qubit GHZ state.
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**MR of 4-qubit GHZ state:**

Corresponding 4-qubit GHZ state in $|2m\rangle$ representation is, $|\psi_{GHZ}\rangle = |2,2\rangle + |2,-2\rangle$. Since $N$ is even, the polynomial equation is given by $Z^4 = 1$ which leads to

$$Z = e^{\frac{2\pi i}{4}(r-\frac{1}{2})}, \quad r = 0, 1, 2, 3 \quad (5.34)$$

We get four distinct spinors or equivalently two distinct axes

$$\left(\frac{\pi}{2}, \frac{\pi}{4}\right), \left(\frac{\pi}{2}, \frac{3\pi}{4}\right), \left(\frac{\pi}{2}, \frac{5\pi}{4}\right), \left(\frac{\pi}{2}, \frac{7\pi}{4}\right) \quad (5.35)$$

Thus,

$$|\psi_{GHZ}\rangle \in D_{1,1,1,1}^4.$$  

Spinors characterizing MR of 4-qubit GHZ state are shown in figure 5.3.
Figure 5.3: MR of the 4-qubit GHZ state.
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**MAR of 4-qubit GHZ state:**

Corresponding density matrix for 4-qubit GHZ state is

$$\rho_{GHZ} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (5.36)

The only non-zero $t^k_q$'s are:

$$t^2_0 = \sqrt{10}, \quad t^4_0 = \frac{1}{\sqrt{14}}, \quad t^4_4 = \frac{\sqrt{5}}{2}, \quad t^4_{-4} = \frac{\sqrt{5}}{2}. \hspace{1cm} (5.37)$$

Since $N$ is even, solving the polynomial equation for $t^k_q; q = 0, 4, -4$ we get

$$Z = e^{\frac{2\pi i}{4} (r - \frac{1}{2})}, \quad r = 0, 1, 2, 3 \hspace{1cm} (5.38)$$

Thus we get two sets of two distinct axes and the axes are given by

$$\left\{ \left( \frac{\pi}{2}, \frac{\pi}{4} \right), \left( \frac{\pi}{2}, \frac{3\pi}{4} \right) \right\}, \left\{ \left( \frac{\pi}{2}, \frac{5\pi}{4} \right), \left( \frac{\pi}{2}, \frac{7\pi}{4} \right) \right\}. \hspace{1cm} (5.39)$$

In this case $t^2 \in D^2_2$ and $t^4 \in D^4_{2,2}$, thus

$$\rho \in \{D^2_2, D^4_{2,2} \}.$$

Axes characterizing MAR of 4-qubit GHZ state are shown in figure 5.4.
Figure 5.4: MAR of $t^2$ and $t^4$ characterizing the 4-qubit GHZ state.
Thus it is evident that MR is not a special case of MAR. One can also note that the basic entity characterizing MR is a spinor and MAR is an axis.