Chapter - 5

Problems for Further Study
This chapter is devoted to a discussion on further possible developments of some results, which we plan to study in our future work.

In chapters 2, 3 and 4 we have seen some properties of derivations, generalized derivations and Jordan derivations in associative rings. Yen[43] studied nonassociative rings with a special derivation. Using this we wish to study other properties of derivations in nonassociative rings.

In section 5.1, we discuss some results on generalized derivation of prime rings related to a Lie ideal \( U \). If \( U \) is a Lie ideal of \( R \) such that \( u^2 \in U \) for all \( u \in U \) and \( f \) is a generalized derivation associated with a derivation \( d \neq 0 \) satisfying either \( [d(x), f(y)] + xy = 0 \) or \( [d(x), f(y)] = xy \) for all \( x,y \in U \), then \( U \subseteq Z \). We wish to try for some more properties of the generalized derivations with commutativity and anticommutativity conditions. In section 5.2, we present some results on orthogonal generalized derivations of semiprime rings. If \((D,d)\) and \((G,g)\) are generalized derivations and \( U \) is a nonzero ideal of a semiprime ring \( R \) such that the left annihilator \( l(u) = 0 \),
then \((D,d)\) and \((G,g)\) are orthogonal implies \((DG, dg)\) is a generalized derivation on \(U\) and \(D(x)G(y) = 0\) for all \(x,y \in U\). We wish to study some results concerning orthogonal generalized derivations of semiprime rings related to left or right ideals. In section 5.3, we discuss some properties of prime and semiprime rings with permuting 3-derivations. We see that, if \(U\) is a nonzero two-sided ideal of a semiprime ring of char. \(\neq 3\) and suppose that there exists a permuting 3-derivation \(A: RXRXR \rightarrow R\) such that the trace is centralizing on \(U\), then the trace of \(A\) is commuting on \(U\). Using these results we wish to study of some properties of higher derivations of semiprime rings.

5.1. Generalized derivations of prime rings related to a Lie ideal

In [5] Ashraf, Ali and Rani investigated the commutativity of a prime ring \(R\) admitting a generalized derivation \(f\) with associated derivation \(d\) satisfying anyone of the following properties (i) \(d(x) of(y) = 0\), (ii) \([d(x), f(y)] = 0\), (iii) \(d(x) of(y) = xoy\), (iv) \(d(x) of(y) + xoy = 0\), (v) \(d(x) of(y) - x\) \(\in Z\) (vi) \(d(x) of(y) + x\) \(\in Z\) (vii) \([d(x), f(y)] = [x, y]\) (viii) \([d(x), f(y)] + [x, y] = 0\) for all \(x, y \in U\), where \(U\) is a nonzero ideal of \(R\). In [12], Bergen, Herstein and Kerr found the relationship between the derivations and Lie ideals of a prime ring, and obtained some useful results.

In this section, we discuss some results on generalized derivations of prime rings related to a Lie ideal \(U\). If \(U\) is a Lie ideal of \(R\) such that \(u^2 \in U\)
for all $u \in U$ and $f$ is a generalized derivation associated with a derivation $d \neq 0$ satisfying either $[d(x), f(y)] + xy = 0$ or $[d(x), f(y)] = xy$ for all $x, y \in U$, then $U \subseteq Z$. 

We know that an additive subgroup $U$ of $R$ is said to be a Lie ideal of $R$ if $[u, r] \in U$ for all $u \in U$, $r \in R$ and an additive function $f: R \to R$ is a generalized derivation if there exists a derivation $d: R \to R$ such that $f(xy) = f(x)y + xd(y)$ holds for all $x, y \in R$.

Throughout this section $R$ will denote a prime ring, $U$ is a Lie ideal and $Z$ its center. For any $x, y \in R$, we denote $(xoy) = xy + yx$.

First we prove the following Lemmas:

**Lemma 5.1.1:** If $U$ is a Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$, then $2uv \in U$ for all $u, v \in U$.

**Proof:** For all $w, u, v \in U$, $uv + vu = (u + v)^2 - u^2 - v^2 \in U$.  

On the other hand,

$uv - vu \in U$.  

By adding two expressions, we have $2uv \in U$ for all $u, v \in U$.  

**Lemma 5.1.2:** If $U \not\subseteq Z$ is a Lie ideal of $R$, then $C_R(U) = Z$.

**Proof:** $C_R(U)$ is both a subring and a Lie ideal of $R$. Since $C_R(U)$ cannot contain a nonzero ideal of $R$, otherwise $U$ centralizes a nonzero ideal of $R$, 

so in \( Z \) by Lemma 1.3. of [26] we conclude that \( C_R(U) \subset Z \). Hence \( C_R(U) = Z \). \( \square \)

**Lemma 5.1.3:** If \( U \) is a Lie ideal of \( R \), then \( C_R([U,U]) = C_R(U) \).

**Proof:** If \([U,U] \not\subset Z\), then by Lemma 5.1.2, \( a \in Z \), so \( a \) centralizes \( U \). On the other hand, if \([U,U] \subset Z\) and \( u \in U, x \in R \), then \( \alpha = [u,[u,x]] \in Z \) and \( \alpha u = [u,[u,ux]] \in Z \). If \( \alpha \neq 0 \), then we get \( u \in Z \), which leads to \( \alpha = 0 \). So \( \alpha = 0 \). Thus \([u,[u,x]] = 0\) for all \( x \in R \). But then, by the sublemma on p.5. of [26] \( u \in Z \). Hence \( U \subset Z \). In both cases we see that \( a \in C_R(U) \). This gives that \( C_R([U,U]) = C_R(U) \). \( \square \)

**Lemma 5.1.4:** Set \( V = \{ u \in U/d(u) \in U \} \). If \( U \not\subset Z \), then \( V \not\subset Z \).

**Proof:** We assume that \( V \subset Z \).

Since \([U,U] \subset U\) and \( d([U,U]) \subset U \), we have \([U,U] \subset V \subset Z \).

Hence \( C_R([U,U]) = R \). From Lemma 5.1.2, \( C_R(U) = Z \). But by Lemma 5.1.3, \( C_R([U,U]) = C_R(U) \). That is \( R=Z \), a contradiction. \( \square \)

**Lemma 5.1.5:** If \( U \not\subset Z \) is a Lie ideal of \( R \) and \( aUb = 0 \), then \( a = 0 \) or \( b = 0 \).

**Proof:** By Lemma 1 of [12], there exists an ideal \( M \) of \( R \) such that \([M,R] \not\subset Z \) but \([M,R] \subset U \). If \( u \in U, m \in M \) and \( y \in R \), then \([mau,y] \in [M,R] \subset U \).

Thus \( 0 = a[mau,y]b = a[ma,y]ub + ama[u,y]b = a(may - yma)ub = amayub \), since \( a[u,y]b \in aUb = 0 \). Thus \( aMaRUb = 0 \). If \( a \neq 0 \), since \( R \) is prime, then we obtain \( Ub = 0 \). So, if \( x \in R, u \in U \), then \((ux - xu) \in U \).
Hence \((ux - xu)b = 0\) and so \(uxb = 0\). In other words, \(uRb = 0\). Since \(U \neq 0\), we get \(b = 0\).

Lemma 5.1.6: If \(U\) is a Lie ideal of a semiprime ring \(R\) and \([U,U] \subseteq Z\), then \(U \subseteq Z\).

**Proof:** If \([U,U] = (0)\), then given \(u \in U\) for any \(x \in R\), so commutes with \(u\). Hence \(u \in Z\), that is \(U \subseteq Z\), the desired conclusion. We suppose then that we can find \(s,u \in U\) so that \(a = su - us \neq 0\). By hypothesis \(a \in Z\), for \(x \in R\), let \(d(x) = xu - ux\). Thus \(d(sx) = a\). By hypothesis \(d^2(x) \in Z\) for all \(x \in R\). Let \(\beta = d^2(x)\). Then \(d^2(sx) \in Z\). However, \(d^2(sx) = d^2(s)x + 2d(s)d(x) + sd^2(x) = 2\alpha d(x) + \beta s\), since \(d^2(s) = 0\), \(s\) being in \(U\). In short, for any \(x \in R\), \(2\alpha d(x) + \beta s \in Z\), so commutes with \(s\). This gives \(2\alpha[sd(x) - d(x)s] = 0\).

Put \(x = su\), \(d(su) = sd(u) + d(s)u = \alpha u\), so \(0 = 2\alpha(\alpha u - \alpha us) = 2\alpha^2 (su - us) = 2\alpha^3\). Since \(R\) is of char. \(\neq 2\), \(\alpha^3 = 0\). However \(\alpha \in Z\) and is nilpotent. Since \(R\) is semiprime, we get the contradiction \(\alpha = 0\). In this way the lemma is proved.

Lemma 5.1.7: If \(d \neq 0\) is a derivation of \(R\), and \(U\) is a Lie ideal of \(R\) such that \(d(U) \subseteq Z\), then \(U \subseteq Z\).

**Proof:** If \(U \not\subseteq Z\), then by Lemma 5.1.3, \(V = [U,U] \not\subseteq Z\). But if \(u,w \in U\), then \(d(uw - wu) = (d(u)w - wd(u)) + (ud(w) - d(w)u) = 0\), since \(d(u), d(w) \in Z\). Thus \(d(V) = 0\) by Lemma 5 of [12] we get the contradiction \(V \subset Z\).

Now we prove the following results:
**Theorem 5.1.1:** Let $R$ be a prime ring with char. $\neq 2$, $U$ a Lie ideal such that $u^2 \in U$ for all $u \in U$, and $d$ a generalized derivation associated with $d \neq 0$. If $d(x)of(y) = 0$ for all $x,y \in U$, then $U \subseteq Z$.

**Proof:** If we assume that $U \not\subseteq Z$. Then $V \not\subseteq Z$ by Lemma 5.1.4. Now we have $d(x)of(y) = 0$ for all $x,y \in U$. If we replace $y$ by $2yz$, by Lemma 5.1.1 and using char. $\neq 2$, then we get $d(x)of(yz) = 0$ for all $x,y,z \in U$.

$\Rightarrow (d(x)oy)d(z) = y[d(x),d(z)] + (d(x)of(y))z = f(y)[d(x),z] = 0$. Now using the hypothesis, the above relation implies $d(x)of(y) = 0$ for all $x,y,z \in U$.

For any $x \in V$, we replace $z$ by $d(x)$ to get

$$(d(x)oy) \hat{d}^2(x) - y[d(x),d^2(x)] = 0.$$ 5.1.3.

Now, we replace $y$ by $2zy$ in 5.1.3. Then $(d(x)of(zy))\hat{d}^2(x) - zy[d(x),d^2(x)] = 0$.

This implies that

$$z(d(x)oy) \hat{d}^2(x) + [d(x),z] y \hat{d}^2(x) - zy[d(x),d^2(x)] = 0.$$ 5.1.4.

By combining 5.1.3 with 5.1.4, we get $[d(x),z] y \hat{d}^2(x) = 0$ for $x \in V$ and $y,z \in U$. In particular, $[d(x),z] y \hat{d}^2(x) = 0$, and hence $[d(x),z] U \hat{d}^2(x) = 0$ for $x \in V$. Then, either $[d(x),z] = 0$ or $\hat{d}^2(x) = 0$ by Lemma 5.1.5. Now let $V_1 = \{x \in V/ [d(x),z] = 0\}$ and $V_2 = \{x \in V/ \hat{d}^2(x) = 0\}$. Then $V_1, V_2$ are both additive subgroups of $V$ and $V_1 \cup V_2 = V$.

Thus, either $V = V_1$ or $V = V_2$ by Lemma 2.6 [37]. If $V = V_1$, then Lemma 2.7 [32] gives $V \subseteq Z$, a contradiction. On the other hand, if $V = V_2$, then $V \subseteq Z$ by Lemma 2.8 [32] again a contradiction.
Theorem 5.1.2: Let $R$ be a prime ring with char. $\neq 2$, $U$ a Lie ideal such that $u^2 \in U$ for all $u \in U$, $f$ a generalized derivation associated with $d \neq 0$. If $[d(x), f(y)] = 0$ for all $x, y \in U$, then $U \subseteq Z$.

Proof: If we assume that $U \not\subseteq Z$, then $V \not\subseteq Z$ by Lemma 5.1.4. By hypothesis, we have

$$[d(x), f(y)] = 0,$$

for all $x, y \in U$.  

If we replace $y$ by $2yz$ in 5.1.5. and using char. $\neq 2$, then

$$f(y)[d(x), z] + y[d(x), d(z)] + [d(x), y] d(z) = 0,$$

for all $x, y, z \in U$.

For any $x \in V$, replacing $z$ by $2zd(x)$ in 5.1.6. and using 5.1.6, we get

$$yz[d(x), d^2(x)] + y[d(x), z] d^2(x) + [d(x), y] zd^2(x) = 0.$$

Again replacing $y$ by $2ty$ in 5.1.7 and using 5.1.7 we get $[d(x), t] yzd^2(x) = 0$ for all $x \in V$ and $y, z, t \in U$. In particular, $[d(x), x] yzd^2(x) = 0$. That is, $[d(x), x] Ud^2(x) = 0$. Then either $[d(x), x] = 0$ or $Ud^2(x) = 0$ (i.e., $d^2(x) = 0$).

Now let $V_1 = \{x \in V | [d(x), x] = 0\}$ and $V_2 = \{x \in V/ Ud^2(x) = 0\}$. Then $V_1, V_2$ are both additive subgroups of $V$ and $V = V_1 \cup V_2 = V$.

Thus, either $V = V_1$ or $V = V_2$ by Lemma 2.6 [37]. If $V = V_1$, then Lemma 2.7 [32] gives $V \subseteq Z$, a contradiction. On the other hand, if $V = V_2$, then $Ud^2(x) = 0$. Hence $d^2(x) Ud^2(x) = 0$ for all $x \in V$. Thus, Lemma 5.1.5. implies that $d^2(x) = 0$ for all $x \in V$. Now, we have obtained $V \subseteq Z$ by Lemma 2.8 [32], again a contradiction. □
**Theorem 5.1.3:** Let $R$ be a prime ring with char. $\neq 2$, $U$ a Lie ideal such that $u^2 \in U$ and $f$ a generalized derivation associated with $d \neq 0$. If $d(x)of(y) = xoy$ for all $x,y \in U$, then $U \subseteq Z$.

**Proof:** We have $d(x)of(y) = xoy$ for all $x,y \in U$. If $f = 0$, then $xoy = 0$ for all $x,y \in U$. If we replace $y$ by $2yz$ and using char. $\neq 2$, then we get $y[x,z] = 0$, for all $x,y,z \in U$. In particular, $[x,z]y[x,z] = 0$ (i.e., $[x,z] U [x,z] = 0$), and hence $[x,z] = 0$ (i.e., $[U,U] = 0$) by Lemma 5.1.5. Then Lemma 5.1.6 implies that the required result. Therefore, we assume that $f \neq 0$, for all $x,y \in U$, we have

$$d(x)of(y) = xoy. \quad 5.1.8.$$ 

If we replace $y$ by $2yz$, then

$$(d(x)oy)d(z) - y[d(x),d(z)] + (d(x)of(y))z - f(y)[d(x),z] = (xoy)z - y[x,z]. \quad 5.1.9.$$ 

By combining 5.1.8 with 5.1.9, we get

$$ (d(x)oy)d(z) - y[d(x),d(z)] - f(y)[d(x),z] + y[x,z] = 0, \quad 5.1.10.$$ 

for all $x,y,z \in U$.

For any $x \in V$, if we replace $z$ by $d(x)$ in 5.1.10, then

$$ (d(x)oy)d^2(x) - y[d(x),d^2(x)] + y[x,d(x)] = 0. \quad 5.1.11.$$ 

Now, replacing $y$ by $2yz$ in 5.1.11, we get

$$(z(d(x)oy) + [d(x),z]y) d^2(x) - zy[d(x),d^2(x)] + zy[x,d(x)] = 0. \quad 5.1.12.$$ 

By combining 5.1.11 with 5.1.12, we get $[d(x),z]y d^2(x) = 0$ for all $x \in V$ and $y,z \in U$. In particular, $[d(x),x]y d^2(x) = 0$. So we get $[d(x),x]U d^2(x) = 0$. 127
The rest of the proof is the same with the end of Theorem 5.1.1, and we get the required result. □

Now using, the similar techniques, we also prove the following, two Theorems.

**Theorem 5.1.4:** Let $R$ be a prime ring with char. $\neq 2$, $U$ a Lie ideal such that $u^2 \in U$ for all $u \in U$, and $f$ a generalized derivation associated with $d \neq 0$. If $d(x)f(y) + xoy = 0$ for all $x, y \in U$, then $U \subseteq Z$.

**Theorem 5.1.5:** Let $R$ be a prime ring with char. $\neq 2$, $U$ a Lie ideal such that $u^2 \in U$ for all $u \in U$, and $f$ a generalized derivation associated with $d \neq 0$. If either $[d(x)f(y)] = [x,y]$ or $[d(x)f(y)] + [x,y] = 0$ for all $x, y \in U$, then $U \subseteq Z$.

**Theorem 5.1.6:** Let $R$ be a prime ring with char. $\neq 2$, $U$ a Lie ideal such that $u^2 \in U$ for all $u \in U$, and $f$ a generalized derivation associated with $d \neq 0$. If $d(x)f(y) - xy \in Z$ for all $x, y \in U$, then $U \subseteq Z$.

**Proof:** We assume that $U \not\subseteq Z$. Then $V \not\subseteq Z$ by Lemma 5.1.4. By hypothesis, we have $d(x)f(y) - xy \in Z$. If $f = 0$, then $xy \in Z$ for all $x, y \in U$. In particular, $[xy,x] = 0$ and hence $x[y,x] = 0$. If we replace $y$ by $2yz$, then $xy[z,x] = 0$ (i.e., $xU[z,x] = 0$). Hence by Lemma 5.1.5, either $x = 0$ or $[z,x] = 0$. But $x = 0$ also implies that $[z,x] = 0$. Hence for any $x, z \in U$, we have $[z,x] = 0$. Then Lemma 5.1.6 gives us the required result. Now we assume that $f \neq 0$. For any $x, z \in U$, we have $d(x)f(y) - xy \in Z$. If we replace $y$ by $2yz$ and using char. $\neq 2$, then we get $d(x)f(yz) - xyz \in Z$, for any
That is, \((d(x)f(y) - xy)z + [d(x),z]yd(z) = 0\), for any \(x,y,z \in U\). For any \(x \in V\), replace \(y\) by \(2d(x)y\) in the above relation to get \([d(x),z]d(x)yd(z) = 0\) (i.e., \([d(x),z]d(x) Ud(z) = 0\)). Then we have \([d(x),z]d(x) = 0\) or \(d(z) = 0\) by Lemma 5.1.5. Now, let \(U_1 = \{z \in U|[d(x),z]d(x) = 0\}\) and \(U_2 = \{z \in U|d(x) = 0\}\).

Then \(U_1, U_2\) are both additive subgroups of \(U\) and \(U_1 \cup U_2 = U\). Thus, either \(U = U_1\) or \(U = U_2\) by Lemma 2.6 [37]. If \(U = U_1\), replace \(z\) by \(2zy\), then we get \([d(x),z]yd(x) = 0\), and hence \([d(x),z] = 0\), (especially, \([d(x),x] = 0\)) or \(d(x) = 0\). For all \(x \in U\), we have \([d(x),x] = 0\). Thus \(U \subseteq Z\) by Lemma 2.7 [32], a contradiction. On the other hand, if \(U = U_2\), then \(d(U) = 0\) and hence \(U \subseteq Z\) by Lemma 5.1.7, again a contradiction. □

The following is proved as in Theorem 5.1.6 with necessary variations.

**Theorem 5.1.7:** Let \(R\) be a prime ring with char. \(\neq 2\), \(U\) a Lie ideal such that \(u^2 \in U\) for all \(u \in U\), and \(f\) a generalized derivation associated with \(d \neq 0\). If \(d(x)f(y) + xy \in Z\) for all \(x,y \in U\), then \(U \subseteq Z\).

**Theorem 5.1.8:** Let \(R\) be a prime ring with char. \(\neq 2\), \(U\) a Lie ideal such that \(u^2 \in U\) for all \(u \in U\), and \(f\) a generalized derivation associated with \(d \neq 0\). If \([d(x), f(y)] = [x,y]\) for all \(x,y \in U\), then \(U \subseteq Z\).

**Proof:** If \(f = 0\), then \([x,y] = 0\). That is \([U,U] = 0\), and hence \(U \subseteq Z\) by Lemma 5.1.6. Now we assume that \(f \neq 0\). Then we have

\[[d(x),f(y)] = [x,y],\]

for all \(x,y \in U\).
If we replace $y$ by $2yz$ in 5.1.13 and using 5.1.13, then

$$f(y)[d(x),z] + y[d(x),d(z)] + [d(x),y]d(z) = y[x,z],$$

for all $x,y,z \in U$.

Now for any $x \in V$, replace $z$ by $2zd(x)$ in 5.1.14 and using 5.1.14 to get

$$y[d(x),z]d^2(x) + yz[d(x),d^2(x)] + [d(x),y]zd^2(x) = yz[x,d(x)].$$  \hspace{1cm} 5.1.15.

Again, replace $y$ by $2ty$ in 5.1.15, to get

$$tyz[d(x),d^2(x)] + ty[d(x),z]d^2(x) + t[d(x),y]zd^2(x) + [d(x),t]yzd^2(x) = t yz[x,d(x)].$$

By combining 5.1.15 with 5.1.16, we have $[d(x),t]yzd^2(x) = 0$. In particular, $[d(x),x]yzd^2(x) = 0$ for all $x \in V$ and $y,z \in U$. The rest of the proof is same with the end of Theorem 5.1.1, and we get the required result. □

**Theorem 5.1.9:** Let $R$ be a prime ring with char. $\neq 2$, $U$ a Lie ideal such that $u^2 \in U$ for all $u \in U$, and $f$ a generalized derivation associated with $d \neq 0$. If $[d(x),f(y)] + [x,y] = 0$ for all $x,y \in U$, then $U \subseteq Z$.

Finally, using the similar techniques as in the above theorems, we prove the following.

**Theorem 5.1.10:** Let $R$ be a prime ring with char. $\neq 2$, $U$ a Lie ideal such that $u^2 \in U$ for all $u \in U$, and $f$ a generalized derivation associated with $d \neq 0$. If $[d(x),f(y)] = xoy$ or $[d(x),f(y)] + xoy = 0$ for all $x,y \in U$, then $U \subseteq Z$. 

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Theorem 5.1.11: Let $R$ be a prime ring with char. $\neq 2$, $U$ a Lie ideal such that $u^2 \in U$ for all $u \in U$, and $f$ a generalized derivation associated with $d \neq 0$. If $[d(x), f(y)] + xy = 0$ or $[d(x), f(y)] = xy$ for all $x, y \in U$, then $U \subseteq Z$.

**Proof:** We assume $U \nsubseteq Z$. Then $V \nsubseteq Z$ by Lemma 5.1.4. By hypothesis, we have

$$[d(x), f(y)] + xy = 0,$$

for all $x, y \in U$.

If we replace $y$ by $2yz$ in 5.1.17 and using char. $\neq 2$, then

$$f(y)[d(x), z] + y[d(x), d(z)] + [d(x), y]d(z) = 0,$$

for all $x, y, z \in U$.

For any $x \in V$, replacing $z$ by $2zd(x)$ in 5.1.18 and using 5.1.18, we get

$$y[d(x), d^2(x)] + [d(x), y]d^2(x) = 0.$$

Again replacing $y$ by $2yz$ in 5.1.19 and using 5.1.19, we get $[d(x), x]d^2(x) = 0$ for all $x \in V$ and $y, z \in U$. In particular, $[d(x), y]d^2(x) = 0$, and hence $[d(x), x]Ud^2(x) = 0$ for $x \in V$. Then, either $[d(x), x] = 0$ or $d^2(x) = 0$ by Lemma 5.1.5. Now let $V_1 = \{x \in V / [d(x), x] = 0\}$ and $V_2 = \{x \in V / d^2(x) = 0\}$.

Then $V_1, V_2$ are both additive subgroups of $V$ and $V_1 \cup V_2 = V$.

Thus, either $V = V_1$ or $V = V_2$ by Lemma 2.6 [37]. If $V = V_1$, then Lemma 2.7 [32] gives $V \subseteq Z$, a contradiction. On the other hand, if $V = V_2$, then $V \subseteq Z$, by Lemma 2.8 [32], again a contradiction. □

We wish to try for some more properties of the generalized derivations with commutativity and anticommutativity conditions.
5.2. Orthogonal generalized derivations of semiprime rings

Bresar and Vukman [16] introduced the notion of orthogonality for a pair $d,g$ of derivations on a semiprime ring and they gave several necessary and sufficient conditions for $d$ and $g$ to be orthogonal.

In this section, we present some results on orthogonal generalized derivations of semiprime rings. If $(D,d)$ and $(G,g)$ are generalized derivations and $U$ is a nonzero ideal of a semiprime ring $R$ such that the left annihilator $l(u) = 0$, then $(D,d)$ and $(G,g)$ are orthogonal implies $(DG,dg)$ is a generalized derivation on $U$ and $D(x)G(y) = 0$ for all $x,y \in U$.

We know that two additive maps $d,g : R \to R$ are orthogonal if $d(x)Rg(y) = 0 = g(y)Rd(x)$ for all $x,y \in R$. Two generalized derivations $(D,d)$ and $(G,g)$ of $R$ are orthogonal if $D(x)RG(y) = 0 = G(y)RD(x)$ for all $x,y \in R$.

Throughout this section $R$ will denote a semiprime ring of char. $\neq 2$, $U$ is a nonzero ideal of $R$ and $Ann(U)$ is annihilator of $U$. Here we note that $U \cap l(U) = 0$ (or $U \cap r(U) = 0$) where $l(U)$ and $r(U)$ denote the left annihilator and right annihilator of $U$, respectively.

We require the following Lemmas of [44]:

**Lemma 5.2.1:** Let $R$ be a semiprime ring of char. $\neq 2$, $U$ a nonzero ideal of $R$ and $a,b$ the elements of $R$. Then the following conditions are equivalent.

(i) $axb = 0$ for all $x \in U$.
(ii) $bxa = 0$ for all $x \in U$.
(iii) $axb + bxa = 0$ for all $x \in U$. 

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Moreover, if one of the three conditions is fulfilled and \( l(l) = 0 \), then 
\[ ab = ba = 0. \]

**Lemma 5.2.2:** Let \( R \) be a semiprime ring and \( U \) be a nonzero ideal of \( R \). 
Suppose that additive mappings \( F \) and \( H \) of \( R \) into itself satisfy \( F(x) \) \( UH(x) = 0 \) for all \( x \in U \). Then \( F(x) UH(y) = 0 \) for all \( x, y \in U \).

Now we prove the following results:

**Lemma 5.2.3:** Let \((D,d)\) and \((G,g)\) be generalized derivations of \( R \) and \( l(U) = 0 \). If \( D(U)UG(U) = 0 \), then \( D(R)RG(R) = 0 \).

**Proof:** By \( 0 = D(x)zG(y) = G(y)zD(x) \) for all \( x, y, z \in U \) and Lemma 5.2.1, we have \( 0 = D(x)g(r) = g(r)D(x) \) and by \( g(r)D(x) = 0 \), we get \( 0 = g(r)d(s) = d(s)g(r) \). Using these relations, we have \( D(s)xg(r) = 0 \) and so by \( 0=D(xz)G(y) \), we obtain \( d(z)G(y) = 0 \). Therefore \( 0 = D(rx)G(sy) = D(r)xG(s)y \), which shows \( D(r)xG(s) = 0 \). If we replace \( x \) by \( r'G(s)sD(r) r' \) for some \( r' \in R \), then we have \( D(r) r' G(s) = 0 \), as desired. \( \Box \)

Moreover, we have the following:

**Lemma 5.2.4:** Let \((D,d)\) and \((G,g)\) be generalized derivations of \( R \) and \( U \) an ideal of \( R \) such that \( l(U) = 0 \). Then the following conditions are equivalent.

(i) For any \( x, y \in U \), the following relations hold.

(i)' \[ D(x)G(y) + G(x)D(y) = 0. \]

(ii)' \[ d(x)G(y) + g(x)D(y) = 0. \]

(ii) \[ D(x)G(y) = d(x)G(y) = 0 \text{ for all } x, y \in U. \]
(iii) \( D(x)G(y) = 0 \) for all \( x, y \in U \) and \( dG = dg = 0 \) for all \( x, y \in U \).

(iv) \((DG, dg)\) is a generalized derivation from \( U \) to \( R \) and \( D(x)G(y) = 0 \)
for all \( x, y \in U \).

**Proof:** (i) \( \Leftrightarrow \) (ii): By (i)', (ii)' Lemmas 5.2.1 and 5.2.2, we have
\[ 0 = D(x)zG(y) = D(x)G(y) \]
and using this \( d(z)G(y) = 0 \). This shows (ii). And the converse is easily obtained by the relations \( D(x)G(y) = G(y)D(x) = 0 \) and Lemma 5.2.1.

(ii) \( \Rightarrow \) (iii): By assumption, \( D(x)zG(y) = d(x)zg(y) = 0 \). Then by Lemma 5.2.3, \( d \) and \( g \) are orthogonal, which shows \( dg = 0 \). Moreover, by \( 0 = d(x)G(y) \) and Lemma 5.2.1, we have \( 0 = d(d(r)sG(y)) = d(r)sdG(y) \) for \( r, s \in R \).

If we take \( r = G(y) \), then we have \( dG(y) = 0 \). Since \( D(x)G(y) = G(y)d(x) = d(x)g(y) = 0 \), using Lemma 5.2.3, we obtain \( dG(r) = 0 \), this gives (iii).

(iii) \( \Rightarrow \) (iv): By \( dG = dg = 0 \), we have
\[ G(x)d(y) + d(x)g(y) = 0 = g(x)d(y) + d(x)g(y). \]

Then by the proof of (i) \( \Rightarrow \) (ii), we see that \( d(x)g(y) = 0 \) and so \( G(x)d(y) = 0 \).

Moreover, by \( 0 = D(x)G(y) \), we have \( D(x)g(z) = 0 \). Therefore \( DG(xy) = DG(x)y \) and thus \((DG, dg = 0)\) is a generalized derivation from \( U \) to \( R \).

(iv) \( \Rightarrow \) (ii): \((DG, dg)\) is a generalized derivation if and only if \( G(x)d(y) + D(x)g(y) = 0 \) \( \Leftrightarrow \) \( g(x)d(y) + d(x)g(y) \). So we obtain \( dg = 0 \). Furthermore by \( 0 = D(x)G(y) \), we get \( D(x)g(y) = 0 \) and by the above relation, we see
\[ G(x)d(y) = 0. \] Therefore \( G(x)zd(y) = 0 \) and by Lemma 5.2.1, we arrive at \( d(y)G(x) = 0 \). This shows (ii). \[ \square \]

**Theorem 5.2.1:** Let \((D,d)\) and \((G,g)\) be generalized derivations of \(R\) and \(U\) be a nonzero ideal of \(R\) such that \(l(U) = 0\). Then following conditions are equivalent.

(i) \((D,d)\) and \((G,g)\) are orthogonal.

(ii) For all \(x,y \in U\), the following relations hold.

(i)' \[ D(x)G(y) + G(x)D(y) = 0. \]

(ii)' \[ d(x)G(y) + g(x)D(y) = 0. \]

(iii) \[ D(x)G(y) = d(x)G(y) = 0 \text{ for all } x,y \in U. \]

(iv) \[ D(x)G(y) = 0 \text{ for all } x,y \in U \text{ and } dG(x) = dg(x) = 0 \text{ for all } x,y \in U. \]

(v) \((DG,dg)\) is a generalized derivation on \(U\) and \(D(x)G(y) = 0\) for all \(x,y \in U\).

**Proof:** (i) \(\Rightarrow\) (ii), (iii), (iv), and (v) and clear by [3]. Since (ii), (iii), (iv) and (v) are equivalent by Lemma 5.2.4, we assume (iii). This implies that \(0 = (D(x)z + xd(z))G(y) = D(x)zG(y)\). Then we have \(D(U)UG(U) = 0\). Thus by Lemma 5.2.3, we have 5.2.1 (iii) \(\Rightarrow\) (i). \[ \square \]

**Remark 5.2.1:** If \((DG,dg)\) is a generalized derivations on \(U\) and \(l(U) = 0\), then \((DG,dg)\) is a generalized derivation on \(R\).

**Proof:** It is easily seen that \((DG,dg)\) is a generalized derivation on \(U\) if and only if \(G(x)d(y) + D(x)g(y) = 0, d(x)g(y) + g(x)d(y) = 0\).
Then by the second relation, we have \( d \) and \( g \) are orthogonal. By the first relation 
\[ 0 = G(x)d(y) + D(x)g(y), \]
we get 
\[ 0 = G(x)zd(y) + D(x)zg(y). \]
Hence replacing \( z \) by \( g(y)z \) in this relation and using the orthogonality of the derivations \( d \) and \( g \), we obtain 
\[ 0 = D(x)g(y)zg(y) \]
which implies that 
\[ D(x)g(y) = G(x)d(y) = 0. \]
Moreover by 
\[ 0 = D(x)g(ry), \]
we get 
\[ 0 = D(x)g(r) \]
for all \( r \in R \). Using this relation we have 
\[ D(s)xg(r) = 0 \]
and similarly we can see that 
\[ D(s)g(r) = G(s)d(r) = 0. \]
Thus we obtain 
\[ DG(rs) = DG(r)s \]
for all \( r,s \in R \) which completes the proof.

\[ \square \]

**Theorem 5.2.2:** Let \((D,d)\) and \((G,g)\) be generalized derivations of \( R \) and 
\( l(U) = 0 \). Then the following conditions are equivalent.

(i) \((DG,dg)\) is a generalized derivation on \( U \).

(ii) \((GD,gd)\) is a generalized derivation on \( U \).

(iii) \( D \) and \( g \) are orthogonal, and \( G \) and \( d \) are orthogonal.

The proof of the Theorem 5.2.2. is clear by Remark 5.2.1. and [3] \( \square \)

**Corollary 5.2.1:** Let \((D,d)\) be a generalized derivations of \( R \) and \( l(U) = 0 \). If 
\( (D^2,d^2) \) is a generalized derivation on \( U \), then \( d = 0 \).

**Proof:** By using the fact that \((D^2,d^2)\) is a generalized derivation on \( U \) implies that \( d \) and \( d \) are orthogonal. Therefore we get \( d = 0 \) by the semiprimeness of \( R \). \( \square \)

**Corollary 5.2.2:** Let \((D,d)\) be a generalized derivations of \( R \) and \( l(U) = 0 \). If 
\( D(x)D(y) = 0 \) for all \( x,y \in U \), then \( D = d = 0 \).
Proof: By the hypothesis, we have $0 = D(x)D(yz) = D(x)D(y)z + D(x)yd(z) = D(x)yd(z)$ for all $x,y,z \in U$. In particular, we see that $d(z)D(x) = 0$ for all $x,z \in U$ by Lemma 5.2.1. If we replace $x$ by $xy$ in the last relation, then we get $0 = d(z)D(x)y + d(z)xd(y) = d(z)xd(y)$ for all $x,y,z \in U$. By [44], we obtain $d(s) Rd(r) = 0$ for all $s,r \in R$. In particular, $d(s) Rd(s) = 0$ for all $s \in R$. Thus $d = 0$ by the semiprimeness of $R$. Then we have $0 = D(xz) D(y) = D(x)zD(y)$ for all $x,y,z \in U$. By Lemma 5.2.3, we arrive at $D(r)RD(s) = 0$ for all $r,s \in R$. In particular, $D(r)RD(r) = 0$ for all $r \in R$ which implies $D = 0$, as desired.

We wish to study some results concerning orthogonal generalized derivations of semiprime rings related to left or right ideals.

5.3. Prime and Semiprime rings with permuting 3-derivations

Posner [36] first studied the theory of centralizing (commuting) maps on prime rings and proved that the existence of a nonzero centralizing derivation on a prime ring $R$ implies that $R$ is commutative. Vukman [39,40] investigated symmetric bi-derivations on prime and semiprime rings.

In this section, we discuss some properties of prime and semiprime rings with permuting 3-derivations. We see that, if $U$ is a nonzero two-sided ideal of a semiprime ring of char. $\neq 3$ and suppose that there exists a
permuting 3-derivation $\Delta: R\times R\times R \to R$ such that the trace is centralizing on $U$, then the trace of $\Delta$ is commuting on $U$.

We know that a map $\Delta: R\times R\times R \to R$ is permuting if the equation
$$\Delta(x_1,x_2,x_3) = \Delta(x_{\pi(1)},x_{\pi(2)},x_{\pi(3)})$$
hold for all $x_1,x_2,x_3 \in R$ and for every permutation $\{\pi(1), \pi(2), \pi(3)\}$. A map $\delta: R \to R$ defined by $\delta(x) = \Delta(x,x,x)$ for all $x \in R$, where $\Delta: R\times R\times R \to R$ is a permuting map, is called the trace of $\Delta$. It is obvious that, in case when $\Delta: R\times R\times R \to R$ is a permuting map which is also 3-additive (i.e., additive in each argument), the trace $\delta$ of $\Delta$ satisfies relation
$$\delta(x+y) = \delta(x) + \delta(y) + 3\Delta(x,x,y) + 3\Delta(x,y,y),$$
for all $x,y \in R$.

Since we have
$$\Delta(o,y,z) = \Delta(o,o,y,z) = \Delta(o,yz) + \Delta(o,y,z),$$
for all $y,z \in R$, we obtain
$$\Delta(o,y,z) = 0$$
for all $y,z \in R$. Hence we get $0 = \Delta(o,y,z) = \Delta(x-x,y,z) = \Delta(x,y,z) + \Delta(-x,y,z)$ and so $\Delta(-x,y,z) = -\Delta(x,y,z)$ for all $x,y,z \in R$. This shows that $\delta$ is an odd function.

A 3-additive map $\Delta: R\times R\times R \to R$ is a 3-derivation if the relations
$$\Delta(x_1x_2,y,z) = \Delta(x_1,y,z)x_2 + x_1\Delta(x_2,y,z),$$
$$\Delta(x,y_1,y_2,z) = \Delta(x,y_1,z)y_2 + y_1\Delta(x,y_2,z)$$
and
$$\Delta(x,y,z_1z_2) = \Delta(x,y,z_1)z_2 + z_1\Delta(x,y,z_2),$$
for all $x,y,z,x_0,y_0,z_i \in R$, $i = 1,2$. If $\Delta$ is permuting, then the above three relations are equivalent to each other.

Throughout this section $R$ will denote a ring and $Z$ its center.
To prove the results we require the following Lemma [35]:

**Lemma 5.3.1**: Let $R$ be a prime ring. Let $d: R \to R$ be a derivation and $a \in R$. If $ad(x) = 0$ holds for all $x \in R$, then we have either $a = 0$ or $d = 0$.

We begin our investigation of permuting 3-derivations with the next result.

**Lemma 5.3.2**: Let $R$ be a noncommutative 3!-torsion free prime ring and let $U$ be a nonzero two-sided ideal of $R$. Suppose that there exists a permuting 3-derivations $\Delta: RXRXR \to R$ such that $\delta$ is commuting on $U$, where $\delta$ is the trace of $\Delta$. Then we have $\Delta = 0$.

**Proof**: We suppose that

$$[\delta(x), x] = 0,$$

for all $x \in U$.

By substituting $x = x + y$ and to linearize the equation 5.3.1, then we get

$$0 = [\delta(x), y] + [\delta(y), x] + 3[\Delta(x, x, y), x] + 3[\Delta(x, y, y), x] + 3[\Delta(x, x, y), y] + 3[\Delta(x, y, y), y],$$

for all $x, y \in U$.

We put $-x$ instead of $x$ in equation 5.3.2, and comparing 5.3.2 with the result, then

$$[\Delta(x, y, y), x] + [\Delta(x, x, y), y] = 0,$$

for all $x, y \in U$,

since $\delta$ is odd. We set $x = x + y$ in equation 5.3.3 & using equations 5.3.1, and 5.3.3, to obtain
Let us write in equation 5.3.4 \(yx\) instead of \(x\). Then we get

\[
0 = [\delta(y),yx] + 3[\Delta(x,y,y),y],
\]

\[
= y[\delta(y),x] + 3\delta(y)[x,y] + 3y[\Delta(x,y,y),y],
\]

\[
= y\{[\delta(y),x] + 3 \{\Delta(x,y,y),y]\} + 3\delta(y) [x,y],
\]

which implies that

\[
\delta(y) [x,y] = 0, \quad 5.3.5.
\]

for all \(x,y \in U\).

Since \(U\) is a nonzero commutative prime ring, it follows from equation 5.3.5 and Lemma 5.3.1 that, for all \(y \in U\) with \(y \not\in Z\), we have \(\delta(y) = 0\) since for every fixed \(y \in U\), a map \(x \mapsto [x,y]\) is a derivation on \(U\).

Now, let \(x \in U\) with \(x \in Z\) and \(y \in U\) with \(y \not\in Z\). Then \(x+y \not\in Z\) and \(-y \in Z\). Thus we have

\[
0 = \delta(x+y) = \delta(x) + 3\Delta(x,x,y) + 3\Delta(x,y,y) \quad \text{and}
\]

\[
0 = \delta(x-y) = \delta(x) - 3\Delta(x,x,y) + 3\Delta(x,y,y),
\]

which shows that

\[
\delta(x) + 3\Delta(x,y,y) = 0. \quad 5.3.6.
\]

By replacing \(y \in U(y \not\in Z)\) by \(2y\) in equation 5.3.6 and using 5.3.6, we obtain that \(\Delta(x,y,y) = 0\) and so the relation 5.3.6 gives \(\delta(x) = 0\) for all \(x \in U\) with \(x \in Z\). Therefore, we conclude that \(\delta(x) = 0\) for all \(x \in U\).

On the other hand, since the relation \(\delta(x+y) = \delta(x) + \delta(y) + 3\Delta(x,x,y) + 3\Delta(x,y,y)\) is fulfilled for all \(x,y \in U\), it follows that
\[ \Delta(x,x,y) + \Delta(x,y,y) = 0, \tag{5.3.7} \]

for all \( x,y \in U. \)

By substituting \( y+z \) for \( y \) in equation 5.3.7 and using 5.3.7, we obtain that \( 2\Delta(x,y,z) = 0 = \Delta(x,y,z) \) for all \( x,y,z \in U. \)

Let us substitute \( rx(r \in R) \) for \( x \) in the above relation \( \Delta(r,y,z) = 0 \) for all \( x,y,z \in U. \) Then we have \( \Delta(r,y,z)x = 0, \) that is, \( \Delta(r,y,z)U = \{0\} \). Since \( R \) is prime, we get \( \Delta(r,y,z) = 0 \) for all \( y,z \in U \) and \( r \in R. \) Also by substituting \( ys(s \in R) \) for \( y \) in this relation, we have \( y\Delta(r,s,z) = 0 \) and so \( U\Delta(r,s,z) = \{0\}. \)

Again, by primeness of \( R \), we obtain that \( \Delta(r,s,z) = 0 \) for all \( z \in U \) and \( r,s \in R. \) Furthermore, we replace \( z \) by \( tz(t \in R) \) in the relation \( \Delta(r,s,z) = 0. \)

Then \( \Delta(r,s,t)z = 0, \) i.e., \( \Delta(r,y,t)U = \{0\}. \) The primeness of \( R \) implies that \( \Delta(r,s,t) = 0 \) for all \( r,s,t \in R \) which completes the proof of the theorem. \( \square \)

We continue with the following result for permuting \( 3 \)-derivations on semiprime rings.

**Theorem 5.3.3:** Let \( R \) be a noncommutative \( 3 \)-torsion free semiprime ring and let \( U \) be a nonzero two-sided ideal of \( R. \) Suppose that there exists a permuting \( 3 \)-derivation \( \Delta:RXRXR \to R \) such that \( \delta \) is centralizing on \( U, \) where \( \delta \) is the trace of \( \delta. \) Then \( \delta \) is commuting on \( U. \)

**Proof:** We assume that

\[ [\delta(x),x] \in Z, \tag{5.3.8} \]

for all \( x \in U. \)

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By linearizing equation 5.3.8 and again using 5.3.8 we obtain

\[ Z \circ [\delta(x),y] + [\delta(y),x] + 3[A(x,x,y),x] + 3[A(x,y,y),x] \]

\[ + 3[A(x,x,y),y] + 3[A(x,y,y),y], \]

for all \( x, y \in U \).

We substitute \(-x\) for \( x \) in equation 5.3.9 and compare 5.3.9 with the result to get

\[ [\Delta(x,y,y),x] + [\Delta(x,x,y),y] \in Z, \]

for all \( x, y \in U \),
since \( R \) is 3-torsion free.

If we replace \( x \) by \( x+y \) in equation 5.3.10 and using 5.3.10, then

\[ [\delta(y),x] + 3[A(x,y,y),y] \in Z, \]

for all \( x, y \in U \).

By taking \( x = y^2 \) in equation 5.3.11 and invoking 5.3.8 shows that

\[ Z \circ [\delta(x),y^2] + 3[A(x,y,y),y] = 8[\delta(x),y]y, \]

for all \( y \in U \).

By commuting with \( \delta(y) \) in equation 5.3.12 gives

\[ 8[\delta(y),y]^2 = 0, \]

for all \( y \in U \).

On other hand, we substitute \( x \) by \( yx \) in equation 5.3.11, then

\[ Z \circ [\delta(y),yx] + 3[A(yx,y,y),y] = y\{[\delta(y),x] + 3[A(x,y,y),y]\} \]

\[ + 3\delta(y)[x,y] + 4[\delta(y),y]x, \]

for all \( x, y \in U \).

Hence we have, for all \( x, y \in U \),
\[ y\{[\delta(y),x] + 3[\Delta(x,y,y),y] + [3\delta(y)[x,y] + 4[\delta(y),y]x,y] = 0 \]

and so we get

\[ 3\delta(y)[[x,y],y] + 7[\delta(y),y][x,y] = 0, \quad 5.3.14. \]

for all \( x,y \in U, \)

according to 5.3.11.

By substituting \( \delta(y)x \) for \( x \) in equation 5.3.14, it follows that

\[ 0 = \delta(y)\{3\delta(y)[[x,y],y] + 7[\delta(y),y][x,y]\} + 6\delta(y)[\delta(y),y][x,y] + 7[\delta(y),y]^2x, \]

for all \( x,y \in U. \)

By equation 5.3.14, implies that

\[ 6\delta(y)[\delta(y),y][x,y] + 7[\delta(y),y]^2x = 0, \quad 5.3.15. \]

for all \( x,y \in U. \)

Let \( x = [\delta(y),y] \) in equation 5.3.15, we arrive at \( 7[\delta(y),y]^3 = 0 \) and so we get

\[ 7[\delta(y),y]^2 R 7[\delta(y),y]^2 = 0. \]

Since \( R \) is semiprime, we deduce that

\[ 7[\delta(y),y]^2 = 0, \quad 5.3.16. \]

for all \( y \in U. \)

Hence, the relations 5.3.13 and 5.3.16 yield \( [\delta(y),y]^2 = 0, \) for all \( y \in U. \)

Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that \( [\delta(y),y] = 0, \) for all \( y \in U. \) This completes the proof of the theorem. □

The following result is an analogue of Posner’s theorem [36].

**Theorem 5.3.4**: Let \( R \) be a 3!-torsion free prime ring and let \( U \) be a nonzero two-sided ideal of \( R. \) Suppose that there exists a nonzero permuting
3-derivation $\Delta: R \rightarrow R$ such that $\delta$ is centralizing on $U$, where $\delta$ be the trace of $\Delta$. Then $R$ is commutative.

**Proof:** We suppose that $R$ is noncommutative. Then it follows from Theorem 5.3.3. that $\delta$ is commuting on $U$. Hence by Lemma 5.3.2. gives $\Delta = 0$ which guarantees the conclusion of the theorem. □

Using the above results on 3-derivations, we wish to study the properties of higher derivations of semiprime rings.