ABSTRACT

In the theory of rings, the study of both associative and nonassociative rings has evoked great interest and assumed importance. In particular, the study of derivations in prime or semiprime rings has yielded many interesting results. These results have applications in other branches.


Throughout this work a ring $R$ is a synonym for an associative ring which does not have a unit element.

In this work we present some properties of derivations, generalized derivations and Jordan derivations in prime rings and semiprime rings. We characterize dependent elements of a commuting derivation $d$ on a semiprime ring $R$ and investigate a decomposition of $R$ using dependent elements of $d$. It is shown that if a left derivation $d$ acts as a homomorphism
or an antihomomorphism on a nonzero right ideal $U$ of a prime ring $R$, then $d = 0$. Also we prove the commutativity of a prime near ring or a prime ring $R$ admitting a generalized derivation $f$ with associated derivation $d$ satisfying certain properties of derivations related with ideals and center $Z$.

The commutator $[x,y]$ is defined as $[x,y] = xy - yx$ and the anticommutator is defined by $xoy = xy + yx$ for all $x, y$ in a ring. A left (right) near ring is a set $N$ with two operations $+$ and $\cdot$ such that $(N, +)$ is a group and $(N, \cdot)$ is a semi group satisfying the left distributive law: $x(y + z) = xy + xz$ (right distributive law: $(x + y)z = xz + yz$) for all $x, y, z \in N$.

A ring $R$ is called prime if $xay = 0$ implies $x = 0$ or $y = 0$ for all $x, a, y$ in $R$. A ring $R$ is called semiprime if $xax = 0$ implies $x = 0$ for all $x, a$ in $R$. An additive mapping $x \rightarrow x^*$ on a ring $R$ is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ hold for all $x, y$ in $R$. A ring equipped with an involution is called a ring with involution or a $^*$-ring. A semiprime $^*$-ring is defined as $xa^*x = 0$ implies $x = 0$ for all $x, a$ in $R$. The center $Z$ of $R$ is defined as $Z = \{z \in R / [z, R] = 0\}$. We define a ring $R$ to be of characteristic $\neq n$ if $nx = 0$ implies $x = 0$ for all $x$ in $R$.

An additive map $d$ from a ring $R$ to $R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y$ in $R$. An derivation $f : R \rightarrow R$ is said to be a right generalized derivation if there exists a derivation $d$ from $R$ to $R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y$ in $R$ and $f$ is said to be a left generalized
derivation if there exists a derivation $d$ from $R$ to $R$ such that $f(xy) = d(x)y + x f(y)$ for all $x, y$ in $R$. It is a generalized derivation if it is both right and left generalized derivation. An additive mapping $d : R \rightarrow R$ is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ for all $x$ in $R$. An additive mapping $G : R \rightarrow R$ is said to be a Jordan generalized derivation if there exists a derivation $D$ from $R$ to $R$ such that $G(x^2) = G(x)x + xD(x)$ for all $x$ in $R$. An additive mapping $D$ from $R$ to itself is called an $u$-derivation if $D(xy) = D(x)u(y) + xD(y)$ hold where $u$ is a homomorphism of $R$, for all $x, y$ in $R$. An additive mapping $G : R \rightarrow R$ is said to be a $u$-generalized derivation if there exists a derivation $D$ from $R$ to $R$ such that $D(xy) = D(x)u(y) + xD(y)$ for all $x, y \in R$ and an additive mapping $G : R \rightarrow R$ is said to be a Jordan $u$-generalized derivation if there exists a derivation $D : R \rightarrow R$ such that $G(x^2) = G(x)u(x) + xD(x)$ for all $x \in R$. An additive mapping $G : R \rightarrow R$ is called a Jordan $u$-*generalized derivation if there exists a derivation $D$ from $R$ to $R$ such that $G(x^2) = u(x)G(x) + D(x)x$ where $u$ is an antihomomorphism in $R$ for all $x \in R$. An additive mapping $D : R \rightarrow R$, where $R$ is a *-ring, is called a*-derivation if $D(xy) = D(x)y^* + xD(y)$ hold for all $x, y \in R$ and is called a Jordan *-derivation if $D(x^2) = D(x)x^* + xD(x)$ hold for all $x$ in $R$. An additive mapping $G : R \rightarrow R$ is said to be a *-generalized derivation if there exists a *-derivation $D : R \rightarrow R$ such that $G(xy) = G(x)y^* + xD(y)$ for all $x, y$ in $R$ and an additive mapping $G : R \rightarrow R$
is said to be a Jordan \*-generalized derivation if there exists a \*-derivation \( D : R \to R \) such that 
\[ G(x^2) = G(x)x^* + xD(x) \]
for all \( x \in R \). An additive mapping \( D : R \to R \), where \( R \) is a \*-ring, is called a Reverse \*-derivation if 
\[ D(xy) = y^*D(x) + D(y)x \]
hold for all \( x,y \in R \) and is called a Reverse Jordan \*-derivation if 
\[ D(x^2) = x^* D(x) + D(x)x \]
hold for all \( x \) in \( R \). An additive mapping \( G : R \to R \) is said to be a Reverse \*-generalized derivation if there exists a reverse \*-derivation \( D : R \to R \) such that 
\[ G(xy) = y^* G(x) + D(y)x \]
for all \( x,y \) in \( R \) and an additive mapping \( G : R \to R \) is said to be a Reverse Jordan \*-generalized derivation if there exists a reverse \*-derivation \( D : R \to R \) such that 
\[ G(x^2) = x^* G(x) + D(x)x \]
for all \( x \) in \( R \). A mapping \( d : R \to R \) is called centralizing if 
\[ [d(x),x] \in Z \]
for all \( x \) in \( R \). In particular if \( [d(x),x] = 0 \) for all \( x \) in \( R \), then \( d \) is called commuting and is called central if \( d(x) \in Z \) for all \( x \) in \( R \).

The first chapter is devoted to present the necessary background. We give a brief survey of the work done by E.C. Posner, I.N. Herstein, T.P. Kelzan, H.E.Bell,W.S. Martindale, M.N. Daif, B.Hvala, M.Bresar, N.Argac, A. Nakajima, E. Albas and J. Vukman.

In chapter 2, we present some results on derivations in prime rings. In section 2.1, first we present some fundamental properties of the derivations, which are of great importance in the rest of our work. We derive the structure of a prime ring \( R \) which has a derivation \( d \neq 0 \) such that
the values of \(d\) commute, that is for which \(d(x)d(y) = d(y)d(x)\) for all \(x, y\) in \(R\). In section 2.2, we prove that if \(d\) is a commuting derivation of a semiprime ring \(R\), then \(a \in D(d)\) where \(D(d)\) is the collection of all dependent elements of \(d\) if and only if \(a \in Z\) and \(d(x^2)a = 0\) or \(d(x)a = 0\) for all \(x \in R\). Also we show that if \(d\) is a derivation of a semiprime ring \(R\), \(U\) is a right ideal of \(R\) and \(d(xr) \in Z\) for all \(x \in U, r \in R\), then \([U, R] d(R) = 0\). Using this we prove that if \(d\) is commuting, then there exist ideals \(U\) and \(V\) of \(R\) such that \(U \oplus V\) is an essential ideal of \(R\), \(U \cap V = \{0\}\), \(d = 0\) on \(U\), \(d(V) \subseteq V\) and \(d\) acts freely on \(V\). In section 2.3, we prove that a mapping \(d\) on a semiprime ring \(R\) is a left derivation if and only if it is a central derivation. It is shown that if a left derivation \(d\) acts as a homomorphism or an antihomomorphism on a nonzero right ideal \(U\) of a prime ring \(R\) or \([d(x), x] = 0\), then \(d = 0\). Also, we prove that if \([d(x), d(y)] = [x, y]\) or \([d(x), d(y)] = 0\), then \(R\) is commutative.

In chapter 3, we prove some results on the generalized derivations in rings. In section 3.1, we extend some results concerning derivations of prime ring \(R\) to the left generalized derivations associated with a derivation \(d\) of \(R\) and a nonzero right ideal \(U\) of \(R\) which is semiprime as a ring. In section 3.2, we prove that if \(d\) is a nonzero derivation of a prime ring \(R\) and \(f\) is a generalized derivation, then \(f\) is a strong commutativity preserving. Using this, we prove that \(R\) is commutative. In section 3.3, we consider a
prime near ring $N$ of char.$\neq 2$ with center $Z$ and two generalized derivations $(f,d)$ and $(g,h)$ of $N$. We prove that if $f([x,y]) = 0$ or $f([x,y]) = \pm [x,y]$ or $f([x,y]) = \pm xy$ or $f(xy) \neq xy \in Z$ or $f(xy) \neq yx \in Z$, then $N$ is a commutative ring. Also we prove that if $af(x) = 0$ for $a \neq 0 \in N$, then $a \in Z$ and hence $d(a) \in Z$.

In chapter 4, we prove some properties of Jordan derivations in semiprime and semiprime $^*$-rings. In section 4.1, we see that if $d$ is a Jordan derivation of a semiprime ring $R$, then $d$ is a quarter derivation. Using this, we show that $d$ is a triple derivation. Also we prove that if $R$ is semiprime ring of char.$\neq 2$ and $d$ acts as a homomorphism on $R$, then $d$ is a central derivation. In section 4.2, we prove that if $G$ is a Jordan $u$-generalized derivation of a semiprime ring $R$ of char.$\neq 2$, then $G$ is a $u$-generalized derivation. Similarly, we show that if $G$ is a Jordan $u$-$^*$generalized derivation of a semiprime ring $R$ of char.$\neq 2$, then $G$ is a $u$-$^*$generalized derivation. We also prove that commutativity of $R$ if $G([x,y]) = 0$. In section 4.3, we show that if $R$ is a semiprime $^*$-ring of char.$\neq 6$ and let $G: R \to R$ be an additive mapping satisfying the relation $G(xyx) = x^*y^* G(x) + x^* D(y)x + D(xy)x$ for all $x,y$ in $R$ and some reverse Jordan $^*$-derivation $D$ of $R$, then $G$ is a Reverse Jordan $^*$-generalized derivation. It is also shown that if $R$ is a semiprime $^*$-ring of char.$\neq 2$ and suppose there exists an additive mapping $G : R \to R$ related with some $u$-derivation $D$ of $R$, where $u$ is a
homomorphism of $R$ such that $G(xx^*) = G(x) u(x^*) + xD(x^*)$ for all $x$ in $R$, then $G$ is a Jordan $u$-generalized derivation.

Chapter 5 is devoted to discuss further possible developments of some results which we wish to study in future. In section 5.1, we discuss some results on generalized derivations of prime rings related to a Lie ideal $U$. If $U$ is a Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$ and $f$ is a generalized derivation associated with a derivation $d \neq 0$ satisfying either $[d(x), f(y)] + xy = 0$ or $[d(x), f(y)] = xy$ for all $x, y \in U$, then $U \subseteq Z$. We wish to try for some more properties of the generalized derivations with commutativity and anticommutativity conditions. In section 5.2, we present some results on orthogonal generalized derivations of semiprime rings. If $(D,d)$ and $(G,g)$ are generalized derivations and $U$ is a nonzero ideal of a semiprime ring $R$ such that the left annihilator $l(u) = 0$, then $(D,d)$ and $(G,g)$ are orthogonal implies $(DG, dg)$ is a generalized derivation on $U$ and $D(x) G(y) = 0$ for all $x, y \in U$. We wish to study some results concerning orthogonal generalized derivations of semiprime rings related to left or right ideals. In section 5.3, we discuss some properties of prime and semiprime rings with permuting 3-derivations. We see that, if $U$ is a nonzero two-sided ideal of a semiprime ring of char.$\neq 3$ and suppose that there exists a permuting 3-derivation $\Delta : R \times R \times R \to R$ such that the trace is centralizing.
on $U$, then the trace of $\Delta$ is commuting on $U$. We wish to study on higher derivations of semiprime rings.

These are the few ideas which arise in course of our study. We do hope several others will emerge as we proceed.

Some of the results of *Chapter 3* are accepted in “Indian Journal of Mathematics and Mathematical Sciences” and “Pacific-Asian Journal of Mathematics”.