Chapter - II

UNSTEADY FLOW OF INCOMPRESSIBLE ELECTRICALLY CONDUCTING SECOND GRADE FLUID THROUGH A COMPOSITE MEDIUM IN A ROTATING PARALLEL PLATE CHANNEL BOUNDED BY A POROUS BED
2.1. Introduction:

It is well known that volume flow rate variation as well as rotation plays an important role in various phenomena in cosmical fluid dynamics, in meteorology, in geophysical fluid dynamics, in gaseous and nuclear reactors etc. Recently, the study of non-Newtonian fluid has gained considerable importance. This is mainly due to their several applications in petroleum industry, manufacturing and processing of foods and paper and many other industrial applications. Besides that the flows such fluids offer varied challenges to applied mathematicians, numerical analysts and modelers in developing suitable algorithms for computing the flows. It is because of the fact that governing flow equations of non-Newtonian fluids are much more complicated and of higher order than the Navier-Stokes equations. But due to complexity of fluids, several constitutive equations of non-Newtonian fluids have been proposed in the literature. Amongst these there is a subclass of non-Newtonian fluids namely the second grade fluids for which one can reasonably hope to obtain analytic solution. Some recent studies in this direction may be mentioned in [1, 3, 15, 16, 18 and 35]. Due to this fact in mind, the fluid model in the present paper is second grade.

Due to their application in industry and technology few problems in fluid mechanics have enjoyed the attention that has been accorded to the flow which involves non-Newtonian fluids. It is well known that mechanics of non-Newtonian fluids present a special challenge to engineers, physicists and mathematicians. The non-linearity can manifest itself in a variety of ways in many fields, such as food, drilling operations and bio-engineering. The Navier-Stokes theory is inadequate for such fluids and no single constitutive equation is available in the literature which exhibits the properties of all
fluids. Because of complex behavior many fluid models have been suggested. Amongst these, the fluids of visco-elastic type have received much attention. In fact interest in visco-elastic fluids goes back almost to 65 years, triggered by the discovery of Mysels [21] and Toms [36] who found that the addition of small amounts of a high molecular weight polymer to a Newtonian fluid in turbulent pipe flows resulted in a dramatic decrease in pressure drop. The second grade fluid model is the simplest subclass of visco-elastic fluids for which one can reasonably hope to obtain the analytic solution. Some typical works on the topic are given in the references [12, 13, 14, 20, 23, 24, 31-34]. Even though considerable progress has been made in our understanding of the flow phenomena, more works are needed to understand the effects of the various parameters involved in the non-Newtonian models and the formulation of an accurate method of analysis for anybody shapes of engineering significance.

Unsteady flow of non-Newtonian fluids, which is of vital interest, has also been investigated by several researchers and scientists. From literatures, the non-Newtonian fluids are principally classified on the basis of their behavior in shear. A fluid with a linear relationship between the shear stress and the shear rate, giving rise to a constant viscosity, is always characterized to be a Newtonian fluid. Based on the knowledge of solutions to Newtonian fluid, the different fluids can be extended, such as Maxwell fluid, Voigt fluid, Oldroyd-B fluid, Rivlin-Ericksen fluid or power-law fluid, or so called second grade fluid. Several authors are now engaged with the equations of motion of second grade fluid and also to the study of the existence and uniqueness of solutions for flows of volume rate variation and rotation case fluids [7, 8, 10, 17, 19, 22, 38 and 39].
Flow of fluids either Newtonian or non-Newtonian through porous media has been attracted considerable research activity (4-6, 25 and 27-29) because of its several important applications notably in the flow of oil through porous rock, the extraction of energy from the geothermal regions, the evaluation of the capability of heat removal from particulate nuclear in a nuclear reactor, the filtration of solids from liquids, flow of liquids through ion exchange beds, drug permeation through human skin, chemical reactors for the economical separation or purification of mixture and so on. Fluid flow through porous pipes has been studied theoretically and experimentally by numerous researchers [9 and 30), because of many applications in fields such as Diffusion technology, Transpiration cooling, Hemodialysis processes, Desalination, fluid control in Nuclear Reactor and numerous other fields. Several problems of interest in Industry involve flow of fluids through composite media consisting partially of porous bed. Examples include use of filtration to purify water and treat sewage, movement of fertilizers in the soil, transition zone between salt water and fresh water in coastal aquifers. Such problems involving the flow past a porous bed is also of immense use in Bio-medical problems involving transport process in lungs and kidneys excetra. This prompted several researches investigate the flow through in composite system with porous lining abutting the boundaries.

In this chapter, we consider the unsteady flow of an incompressible electrically conducting second grade fluid in a rigidly rotating parallel plate channel bounded below by a sparsely packed porous bed. The flow in nonporous region is governed by the equation of motion derived using the constitutive equation for the stress in compressible second order fluid, while the Brinkman’s model equation has been used for the
momentum equation in the porous bed. In the undisturbed state both the fluid and the plates are in rigid rotation with same angular velocity about the normal to the plates and at $t > 0$ the fluid is driven by the constant pressure gradient-parallel to the channel wall and in addition, the lower plate perform non-tortional oscillation in its own plane. Exact solution of the velocity in the clean fluid and the porous medium consists of steady state and transient state. The time required for the transient state to decay is evaluated and the ultimate quasi steady state solution has been derived analytically and its behaviour is computationally discussed with reference to the governing parameter. The shear stress on the boundary is obtained analytically and its behaviour is computationally discussed.
2.2. Formulation and Solution of the problem:

We consider the unsteady flow of an incompressible electrically conducting second grade fluid through a composite medium in a rotating parallel plate channel with upper rigid plate bounding a clean fluid and the lower plate bounding below a sparsely packed porous bed. The entire configuration is subjected to a uniform transverse magnetic field normal to the channel. In the initial undisturbed state both the fluid and the plates are in rigid rotation with the same angular velocity $\Omega$ about the normal to the plates and at $t > 0$ the fluid is driven by a constant pressure gradient parallel to the channel walls and in addition the lower plate perform non-torsional oscillations in its own plane.

We choose a Cartesian system $0(x, y, z)$ such that the boundary walls are at $z=0$ and $z=1$, z-axis being the axis of rotation of the plates. The fluid medium consists of two zones namely zone 1 and zone 2. Zone 1 consists of clean fluid governed by equations of motion, which are derived using constitutive equations for the stresses in compressible second order fluid while zone 2 corresponds to the flow through porous bed governed by Brinkman's equations. At the interface the fluid satisfies the continuity condition of velocity and shear stress. Since the plates extends to infinity along x and y directions, all the physical quantities except the pressure depend on z and t alone.

An incompressible simple fluid is defined as a material whose state of present stress is determined by the history of the deformation gradient without a preferred reference configuration [Truesdell. (37)]. Its constitutive equation can be written in the form of a functional.

$$T(t) = -pI + \sum_{j=0}^{\infty} F'_{ij}(S), \quad (2.2.1)$$

Where $pI$ is the undetermined part of the stress tensor and $F$ is the deformation gradient.
The constitutive equation for the stress $T$ in an incompressible fluid of second grade is given by

$$T(t) = -pI + \mu A_i + \alpha_1 A_2 + \alpha_2 A_i$$  \hspace{1cm} (2.2.2)

Where $\mu$ is the dynamic viscosity $\alpha_1$, $\alpha_2$ are the normal stress moduli and the kinematical tensors $A_i$ and $A_2$ are defined through [Rivlin et.al. (26)].

$$A_i = (\text{Grad } V) + (\text{grad } V)^T,$$

$$A_2 = \frac{dA_i}{dt} + A_i (\text{Grad } V) + (\text{grad } V)^T A_i,$$  \hspace{1cm} (2.2.3)

Where $V$ is the velocity, grad the gradient operator and $d/dt$ the material time derivative. Dunn and Fosdick [26] found that if the fluid modeled by (2.2.2) is to be compatible with thermodynamics, in the sense that all motions of the fluid meet the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid takes its minimum value in equilibrium, then the material moduli must satisfy

$$\mu \geq 0, \, \alpha_1 \geq 0, \, \alpha_1 + \alpha_2 = 0$$  \hspace{1cm} (2.2.4)

This, then, was shown to give to the theory a rather well behaved and pleasant stability and boundedness structure. It was also shown that if $\alpha_1$ was taken negative, the remainder of (2.2.4) being preserved, then in quite arbitrary flows instability and unboundedness were unavoidable. However, it is well known that for most non-Newtonian fluids of current rheological interest, conclusions (2.2.4) are contradicted by experiments. Fosdick and Rajagopal [16] used.

$$\mu \geq 0, \, \alpha_1 \leq 0, \, \alpha_1 + \alpha_2 \neq 0,$$  \hspace{1cm} (2.2.5)

Which were supposedly obtained by data reduction from experiments for those fluids which the experimentalists presumed to be constitutively described by (2.2.2) as a
second grade fluid, and they showed that such values for the material moduli led to anomalous behavior, thus questioning whether the fluid under consideration in the experiments could be described as a second grade fluid.

The unsteady hydro magnetic flow in a rotating co-ordinate system is governed by the equation of motion, continuity equation and the Maxwell equations in the form.

\[
\rho \left( \frac{\partial V}{\partial t} + (V \cdot \nabla) V + 2\Omega \times V + \Omega \times (\Omega \times r) \right) = \nabla \cdot T + J \times B \tag{2.2.6}
\]

\[
\nabla \cdot V = 0 \tag{2.2.7}
\]

\[
\nabla \cdot B = 0 \tag{2.2.8}
\]

\[
\nabla \times B = \mu_m J \tag{2.2.9}
\]

\[
\nabla \times E = -\frac{\partial B}{\partial t} \tag{2.2.10}
\]

Where, J is the current density, B is the total magnetic field, E is the total electric field, \( \mu_m \) is the magnetic permeability and r is radial co-ordinate given by \( r^2 = x^2 + y^2 \).

Making use of the governing equation (2.2.6), the constitutive equations and keeping in view of the flow configuration of the problem. The unsteady hydro magnetic equations governing the incompressible electrically conducting second grade fluid in zone 1 under the influence of transverse magnetic field with reference to a frame rotating with a constant angular velocity \( \Omega \) are

\[
\frac{\partial u}{\partial t} - 2\Omega v = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial z^2} + \frac{a_i}{\rho} \frac{\partial^3 u}{\partial z^2 \partial t} - \frac{\sigma \mu_e H_0^2}{\rho} u \tag{2.2.11}
\]

\[
\frac{\partial v}{\partial t} + 2\Omega u = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial z^2} + \frac{a_i}{\rho} \frac{\partial^3 v}{\partial z^2 \partial t} - \frac{\sigma \mu_e H_0^2}{\rho} v \tag{2.2.12}
\]

The Brinkman-equations governing the flow through porous medium with respect to the rotating frame zone 2.

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\[
\frac{\partial u_p}{\partial t} - 2\Omega v_p = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v_{\text{eff}} \frac{\partial^2 u_p}{\partial z^2} + \frac{a_1}{\rho} \frac{\partial^3 u_p}{\partial z^2 \partial t} - \frac{\sigma \mu_e^2 H_0^2}{\rho} u_p - \frac{v}{k} u_p \tag{2.2.13}
\]

\[
\frac{\partial v_p}{\partial t} + 2\Omega u_p = -\frac{1}{\rho} \frac{\partial p}{\partial y} + v_{\text{eff}} \frac{\partial^2 v_p}{\partial z^2} + \frac{a_1}{\rho} \frac{\partial^3 v_p}{\partial z^2 \partial t} - \frac{\sigma \mu_e^2 H_0^2}{\rho} v_p - \frac{v}{k} v_p \tag{2.2.14}
\]

where \((u, v)\) and \((u_p, v_p)\) are velocity components along \(O(x, y)\) directions respectively. \(\rho\) the density of the fluid, \(\sigma\) the conductivity of the medium, \(\mu_e\) the magnetic permeability, \(v\) the coefficient of kinematic viscosity, \(v_{\text{eff}}\) the coefficient of effective kinematic viscosity, \(k\) the permeability of the medium, \(H_0\) is the applied magnetic field. Hence \(u, v\) and \(u_p, v_p\) are function of \(z\) and \(t\) alone and hence the respective equations of continuity are trivially satisfied.

Let \(q = u + iv, q_p = u_p + iv_p\) and \(\xi = x - iy\)

Now combining equations (2.2.11) and (2.2.12), we obtain

\[
\frac{\partial q}{\partial t} + 2i\Omega q = -\frac{1}{\rho} \frac{\partial p}{\partial \xi} + v_{\text{eff}} \frac{\partial^2 q}{\partial z^2} + \frac{a_1}{\rho} \frac{\partial^3 q}{\partial z^2 \partial t} - \frac{\sigma \mu_e^2 H_0^2}{\rho} q \tag{2.2.15}
\]

and combining equations (2.2.13) and (2.2.14), we obtain

\[
\frac{\partial q_p}{\partial t} + 2i\Omega q_p = -\frac{1}{\rho} \frac{\partial p}{\partial \xi} + v_{\text{eff}} \frac{\partial^2 q_p}{\partial z^2} + \frac{a_1}{\rho} \frac{\partial^3 q_p}{\partial z^2 \partial t} - \frac{\sigma \mu_e^2 H_0^2}{\rho} q_p - \frac{v}{k} q_p \tag{2.2.16}
\]

The boundary and initial conditions are

\[
q_p = ae^{i\omega t} + be^{-i\omega t}, \quad z = 0 \tag{2.2.17}
\]

\[
q = 0, \quad t \neq 0, \quad z = 1 \tag{2.2.18}
\]

\[
q = 0, \quad q_p = 0, \quad t \leq 0, \quad \text{for all } z \tag{2.2.19}
\]

The interfacial conditions are

\[
\begin{align*}
q &= q_p, \\
\frac{\nu}{\partial z} &= v_{\text{eff}} \frac{\partial q_p}{\partial z} \\
\end{align*}
\]

at \(z = h\) \tag{2.2.20}
We introduce the following non-dimensional variables are

\[ z^* = \frac{z}{l}, q^* = \frac{q}{v}, q^*_p = \frac{q_p}{v}, t^* = \frac{tv}{l^2} \]

\[ \omega^* = \frac{\omega l^2}{v}, \xi^* = \frac{\xi}{l}, p^* = \frac{pl^2}{\rho v^2}, h^* = \frac{h}{l} \]

Using non-dimensional variables the governing equations are (dropping asterisks in all forms)

\[
\frac{\partial q}{\partial t} + 2iE^{-1} q = -\frac{\partial p}{\partial \xi} + \frac{\partial^2 q}{\partial z^2} + a\frac{\partial^3 q}{\partial z^2 \partial t} - M^2 q
\]  
(2.2.21)

\[
\frac{\partial q^*_p}{\partial t} + 2iE^{-1} q^*_p = -\frac{\partial p}{\partial \xi} + \frac{\partial^2 q^*_p}{\partial z^2} + a\frac{\partial^3 q^*_p}{\partial z^2 \partial t} - M^2 q^*_p - D^{-1} q^*_p
\]  
(2.2.22)

where,

\[ M^2 = \frac{\sigma \mu e^2 H_0 l^2}{\rho v} \] is the Hartmann number

\[ D^{-1} = \frac{l^2}{k} \] is the Inverse Darcy Parameter

\[ E = \frac{v}{\Omega l^2} \] is the Ekman number

\[ \alpha = \frac{\alpha_1}{pl^2} \] is the second grade fluid parameter

The corresponding initial and boundary conditions are
\[ q_p = ae^{-\omega t} + be^{-i\omega t}, \quad t > 0, \quad z = 0 \]  
(2.2.23)

\[ q = 0, \quad t > 0, \quad z = l \]  
(2.2.24)

\[ q = q_p = 0, \quad t \leq 0, \quad \text{for all } z \]  
(2.2.25)

The interfacial conditions are
\[ \begin{aligned}
q &= q_p, \\
\frac{dq}{dz} &= \beta \frac{dq_p}{dz}, \\
\end{aligned} \quad z = h \]  
(2.2.26)

Taking Laplace transforms of equations (2.2.21) and (2.2.22) using initial condition (2.2.25) the governing equations in terms of the transformed variable in zone 1 reduces to
\[ \left(1 + s\alpha \right) \frac{d^2 \tilde{q}}{dz^2} - (M^2 + 2iE^{-1} + s) \tilde{q} = -\frac{P}{s} \]  
(2.2.27)

The relevant transformed boundary condition is
\[ \tilde{q} = 0, \quad z = 1, \]  
(2.2.28)

Likewise the governing equation in zone 2 is
\[ (\beta + s\alpha) \frac{d^2 \tilde{q}_p}{dz^2} - (\Delta^2 + 2iE^{-1} + D^{-1} + s) \tilde{q}_p = -\frac{P}{s} \]  
(2.2.29)

the corresponding transformed condition is
\[ q_p = \frac{a}{s - i\omega} + \frac{b}{s + i\omega}, \quad z = 0 \]  
(2.2.30)

The transformed interfacial conditions are
\[ \begin{aligned}
\tilde{q} &= \tilde{q}_p, \\
\frac{d\tilde{q}}{dz} &= \beta \frac{d\tilde{q}_p}{dz}, \\
\end{aligned} \quad z = h \]  
(2.2.31)

(2.2.32)

Solving equation (2.2.27) subjected to the condition (2.2.28), we get
\[ A \cosh \lambda_1 + B \sinh \lambda_1 + \frac{P}{\lambda_1^2 s(1 + s\alpha)} = 0 \]  
(2.2.33)
Solving (2.2.29) subjected to the condition (2.2.30)
\[
C + \frac{p}{\lambda_2^2 s(\beta + \alpha)} = a \frac{s - i\omega}{s - i\omega} + a \frac{s + i\omega}{s + i\omega}
\]  
(2.2.34)

Making use of the interfacial conditions (2.2.31) and (2.2.32)
\[
ACosh\lambda_1 h + B \text{Sinh}\lambda_1 h + \frac{p}{\lambda_1^2 s(1 + s\alpha)} = C \text{Cosh}\lambda_1 h + D \text{Sinh}\lambda_1 h + \frac{p}{\lambda_2^2 s(\beta + s\alpha)}
\]  
(2.2.35)
\[
A\lambda_1 \text{Sinh}\lambda_1 h + B\lambda_1 \text{Cosh}\lambda_1 h
\]
\[
= \beta \left[ C\lambda_2 \text{Sinh}\lambda_2 h + D\lambda_2 \text{Cosh}\lambda_2 h \right]
\]  
(2.2.36)

Solving equations (2.2.33), (2.2.34), (2.2.35) and (2.2.36), we obtain the constants \(A, B, C\) and \(D\) involved in the variable and are substituting in the following equations
\[
\bar{q} = A \text{Cosh}\lambda_1 z + B \text{Sinh}\lambda_1 z + \frac{p}{\lambda_1^2 s(1 + s\alpha)}
\]  
(2.2.37)
\[
\bar{q}_p = C \text{Cosh}\lambda_2 z + D \text{Sinh}\lambda_2 z + \frac{p}{\lambda_2^2 s(\beta + s\alpha)}
\]  
(2.2.38)

where,
\[
A_1 = \left\{ \frac{a}{s - i\omega} + \frac{p}{\lambda_2^2 s(\beta + s\alpha)} \right\} \text{Cosh}\lambda_2 h + \frac{p}{\lambda_1^2 s(1 + s\alpha)} \text{Sinh}\lambda_1 h
\]
\[
+ \frac{p}{s} \left( \frac{1}{\lambda_2^2 (\beta + s\alpha)} - \frac{1}{\lambda_1^2 (1 + s\alpha)} \right) \frac{\lambda_2}{\lambda_1} \beta \text{Cosh}\lambda_2 h
\]
\[
\left\{ \frac{\lambda_2}{\lambda_1} \left( \frac{a}{s - i\omega} + \frac{b}{s + i\omega} - \frac{p}{\lambda_2^2 s(\beta + s\alpha)} \right) \right\} \text{Sinh}\lambda_2 h + \text{Cosh}\lambda_1 h
\]
\[
\left\{ \frac{p}{\lambda_2^2} \frac{\lambda_2}{\lambda_1} \left( \frac{a}{s - i\omega} + \frac{b}{s + i\omega} - \frac{p}{\lambda_2^2 s(\beta + s\alpha)} \right) \right\} \text{Sinh}\lambda_1 h
\]

\[
A_2 = \frac{\lambda_2}{\lambda_1} \beta \text{Cosh}\lambda_2 h \cdot \text{Sinh}(\lambda_1 (1 - h)) + \text{Sinh}\lambda_2 h \cdot \text{Sinh}(\lambda_1 (1 - h))
\]
\[ A = \frac{A_1}{A_2} \text{Sinh}\lambda_1 \]

\[ B = \frac{1}{\text{Sinh}\lambda_1} \left\{ -\frac{P}{\lambda_1^2 s(1 + s\alpha)} - \text{Cosh}\lambda_1 \cdot \text{Sinh}\lambda_1 \cdot \frac{A_1}{A_2} \right\} \]

\[ C = \frac{a}{s - i\omega} + \frac{a}{s + i\omega} - \frac{p}{\lambda_2^2 s(\beta + s\alpha)} \]

\[ D = \frac{1}{\text{Sinh}^2\lambda_2 h} \left\{ \text{Sinh}(\lambda_1 (1 - h)), \frac{A_1}{A_2} - \left( \frac{a}{s - i\omega} + \frac{b}{s + i\omega} - \frac{p}{\lambda_2^2 s(\beta + s\alpha)} \right) \text{Cosh}\lambda_2 h \right. \]

\[ \left. - \frac{p}{\lambda_1^2 s(1 + s\alpha)} - \frac{p}{s} \left( \frac{1}{\lambda_2^2 (\beta + s\alpha)} - \frac{1}{\lambda_1^2 (1 + s\alpha)} \right) \right\} \]

Taking inverse Laplace transforms to the equations (2.2.37) and (2.2.38) on both sides. We obtain

\[ q = \left\{ -\frac{P}{d_1 d_7} + \frac{P}{d_3 d_7} \text{Cosh}(d_2 h), \frac{P}{d_2 d_7} \text{Sinh}(d_1 h) \right\} \text{Cosh}(d_2 h) + \]

\[ + \frac{P}{d_2} \beta \text{Cosh}(d_2 h) \cdot \text{Sinh}(d_1 h), - \frac{P}{d_1 d_7} \beta \text{Cosh}(d_2 h) \right\} \text{Sinh}(d_1 (1 - z)) - \]

\[ - \frac{P}{d_1^2 d_7} \text{Sinh}(d_1 h) + \frac{P}{d_1^2} \text{Sinh}(d_1 h) + a \left\{ \frac{\beta d_4}{d_8} \text{Sinh}(d_1 (1 - z)) \right\} e^{i\omega t} + b \left\{ \frac{\beta d_6}{d_9} \text{Sinh}(d_3 (1 - z)) \right\} e^{-i\omega t} + \]

\[ + \left\{ \frac{p \beta D^{-1/2} (1 - h)}{d_1^2 (d_1^2 \alpha + 1)d_1^2} + \frac{p}{d_1^2} (D^{-1/2} h) \right\} e^{i\omega t} + b \left\{ \frac{p}{d_1^2 (d_1^2 \alpha + 1)} \right\} (1 - z) e^{-(\omega t + 2\beta^{-1})} \]

(2.2.39)

and

\[ q_p = \frac{P}{d_2^2 \beta} - \frac{P}{d_2 d_7} \text{Sinh}(d_2 h) \text{Cosh}(d_1 (1 - h)) \text{Sinh}(d_2 h) + \]

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\[ + \frac{P \beta d_2 \cosh(d, h) \sinh(d, (1 - h)) \sinh(d_4 z)}{d_4 \sinh(d_4 z)} + \]
\[ - \frac{P \beta d_2 \cosh(d, h) \sinh(d, (1 - h)) \sinh(d_2 h)}{d_7 \sinh(d_7 h)} + \]
\[ + \frac{P \cosh(d_4 h) \sinh(d, (1 - h)) \sinh(d_4 z)}{d_2 \sinh(d_2 h)} + \]
\[ + \frac{P \sinh(d_4 h) \sinh(d, (1 - h)) \sinh(d_4 z)}{d_7 \sinh(d_7 h)} + \]
\[ + \frac{P \beta d_2 \cosh(d, h) \sinh(d, (1 - h)) \sinh(d_2 h)}{d_4 \sinh(d_4 h)} + \]
\[ + \frac{P \cosh(d_4 h) \sinh(d_4 z)}{d_7 \sinh(d_7 h)} - \frac{P \sinh(d_4 h) \sinh(d_4 z)}{d_7 \sinh(d_7 h)} - \frac{P \sinh(d_4 h) \sinh(d_4 z)}{d_7 \sinh(d_7 h)} + \frac{P}{d_7 ^2 \beta} \]
\[ + a \left\{ \cosh(d_4 z) + \frac{\beta d_4 \cosh^2 (d_4 h) \sinh(d_4 (1 - h)) \sinh(d_4 z)}{d_4 \sinh(d_4 h)} - \frac{\beta d_4 \sinh(d_4 h) \sinh(d_4 (1 - h)) \sinh(d_4 z)}{d_4 \sinh(d_4 h)} \right\} e^{i a t} \]
\[ + b \left\{ \cosh(d_6 z) + \frac{\beta d_6 \cosh^2 (d_6 h) \sinh(d_6 (1 - h)) \sinh(d_6 z)}{d_9 \sinh(d_9 h)} - \frac{\beta d_6 \sinh(d_6 h) \sinh(d_6 (1 - h)) \sinh(d_6 z)}{d_9 \sinh(d_9 h)} \right\} e^{-i a t} \]
\[ + \left\{ - \frac{P \beta D^{-1/2} \cosh(D^{1/2} h) \sinh(D^{1/2} z)}{d_7 ^2 (d_7 ^2 \alpha + 1) d_10 \sinh(D^{1/2} h)} + \right\} \]
\[ + \frac{P \beta D^{-1/2} \cosh(D^{1/2} h) \sinh(D^{1/2} z)}{d_7 ^2 (d_7 ^2 \alpha + 1) d_10 \sinh(D^{1/2} h)} + \frac{P \sinh(D^{1/2} z)}{d_7 ^2 (d_7 ^2 \alpha + 1) d_10 \sinh(D^{1/2} h)} + \]
\[ + \frac{P \beta D^{-1/2} \cosh^2(D^{1/2} h) \sinh(D^{1/2} z)}{d_7 ^2 (d_7 ^2 \alpha + 1) d_10 \sinh(D^{1/2} h)} + \]
The shear stresses on the upper plate and lower plate are given by

\[ \tau_u = \left( \frac{dq}{dz} \right)_{z=1} \quad \text{and} \quad \tau_l = \left( \frac{dq_p}{dz} \right)_{z=0} \]

(These forms are mentioned in the appendix)
2.3. Associated boundary layer discussion:-

The solutions for the combined velocity \( q \) and \( q_p \) consists of three kinds of terms:

1. Steady state
2. The quasi-steady state terms associated with non-torsional oscillations in the boundary
3. The transient terms involving exponentially varying time dependence.

From the velocity expressions, it follows that the transient components in the velocity in
the clean fluid region decays in dimensionless time
\[
t > \frac{1}{(M^4 + 4E^{-2})^{\frac{1}{2}}},
\]
while in the porous bed these transient components decay in the
\[
t > \max\left\{ \frac{1}{((M^2 + D^{-1})^2 + 4E^{-2})^{\frac{1}{2}}} \right\}.
\]

The decay time for hydro magnetic transient velocity \( (M^2 \neq 0) \) is always less than that of
hydro dynamic transient solution \( (M^2=0) \) in both regions. Thus is composite medium in
the presence of clean fluid the time of decay of the transient velocity continues to be
higher although the time of decay involving the porous parameter is comparatively less.

When the transient terms decay in the equations (2.2.39) and (2.2.40), the steady and the
oscillatory solutions in the clean fluid and porous bed are given by (i.e., when the
transient terms decay in the limit \( t \to \infty \))

\[
(q)_{\text{steady}} = \left\{ -\frac{P}{d_2 d_7} + \frac{P \cosh(d_2 h)}{d_2 d_7} - \frac{P \sinh(d_2 h) \cosh(d_1 h)}{d_2 d_7 \sinh(d_1)} + \frac{P d_2 \beta \cosh(d_2 h) \sinh(d_1 h)}{d_1^2 d_7 \sinh(d_1)} - \frac{P \sinh(d_1 z)}{d_1^2 d_7 \sinh(d_1)} + \frac{P}{d_1^2 d_7 \sinh(d_1)} \right\} \sinh(d_1(1-z)) - (2.3.1)
\]
\[ (q)_{\text{Oscillatory}} = a \left\{ \frac{\beta d_4 \sinh(d_4(1-z))}{d_8} \right\} e^{i\omega t} + b \left\{ \frac{\beta d_5 \sinh(d_5(1-z))}{d_9} \right\} e^{-i\omega t} \]

\[ (q_p)_{\text{steady}} = -\frac{P \cosh(d_2 z)}{d_2^2 \beta} - \frac{P \cosh^2(d_4 h) \sinh(d_1(1-h)) \sinh(d_2 z)}{d_4 d_1 \sinh(d_1) \sinh(d_2 h)} + \]

\[ + \frac{P \beta d_2 \cosh(d_2 h) \sinh(d_1(1-h)) \sinh(d_2 z)}{d_2^2 d_1 \sinh(d_1) \sinh(d_2 h)} + \]

\[ + \frac{P \cosh(d_2 h) \sinh(d_1(1-h)) \sinh(d_2 z)}{d_2 d_1 \sinh(d_1) \sinh(d_2 h)} + \]

\[ + \frac{P \sinh(d_2 h) \sinh(d_1(1-h)) \sinh(d_2 z)}{d_2 d_1 \sinh(d_1) \sinh(d_2 h)} - \]

\[ - \frac{P \beta d_2 \cosh(d_2 h) \sinh(d_1(1-h)) \sinh(d_2 z)}{d_2 d_1 \sinh(d_1) \sinh(d_2 h)} + \]

\[ + \frac{P \cosh(d_1 h) \sinh(d_2 z)}{d_2^2 \beta \sinh(d_2 h)} - \frac{P \sinh(d_1 h) \sinh(d_2 z)}{d_2^2 \sinh(d_1) \sinh(d_2 h)} - \frac{P \sinh(d_2 z)}{d_2^2 \beta} + \frac{P}{d_2^2 \beta} \]

\[ (q_p)_{\text{Oscillatory}} = a \left\{ \cosh(d_4 z) + \frac{\beta d_4 \cosh^2(d_4 h) \sinh(d_1(1-h)) \sinh(d_2 z)}{d_8 \sinh(d_4 h)} \right\} e^{i\omega t} \]

\[ - \frac{\beta d_4 \sinh(d_4 h) \sinh(d_1(1-h)) \sinh(d_2 z)}{d_8 \sinh(d_4 h)} - \frac{\cosh(d_4 h) \sinh(d_2 z)}{\sinh(d_4 h)} \right\} e^{-i\omega t} \]

\[ + b \left\{ \cosh(d_6 z) + \frac{\beta d_6 \cosh^2(d_6 h) \sinh(d_5(1-h)) \sinh(d_6 z)}{d_9 \sinh(d_6 h)} \right\} e^{i\omega t} \]

\[ - \frac{\beta d_6 \sinh(d_6 h) \sinh(d_5(1-h)) \sinh(d_6 z)}{d_9 \sinh(d_6 h)} - \frac{\cosh(d_6 h) \sinh(d_6 z)}{\sinh(d_6 h)} \right\} e^{-i\omega t} \]

\[ (2.3.2) \]

\[ (2.3.3) \]

\[ (2.3.4) \]
The hyperbolic terms in the solution (2.3.1) represent the steady hydro magnetic boundary layer flow which consists of the Ekman-Hartmann layer on the upper plate \( z = 1 \). The non-dimensional thickness of the Ekman-Hartmann layer is of order

\[
a_1^{-1} = \left\{ \left( M^4 + \frac{4}{E^2} \right)^{\frac{k}{2}} + M^2 \right\}^{\frac{k}{2}} \text{ since } E \ll 1, \text{ and } M \sim O(1).
\]

In the non-magnetic case (\( M \to 0 \)) the solution (2.3.1) reduces to the steady hydro magnetic boundary layer flow with the Ekman layer of thickness of order \( E^{3/2} \). Since \( E \ll 1 \) and \( M \sim O(1) \), it follows that the thickness of this layer which of order \( E^{3/2} \) sufficiently reduces to a layer of order \( \frac{1}{M} \left[ 1 - \frac{1}{2M^4E^2} \right] \) in the presence of magnetic field. Thus the thickness of the Ekman-Hartman layer decreases with increase in the strength of the magnetic field. The equations (2.3.3) and (2.3.4) represent the steady oscillatory hydro magnetic boundary layer flow which consists of the Ekman-Darcy-Hartmann layer and Stokes-Ekman-Hartmann layer on the lower plate \( z = 0 \). The non-dimensional thickness of the Ekman-hartmann layer is of order,

\[
a_2^{-1} = \left\{ \left( \frac{M^2 + D^{-1}}{\beta} \right)^2 + \frac{4}{\beta^2E^2} \right\}^{\frac{k}{2}} + \left( \frac{M^2 + D^{-1}}{\beta} \right)^{\frac{k}{2}}
\]

This layer can be regarded as Ekamn-Darcy layer modified by the magnetic field (or) as an Ekman-Hartmann layer modified by the porosity of the Medium. When \( E^{-1} \gg D^{-1} \gg M \) i.e., when coriolis force dominates over the Darcy and Lorentz force, the thickness of the layer is \( E^{3/2} \), which coincides with the Ekmann layer. However when \( E \ll D \) and \( M \ll E^{-1} \) or \( D^{-1} \) this modified Darcy-Ekman layer is of thickness
When \( D^{-1} \sim O(E^{-1}) \) and \( M \ll E^{-1} \) or \( D^{-1} \), the thickness of the boundary layer sufficiently reduces into comparison to the usual Ekman layer. However, if \( D \ll E \) such that \( DE^{-1} \ll 1 \), we notice that a layer of thickness \( (D\beta)^{1/2} (1 - \frac{1}{4} DE^{-1}) \) is formed near the lower plate abutting the porous medium. This is true even in the presence or absence of transverse magnetic field, although the thickness of the layer near the lower plate reduces further in the presence of a magnetic field.

We now discuss the nature of oscillatory terms arising due to the non-torsional oscillations of the lower plate in its own plane for \( t > 0 \). We shall consider the following special cases related to the frequency of the oscillatory plate \( z = 0 \) namely viz.,

1) Non-oscillatory plate
2) Low-frequency oscillations \( (\omega << 1) \)
3) High frequency oscillations \( (\omega >> 1) \)
4) Intermediate oscillations \( (\omega \sim O(1)) \)

The non-torsional oscillations of the lower plate influence the boundary layers formed both near the upper and lower plates and the resultant layers may be termed as modified Stokes-Ekman-Darcy-Hartmann layers. To begin with we discuss the nature of the layers related to different frequencies of non-torsional oscillations formed near the upper plate bounding clean fluid region. In the non-oscillatory case, the behavior of these layers for different various in the other parameters has already been discussed above. The
Oscillations of the boundary plate give rise to a layer near the upper plate bounding the non-porous region whose thickness of order, \( \max \left\{ \frac{a_3}{a_1^2 + b_1^2}, \frac{a_5}{a_2^2 + b_2^2} \right\} \)

\[
\begin{align*}
a_3 &= \frac{1}{\sqrt{2}} \left( \left( \frac{M^2 + \omega \alpha (2E^{-1} + \omega)}{1 + \omega^2 \alpha^2} \right)^2 + \left( \frac{2E^{-1} + \omega - M^2 \omega \alpha}{1 + \omega^2 \alpha^2} \right)^2 \right)^{1/2} + \frac{M^2 + \omega \alpha (2E^{-1} + \omega)}{1 + \omega^2 \alpha^2} \\
b_3 &= \frac{1}{\sqrt{2}} \left( \left( \frac{M^2 + \omega \alpha (2E^{-1} + \omega)}{1 + \omega^2 \alpha^2} \right)^2 + \left( \frac{2E^{-1} + \omega - M^2 \omega \alpha}{1 + \omega^2 \alpha^2} \right)^2 \right)^{1/2} - \frac{M^2 + \omega \alpha (2E^{-1} + \omega)}{1 + \omega^2 \alpha^2} \\
a_5 &= \frac{1}{\sqrt{2}} \left( \left( \frac{M^2 - \omega \alpha (2E^{-1} - \omega)}{1 + \omega^2 \alpha^2} \right)^2 + \left( \frac{2E^{-1} - \omega + M^2 \omega \alpha}{1 + \omega^2 \alpha^2} \right)^2 \right)^{1/2} + \frac{M^2 - \omega \alpha (2E^{-1} - \omega)}{1 + \omega^2 \alpha^2} \\
b_5 &= \frac{1}{\sqrt{2}} \left( \left( \frac{M^2 - \omega \alpha (2E^{-1} - \omega)}{1 + \omega^2 \alpha^2} \right)^2 + \left( \frac{2E^{-1} - \omega + M^2 \omega \alpha}{1 + \omega^2 \alpha^2} \right)^2 \right)^{1/2} - \frac{M^2 - \omega \alpha (2E^{-1} - \omega)}{1 + \omega^2 \alpha^2}
\end{align*}
\]

For low frequency oscillations (\( \omega \ll 1 \)), so that \( \omega \alpha \ll 1 \), we assume \( E^{-1} \alpha \sim O(1) \).

Then the boundary layer is of order

\[
O \left( 1 + \frac{M^2 + 2E^{-1} \omega \alpha}{2(M^2 + 2E^{-1} \omega \alpha)^2 + (2E^{-1} - M^2 \omega \alpha)^2} \right)
\]

In case \( M^2 \gg E^{-1} \omega \alpha \) and \( M^2 \omega \alpha \sim O(1) \), this thickness of layer reduces to

\[
1 + \frac{M^2}{2(M^4 + \omega^2 \alpha^2) + (4E^{-2} - M^2)}, \text{ this is approximately } O \left( 1 + \frac{M^2}{2(M^4 - M^2 + 4E^{-2})} \right)
\]

For high frequency oscillations (\( \omega \gg 1 \)); In this case (\( \omega \gg 1 \)), assuming \( E^{-1} \omega \alpha \gg M^2 \) the layer reduces to

\[
O(1 + \frac{2E^{-1} \omega \alpha - \omega^2 \alpha^2}{2(4E^{-2} \omega^2 \alpha^2 + 2(E^{-1} - M^2 \omega \alpha)^2)})
\]
Finally for intermediate oscillations i.e., $\omega \sim O(1)$, in case $M^2 > > E^{-1} \omega \alpha$ and $M^2 \omega \alpha \sim O(2E^{-1})$

1) the thickness of the layer reduces to $O \left( 1 + \frac{1}{2M^2} \right)$

$$\left( \because 1 + \frac{\omega^2 \alpha + M^2}{M^4 + (M^4 \omega^2 \alpha^2)} = 1 + \frac{\omega^2 \alpha + M^2}{M^4 (1 + \omega^2 \alpha^2)} \sim 1 + \frac{M^2}{2M^4} \right)$$

Similar discussion may be made related to the oscillatory boundary layers near $z = 0$ bounding the porous medium.

Case 1. In the non-oscillatory case the behavior of boundary layers for different variations in the other parameters has already been discussed above. The oscillations of the boundary plate give rise to a layer near the lower plate bounding the porous medium whose thickness of order i.e., $O \left( \max \left( \frac{a_4}{a_4^2 + b_4^2}, \frac{a_6}{a_6^2 + b_6^2} \right) \right)$.

Where,

$$a_4 = \frac{1}{\sqrt{2}} \left[ \left( \frac{(M^2 + D^{-1}) \beta + \omega \alpha (2E^{-1} + \omega)}{\beta^2 + \omega^2 \alpha^2} \right)^2 + \left( \frac{\beta (2E^{-1} + \omega) - \omega \alpha (M^2 + D^{-1})}{\beta^2 + \omega^2 \alpha^2} \right)^2 \right]^{1/2}$$

$$+ \left[ \frac{(M^2 + D^{-1}) \beta + \omega \alpha (2E^{-1} + \omega)}{\beta^2 + \omega^2 \alpha^2} \right]^{1/2}$$

$$b_4 = \frac{1}{\sqrt{2}} \left[ \left( \frac{(M^2 + D^{-1}) \beta + \omega \alpha (2E^{-1} + \omega)}{\beta^2 + \omega^2 \alpha^2} \right)^2 + \left( \frac{\beta (2E^{-1} + \omega) - \omega \alpha (M^2 + D^{-1})}{\beta^2 + \omega^2 \alpha^2} \right)^2 \right]^{1/2}$$

$$- \left[ \frac{(M^2 + D^{-1}) \beta + \omega \alpha (2E^{-1} + \omega)}{\beta^2 + \omega^2 \alpha^2} \right]^{1/2}$$
\[ a_6 = \frac{1}{\sqrt{2}} \left[ \left( \frac{1}{\beta^2 + \omega^2 \alpha^2} \right) \left( \frac{1}{\beta^2 + \omega^2 \alpha^2} \right) + \left( \frac{\beta(2E^{-1} - \omega) + \omega \alpha (M^2 + D^{-1})}{\beta^2 + \omega^2 \alpha^2} \right)^2 \right]^{1/2} \]

\[ + \left( \frac{(M^2 + D^{-1}) \beta - \omega \alpha (2E^{-1} - \omega)}{\beta^2 + \omega^2 \alpha^2} \right)^2 \]

\[ b_6 = \frac{1}{\sqrt{2}} \left[ \left( \frac{1}{\beta^2 + \omega^2 \alpha^2} \right) \left( \frac{1}{\beta^2 + \omega^2 \alpha^2} \right) + \left( \frac{\beta(2E^{-1} - \omega) + \omega \alpha (M^2 + D^{-1})}{\beta^2 + \omega^2 \alpha^2} \right)^2 \right]^{1/2} \]

\[ - \left( \frac{(M^2 + D^{-1}) \beta - \omega \alpha (2E^{-1} - \omega)}{\beta^2 + \omega^2 \alpha^2} \right)^2 \]

Case. 2. Low frequency oscillations \((\omega \ll 1)\).

For low frequency oscillation \((\omega \ll 1)\), so that \(\omega \alpha \ll 1\). We assume \(E^{-1} \omega \alpha \sim O(1)\). Then the boundary layer is of order

\[ O \left( 1 + \frac{M_i^2 + 2E^{-1} \omega \alpha}{2 \{ (M_i^2 + 2E^{-1} \omega \alpha)^2 + (2E^{-1} - M_i^2 \omega \alpha)^2 \} \} \right) \]

where \(M_i^2 = M^2 + D^{-1}\)

In case \(M_i^2 \gg E^{-1} \omega \alpha\), the thickness of layer reduces to

\[ O \left( 1 + \frac{M_i^2}{2 \{ M_i^4 + (2E^{-1} - M_i^2 \omega \alpha)^2 \} \} \right) \]

This is

\[ O \left( 1 + \frac{M_i^2}{2 \{ M_i^4 - M_i^2 + 4E^{-2} \} \} \right) \]

approximately

Case. 3. High frequency oscillations \((\omega \gg 1)\)
For high frequency oscillations, $\omega >> 1$ we assume $E^{-1} \omega \alpha >> M_1^2$, the layer reduces to

$$\mathcal{O}\left(1 + \frac{2E^{-1}\omega\alpha - \omega^2\alpha^2}{2(4E^{-2}\omega^2\alpha^2 + 2(E^{-1} - M_1^2\omega\alpha)^2)}\right)$$

Case 4. Intermediate oscillations ($\omega \sim O(1)$)

In this case $M_1^2 >> E^{-1} \omega\alpha$ and $M_1^2\omega\alpha \sim O(2E^{-1})$ the thickness of the layer reduces to $\mathcal{O}\left(1 + \frac{1}{2M_1^2}\right)$. 
2.4. Results and Discussion:

The computational analysis has been carried out to discuss the behaviour of velocity components in both clean and porous regions as well as shear stresses in the rotating parallel plate channel with reference to variations in the governing parameters namely viz. $E$ the Ekman number, $M$ the Hartmann number, $D^{-1}$ the inverse Darcy parameter and $\alpha$ the second grade fluid parameter. This analysis has also been taken up in two cases: 1. when the thickness of the porous bed in the composite medium is relatively small 2. the thickness is relatively large. The flow takes place in between the rotating plates in planes parallel to the boundaries. The velocity components relative to the rotating frame are obtained solving Neviier-Stokes equation (boundary layer type) in the clean fluid region and the Brinkman equations in porous bed. The velocity in the clean and porous regions is matched using interfacial conditions. We may note in general that the thickness of the porous bed plays a significant role deciding the flow features in the composite medium. The profiles are drawn for the velocity components in the composite medium with the expressions for $u$ and $v$ chosen from the clean fluid region while the relative expressions for velocity component $u_p$ and $v_p$ are chosen from the porous region. For computational purpose we fix the axial pressure gradient $P$, $\omega$ as well as 'a and b', where the frequency oscillation $\omega$, $a$ and $b$ the constants related to non-torsional oscillations of the boundary. This can be carried out for all chapters.

The behaviour of the velocity component $u$ for the different variations in the governing parameters namely $E$ the Ekman number, $M$ the Hartmann number, $D^{-1}$ the inverse Darcy parameter and $\alpha$ the second grade fluid parameter may be analyzed from figures (1-4) while the figures (5-8) depict the behaviour of the velocity component $v$. 54
The flow feature for variation in each parameter may be understood by the behaviour variations of resultant velocity. We may note that (fig. 1) the magnitude of the velocity component $u$ enhances in the entire composite medium with increase in the Ekman number $E$. However the velocity $v$ exhibits a transitional behaviour with its magnitude increase in the porous medium as well as near the lower boundary abutting the clean fluid region, where as reducing in the vicinity of the interfacial layer ($0.3 \leq z \leq 0.4$) (fig. 5). When increasing the Hartmann parameter $M$ enhances $v$ in the porous bed and reduces in clean fluid region (fig. 6). The velocity component however shows $u$ depreciation in the porous bed as well as the vicinity of the interfacial layer although it enhances with $M$ in the entire clean fluid region (fig. 2). The influence of the permeability of the porous bed on the velocity component $u$ may observe from figure (3). We find that the magnitude of the velocity component $u$ reduces with decrease in the permeability in the entire medium except in the layer ($0.5 \leq z \leq 0.7$) in the clean fluid region above the interface, where it experiences a slight enhancement. The behavior of $v$ with increase $D^{-1}$ may be noted from (fig. 7), we find that $v$ reduces in the region ($0 \leq z \leq 0.5$) including the porous bed and enhances in the clean fluid region outside the same layer. A similar behaviour of $u$ with variation in the second grade fluid parameter $\alpha$ may be noticed from (fig. 4), we find that the magnitude of $u$ enhances with in the entire region except in the layer ($0.5 \leq z \leq 0.7$), where it experiences a light depreciation. We also note that $v$ retards in the porous bed as well as near the boundary ($0.7 \leq z \leq 0.8$), while it enhances within the layer ($0.3 \leq z \leq 0.6$) in the vicinity of the interface.

Next we consider the behavior of the velocity profiles for variations in the governing parameters when the thickness of porous bed is sufficiently large ($h=0.6$). The
fig (9-12) corresponds to the profiles of $u$ and the figures (13-16) correspond to the profiles of $v$. We observe (fig. 9) that the magnitude of $u$ increases with increase in $E$ in the entire composite medium. But the magnitude of $v$ reduces in the clean fluid region while it increases in the porous bed (fig. 13). With reference to Hartmann parameter $M$, we find that an increase in $M$ retards $u$ in the entire medium (fig. 10), where as it enhances everywhere in the composite medium (fig. 14). It is interesting to note that lesser the permeability of the porous bed lower the velocity with its magnitude in the entire region (fig. 11). This retardation with reference to increase in $D^{-1}$ is also noticed in $v$ in the entire region (fig. 15). This is in contrast to behaviour of $u$ and $v$ with reference to variation in $\alpha$ where both $u$ and $v$ enhances in the entire region with increase in $\alpha$ (fig. 12&16).

The shear stresses on the upper and lower plates have been calculated for different variations in $E$, $M$, $D^{-1}$ and $\alpha$ are tabulated in the tales (I-IV). On the upper plate both $\tau_x$ and $\tau_y$ enhances with increase in $E$, while on the lower plate $\tau_x$ increases and $\tau_y$ reduces with $E$. In contrast an increase in $M$ increases $\tau_x$ and $\tau_y$ on the lower plate while increases $\tau_x$ and reduces $\tau_y$ on the upper plate. The influence of permeability of the porous bed on the shear stresses is evident from the fact that lesser the permeability lower the shear stresses $\tau_x$ and $\tau_y$ either of the boundaries. The variation of the stresses with reference to the second grade fluid parameter is similar to that of variation in $M$ with stresses increase in magnitude on the lower plate with increase in $\alpha$, while $\tau_x$ increases and $\tau_y$ reduces on the upper plate.
The velocity profiles when the lower plate execute non-torsional oscillations

Fig. 1: The velocity profile for $u$ with $E$.
$a=b=1, \omega = \frac{\pi}{2}, \alpha = 0.25, D^{-1}=2000, M=2$

Fig. 2: The velocity profile for $u$ with $M$.
$a=b=1, \omega = \frac{\pi}{2}, \alpha = 0.25, D^{-1}=2000, E=0.01$
Fig. 3: The velocity profile for $u$ with $D^{-1}$

$a=b=1$, $\omega = \frac{\pi}{2}$, $\alpha = 0.25$, $M=2$, $E=0.01$

Fig. 4: The velocity profile for $u$ with $\alpha$.

$a=b=1$, $\omega = \frac{\pi}{2}$, $M=2$, $D^{-1}=2000$, $E=0.01$
Fig. 5: The velocity profile for $v$ with $E$.
$a=b=1, \omega=\frac{\pi}{2}, \alpha=0.25, D^{-1}=2000, M=2$

Fig. 6: The velocity profile for $v$ with $M$
$a=b=1, \omega=\frac{\pi}{2}, \alpha=0.25, D^{-1}=2000, E=0.01$
Fig. 7: The velocity profile for $v$ with $D^{-1}$

$a = b = 1$, $\omega = \frac{\pi}{2}$, $\alpha = 0.25$, $M = 2$, $E = 0.01$

Fig. 8: The velocity profile for $v$ with $\alpha$.

$a = b = 1$, $\omega = \frac{\pi}{2}$, $M = 2$, $D^{-1} = 2000$, $E = 0.01$
Fig. 9: The velocity profile for \( u \) with \( E \).

\[ a=b=1, \, \omega = \frac{\pi}{4}, \, \alpha = 0.25, \, D^{-1}=2000, \, M=2 \]

Fig. 10: The velocity profile for \( u \) with \( M \).

\[ a=b=1, \, \omega = \frac{\pi}{4}, \, \alpha = 0.25, \, D^{-1}=2000, \, E=0.01 \]
Fig. 11: The velocity profile for $u$ with $D^{-1}$

$a=b=1, \omega=\frac{\pi}{4}, \alpha = 0.25, M=2, E=0.01$

Fig. 12: The velocity profile for $u$ with $\alpha$.

$a=b=1, \omega=\frac{\pi}{4}, M=2, D^{-1}=2000, E=0.01$
Fig. 13: The velocity profile for $v$ with $E$.

$a=b=1$, $\omega=\frac{\pi}{4}$, $\alpha=0.25$, $D^{-1}=2000$, $M=2$

Fig. 14: The velocity profile for $v$ with $M$

$a=b=1$, $\omega=\frac{\pi}{4}$, $\alpha=0.25$, $D^{-1}=2000$, $E=0.01$
The shear stresses ($\tau_x$) on the upper plate

$$a=b=1,\ t=1,\ \omega = \frac{\pi}{4}$$

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The shear stresses \( \tau_y \) on the upper plate

\[ a=b=1, \ t=1, \ \alpha = \frac{\pi}{4} \]
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The shear stresses ($\tau_z$) on the lower plate

$$a=b=1, \ t=1, \ \omega = \frac{\pi}{4}$$
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The shear stresses (\(\tau_y\)) on the lower plate

\[ a=b=1, t=1, \omega = \frac{\pi}{4} \]
2.5. Conclusions:

1. The velocity in the clean and porous regions is matched using interfacial conditions. The thickness of the porous bed plays a significant role deciding the flow features in the composite medium.

2. When the thickness of the porous bed is small, the resultant velocity exhibits a transitional behaviour with its magnitude increase in the porous medium as well as near the lower boundary abutting the clean fluid region, whereas reducing in the vicinity of the interfacial layer with increase in Eckman number.

3. When increasing the Hartmann parameter M, the resultant velocity enhances in the porous bed and reduces in clean fluid region.

4. The resultant velocity reduces with increase in inverse Darcy parameter.

5. The resultant velocity increases with increase in α

6. When the thickness of the porous bed is large, the magnitude of resultant velocity increases with increase in E and α in the entire composite medium.

7. Lesser the permeability of the porous bed lower the velocity with its magnitude in the entire region.

8. The stresses on the lower plate are very small compared to its values on the upper plate.
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Appendix:

\[ \beta = \frac{v_{\text{eff}}}{v} \]
\[ \lambda_1^2 = \frac{M^2 + 2iE^{-1} + s}{1 + \alpha} \quad \lambda_2^2 = \frac{M^2 + 2iE^{-1} + D^{-1} + s}{\beta + \alpha} \]
\[ d_1 = \sqrt{M^2 + 2iE^{-1}} \]
\[ d_2 = \sqrt{M^2 + 2iE^{-1} + D^{-1}} \]
\[ d_3 = \sqrt{M^2 + 2iE^{-1} + i\omega} \quad d_4 = \sqrt{M^2 + 2iE^{-1} + i\omega} \]
\[ d_5 = \sqrt{M^2 + 2iE^{-1} - i\omega} \quad d_6 = \sqrt{M^2 + 2iE^{-1} - i\omega} \]
\[ d_7 = d_2 \beta \cosh(d_2 h) \sinh(d_2(1-h)) + d_5 \sinh(d_2 h) \cosh(d_2(1-h)) \]
\[ d_8 = d_4 \beta \cosh(d_4 h) \sinh(d_4(1-h)) + d_5 \sinh(d_4 h) \cosh(d_4(1-h)) \]
\[ d_9 = d_6 \beta \cosh(d_6 h) \sinh(d_6(1-h)) + d_5 \sinh(d_6 h) \cosh(d_6(1-h)) \]
\[ d_{10} = D^{-1/2} \beta(1-h) \cosh(D^{-1/2} h) \sinh(D^{-1/2} h) \]

\[ \tau_\text{U} = \left\{ \begin{array}{l}
\frac{-P}{d_4} + \frac{P \cosh(d_2 h)}{d_2 d_7} - \frac{P \sinh(d_2 h) \cosh(d_2 h)}{d_5 d_7 \sinh(d_1 h)} \\
+ \frac{P d_2 \beta \cosh(d_2 h) \sinh(d_1 h)}{d_1^2 d_7 \sinh(d_1 h)} - \frac{P \beta d_2 \cosh(d_2 h)}{d_1^2 d_7} (-d_1) \\
- \frac{P d_1 \cosh(d_1)}{d_1^2 d_5 \sinh(d_1 h)} + a \left\{ \frac{\beta d_4 (-d_3)}{d_4} \right\} e^{-it} + b \left\{ \frac{\beta d_6 (-d_5)}{d_9} \right\} e^{-i\omega t} \\
- \left\{ \frac{P \beta D^{-1/2} (1-h) \cosh(D^{-1/2} h)}{d_1^2 (d_1^2 \alpha + 1) d_{10}} \right\} + \frac{P \sinh(D^{-1/2} h)}{d_1^2 (d_1^2 \alpha + 1) d_{10}} - \frac{p}{d_1^2 (d_1^2 \alpha + 1)} \right\} e^{-\left(M^2 + 2iE^{-1}\right) t} \]

and

\[ \tau_\text{L} = - \left\{ \frac{P d_2 \cosh(d_2 h)}{d_2 d_7 \sinh(d_2 h)} + \frac{P \beta d_2^2 \cosh(d_2 h) \sinh(d_2 h)}{d_1^2 d_7 \sinh(d_1 h) \sinh(d_2 h)} - \frac{P \beta d_2^2 \cosh(d_2 h)}{d_1^2 d_7 \sinh(d_2 h)} \right\} \]