Chapter 2

Majorana Representation (MR) and classification of Symmetric Pure States

2.1 Introduction

A way of representing the pure states in terms of symmetric states of $2j = N$ spin-$1/2$ particles was introduced by Majorana (1932). This representation, known as the Majorana Representation (MR), makes it possible to express spin-$j$ states geometrically as $2j$ points on the Bloch sphere.

A pure spin $j = \frac{N}{2}$ quantum state can be represented as a symmetrized combination of $N$ constituent spinors [Usha Devi (2012)], i.e.,

$$|\psi_{\text{sym}}\rangle = \mathcal{N} \sum_{\hat{P}} \hat{P} |\epsilon_1, \epsilon_2, \ldots, \epsilon_N\rangle$$  \hspace{1cm} (2.1)

where

$$\epsilon_r = \cos\left(\frac{\beta_r}{2}\right)|0\rangle + \sin\left(\frac{\beta_r}{2}\right)e^{+i\alpha_r}|1\rangle,$$

and $r = 0, 1, \ldots, N$ denote the spinors constituting the symmetric state $|\psi_{\text{sym}}\rangle$; $\hat{P}$ corresponds to the set of all $N!$ permutations of the spinors and $\mathcal{N}$ corresponds to an overall normalization factor.

Majorana’s geometric realization of a spin-$j$ state has been used extensively in diverse branches of physics [Bloch & Rabi (1945); Dennis (2004); Zimba (2006)] and in quantum information science [Aulbach et al. (2010a); Bastin et al. (2009, 2010); Markham (2011); Martin et al. (2010); Usha Devi et al. (2010)] particularly. It has been used to study the quantumness of pure quantum states in some aspects [Giraud et al. (2010); Zimba (2006)] as
well as to characterize the entanglement of permutationally symmetric states under SLOCC [Bastin et al. (2009); Markham (2011); Mathonet et al. (2010)].

Two states which are SLOCC equivalent can be used to perfume the same quantum computational tasks [Dur et al. (2000)]. For any two $N$-partite states $|\psi\rangle$ and $|\psi'\rangle$, it is well known from [Dur et al. (2000)] that $|\psi\rangle$ is equivalent to $|\psi'\rangle$ under SLOCC if and only if

$$|\psi\rangle = A_1 \otimes A_2 \otimes \ldots \otimes A_N |\psi'\rangle$$

(2.2)

where $A_1 \otimes A_2 \otimes \ldots \otimes A_N$ are Invertible Local Operators (ILO). When two states $|\psi\rangle$ and $|\psi'\rangle$ are symmetric with respect to the permutations of the parties, it is sufficient to look for a symmetric ILO. i.e;

$$|\psi_{sym}\rangle = A \otimes A \otimes \ldots \otimes A |\psi'_{sym}\rangle$$

(2.3)

where the same ILO acts on each qubit.

Bastin et al. (2009) made use of MR for classification of symmetric $N$-qubit pure states into SLOCC inequivalent classes. The classification is based on the parameters namely, degeneracy number and degeneracy configuration [Bastin et al. (2009)]. In other words, two symmetric states $|\psi_{sym}\rangle = \mathcal{N} \sum_p \hat{P} |\epsilon_{i_1}, \epsilon_{i_2}, \ldots \epsilon_{i_N}\rangle$ and $|\psi'_{sym}\rangle = \mathcal{N} \sum_p \hat{P} |\epsilon'_{i_1}, \epsilon'_{i_2}, \ldots \epsilon'_{i_N}\rangle$ ($i = 1, 2, \ldots N$) belong to the same SLOCC class if and only if there exists a single ILO, $A$ which converts each of the spinors $|\epsilon_i\rangle$ to $|\epsilon'_i\rangle$. Here we briefly study SLOCC classification of Bastin et al. (2009). As examples we consider some well known two and three qubit symmetric states.

### 2.2 Majorana representation of symmetric pure states

We now study MR of the most general symmetric $N$-qubit pure states $|\psi_{sym}^N\rangle$ or spin-$j$ pure state $|\psi^j\rangle$ given by

$$|\psi^j\rangle = \sum_{m=-j}^{+j} a_m |jm\rangle$$

(2.4)
using the algebra of Wigner D-matrices. Let us consider a rotation $R(\phi, \theta, 0)$ of the frame of reference such that the expansion coefficient $a_{-j}$ in the rotated frame vanishes i.e;

$$(a_{-j})^R = 0 = \langle j - j | R^{-1}(\phi, \theta, 0) | \psi_j \rangle = \sum_m a_m (j - j | R^{-1}(\phi, \theta, 0) | jm)$$

$$= \sum_m a_m D^*_{m-j}(\phi, \theta, 0) = \sum_m a_m (-1)^{(j+m)} D^j_{-m,j}(\phi, \theta, 0), \quad (2.5)$$

where $D^j_{m', m}(\phi, \theta, 0)$ are the matrix elements of Wigner rotation matrices, given by [Rose (1957)]

$$D^j_{m', m}(\alpha \beta \gamma) = e^{-im'\alpha} e^{-im\gamma} \sum_s (-1)^s \sqrt{(j + m)! (j - m)! (j + m')! (j - m')!}$$

$$\times \left( \cos \frac{\beta}{2} \right)^{2j + m - m' - 2s} (-1)^{m' - m + 2s} \left( \sin \frac{\beta}{2} \right)^{m' + m - 2s}. \quad (2.6)$$

Thus

$$D^j_{-m,j}(\phi, \theta, 0) = e^{im\phi} (-1)^j m \sqrt{2j \binom{2j}{j+m}} \left( \cos \frac{\theta}{2} \right)^{j-m} (-1)^j m \left( \sin \frac{\theta}{2} \right)^{j+m}, \quad (2.7)$$

where $s = j + m$ and $2j \binom{2j}{j+m}$ is the Binomial Coefficient. (Eq. 2.7) becomes

$$\mathcal{A} \sum_{m=-j}^{+j} (-1)^j \sqrt{2j \binom{2j}{j+m}} a_m Z^{j+m} = 0 \quad (2.8)$$

where

$$Z = \tan \left( \frac{\theta}{2} \right) e^{i\phi}$$

and the overall coefficient

$$\mathcal{A} = \cos^{2j} \left( \frac{\theta}{2} \right) e^{-i\phi j}.$$

The Majorana polynomial $P(z)$ is given by

$$P(Z) = \sum_{m=-j}^{+j} (-1)^j \sqrt{2j \binom{2j}{j+m}} a_m Z^{j+m} = 0, \quad (2.9)$$

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for $\theta \neq \pi$.

Equivalently, from (Eq. 2.7) we can also have

$$A' \sum_{m=-j}^{+j} (-1)^{j+m} \sqrt{2j} C_{j+m} a_m Z'^{-m} = 0$$

(2.10)

where

$$A' = \sin^{2j} \left( \frac{\theta}{2} \right) e^{i\phi j}$$

and

$$Z' = \frac{1}{Z} = \cot \left( \frac{\theta}{2} \right) e^{-i\phi}.$$

We thus obtain,

$$P(Z') = \sum_{m=-j}^{+j} (-1)^{j-m} \sqrt{2j} C_{j+m} a_m Z'^{-m} = 0$$

(2.11)

for $\theta \neq 0$.

Solving either of the polynomial equations, one gets $2j$ solutions namely

$$\{ (\theta_1, \phi_1), (\theta_2, \phi_2), \ldots, (\theta_{2j}, \phi_{2j}) \}$$

in general. Thus every pure spin-$j$ state $|\psi^j\rangle$ or the corresponding symmetric state $|\psi^j_{sym}\rangle$ can be represented by a constellation of $2j$ points on the Bloch sphere or

$$|\psi^N_{sym}\rangle = \mathcal{N} \sum_{\hat{P}} \hat{P}(\epsilon_1, \epsilon_2, \ldots, \epsilon_{2j}),$$

(2.12)

where

$$|\epsilon_k\rangle = \cos(\theta_k/2)e^{-i\phi_k/2}|0\rangle + \sin(\theta_k/2)e^{i\phi_k/2}|1\rangle, \quad k = 0, 1, \ldots, 2j$$

(2.13)

refer to the $N$ spinors constituting the symmetric state $|\psi^N_{sym}\rangle$; $\hat{P}$ corresponds to the set of $(2j)!$ permutations of the spinors and $\mathcal{N}$ corresponds to an overall normalization factor.
2.3 Classification of symmetric Pure States based on Majorana representation

Entanglement classification of qubits based on LU [Kraus (2010a); Kraus (2010b)], SLOCC [Aulbach (2010c); Kolenderski (2010); Markham (2011); Mathonet et al. (2010)] and Local Operation and Classical Communication (LOCC) [Hayden et al. (2000)] has gained importance in recent times. The SLOCC classification of the permutationally symmetric $N$-qubit states makes use of the elegant, geometrical representation of the spin-$j$ states given by Majorana. According to SLOCC classification of symmetric pure states by Bastin et al. (2009), the number of identical spinors $|\epsilon_i\rangle$ in (Eq. 2.12) is called the degeneracy number. Further, the degeneracy configuration $D_{\{n_i\}}$ of a symmetric state $|\psi_{\text{sym}}^N\rangle$ is defined such that $\{n_i\}$ is the set of degeneracy numbers ordered in decreasing order by convention. The number of $n_i$’s defines the diversity degree of the symmetric state. For example, if all the $N$ spinors of a symmetric $N$-qubit pure state are identical, then

$$|\psi\rangle = \mathcal{N}|\epsilon\epsilon\epsilon\ldots\rangle$$

and the state is said to have the degeneracy configuration $D_N$ and diversity degree $d = 1$. Thus

$$|\psi\rangle \in D_N$$

and the state belongs to separable class.

Similarly, if all except two spinors are identical, there are two cases.

(1) If the two remaining spinors are identical, then

$$|\psi\rangle = \mathcal{N} \sum_P \hat{P}|\epsilon'\epsilon'\epsilon\ldots\rangle$$

and the state has the degeneracy configuration of $D_{N-2,2}$ with $d = 2$. Thus

$$|\psi\rangle \in D_{N-2,2}.$$
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(2) If the two remaining spinors are not identical, then

$$|\psi'\rangle = \mathcal{N} \sum_P \hat{P} |e'e''\epsilon...\rangle$$

and the state has the degeneracy configuration of $\mathcal{D}_{N-2,1,1}$ with $d = 3$. Thus

$$|\psi\rangle \in \mathcal{D}_{N-2,2}.$$ 

Therefore a separable symmetric $N$-qubit pure state has the degeneracy configuration of $\mathcal{D}_N$ and $d = 1$. 


2.3.1 Some examples

Bell State: Consider

$$|\psi_{Bell_1}\rangle = |10\rangle \equiv \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$$

which is a symmetric state.

Solving one of the polynomial equations (2.9) or (2.11) we get

$$Z = 0,$$

which shows that $\theta = 0$ and $\theta = \pi$. Thus $d = 2$ and $|\psi_{Bell_1}\rangle \in D_{1,1}$.

![Figure 2.1: MR of the Bell state $|\psi\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$.](image)
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**Bell State:** Consider

\[
|\psi_{\text{Bell}_2}\rangle = \frac{|11\rangle + |1 \iff 1\rangle}{\sqrt{2}} = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}.
\]

which is a symmetric state.

Solving the polynomial equation we get

\[Z^2 + 1 = 0.\]

Or

\[Z = e^{\pm i\pi/2}.\]

The corresponding axes are

\[(\pi/2, \pi/2), \ (\pi/2, 3\pi/2).\]

Thus \(d = 2\) and \(|\psi_{\text{Bell}_2}\rangle \in D_{1,1}\).

![Figure 2.2: MR of the Bell state \(|\psi\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}\).](image)
Bell State: Consider

\[ |\psi_{\text{Bell}_3}\rangle = \frac{|11\rangle - |1 - 1\rangle}{\sqrt{2}} \equiv \frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}}. \]

which is a symmetric state.

Solving the polynomial equation we get

\[ Z^2 - 1 = 0, \]

which shows that the corresponding axes are

\( \left( \frac{\pi}{2}, 0 \right), \left( \frac{\pi}{2}, \pi \right). \)

Thus \( d = 2 \) and \( |\psi_{\text{Bell}_3}\rangle \in D_{1,1}. \)

Figure 2.3: MR of the Bell state \( |\psi\rangle = \frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}}. \)
**W State:** Consider

\[ |\psi_W\rangle = |3/2 - 1/2\rangle = \frac{|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle}{\sqrt{3}}. \]

which is a symmetric state.

Solving the polynomial equation we get

\[ Z_{1,2,3} = 0. \]

The corresponding axes are

\[ \theta_1 = 0 \text{ and } \theta_{2,3} = \pi. \]

Thus \( d = 2 \) and \( |\psi_W\rangle \in \mathcal{D}_{2,1}. \)

**Figure 2.4:** MR of the W state \( |\psi\rangle = \frac{|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle}{\sqrt{3}}. \)
**GHZ State:** Consider

\[ |\psi_{GHZ}\rangle = \frac{\left|\frac{3}{2}, \frac{3}{2}\right\rangle + \left|\frac{3}{2}, -\frac{3}{2}\right\rangle}{\sqrt{2}} \equiv \frac{|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}. \]

which is a symmetric state.

Solving the polynomial equation we get

\[ Z^3 - 1 = 0. \]

Or

\[ Z = e^{\frac{2\pi ir}{3}}, \quad r = 0, 1, 2, \]

which shows that the corresponding axes are

\[ \left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, \frac{2\pi}{3}\right), \left(\frac{\pi}{2}, \frac{4\pi}{3}\right). \]

Thus \( d = 3 \) and \( |\psi_{GHZ}\rangle \in D_{1,1,1} \).

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**Figure 2.5:** MR of the GHZ state \( |\psi\rangle = \frac{|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}. \)
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Such a classification based on MR is valid for symmetric pure states only. Therefore we propose a novel scheme for the most general symmetric $N$-qubit pure as well as mixed states based on an equally elegant Multiaxial Representation (MAR) of the density matrix.