Chapter — 3

Generalized Derivations of Prime and Semiprime Rings
Bresar [14] defined generalized derivation of rings. Hvala [29] studied the properties of generalized derivations in prime rings. In this chapter, we present some properties of prime or semiprime rings and near rings with generalized derivations.

In section 3.1, we extend some results concerning derivations of a prime ring $R$ to the left generalized derivations associated with a derivation $d$ of $R$ and a nonzero right ideal $U$ of $R$ which is semiprime as a ring. In section 3.2, we prove that if $d$ is a nonzero derivation of a prime ring $R$ and $f$ is a generalized derivation, then $f$ is a strong commutativity preserving. Using this, we prove that $R$ is commutative. In section 3.3, we consider a prime near ring $N$ of char. $\neq 2$ with center $Z$ and two generalized derivations $(f,d)$ and $(g,h)$ of $N$. We prove that if $f([x,y]) = 0$ or $f([x,y]) = \pm [x,y]$ or $f([x,y]) = \pm xy$ or $f(xy) \pm xy \in Z$ or $f(xy) \pm yx \in Z$ or $f(x)eZ$, then $N$ is a commutative ring. Also we prove that if $af(x) = 0$ for $a \neq 0 \in N$, then $a \in Z$ and hence $d(a) \in Z$. 

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3.1 On right ideals of prime rings with left generalized derivations

Golbasi [21] extended some well known results concerning derivations of prime rings to the right generalized derivations and a nonzero left ideal of a prime ring which is semiprime as a ring.

In this section, we extend some results concerning derivations of a prime ring $R$ to the left generalized derivations associated with a derivation $d$ of $R$ and a nonzero right ideal $U$ of $R$ which is semiprime as a ring. We prove that if $f$ is a generalized derivation of a prime ring $R$, $U$ is a noncommutative right ideal of $R$ and $[x,f(x)]=0$ or $f(U) \subseteq Z$, then there exists $q \in Q_{r}(R)$ such that $f(x) = qx$, for all $x \in R$.

We know that an additive map $f$ from a ring $R$ to $R$ is a right generalized derivation if there exists a derivation $d$ of $R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$ and $f$ is a left generalized derivation if there exists a derivation $d$ of $R$ such that $f(xy) = d(x)y + xf(y)$ for all $x, y \in R$. Finally, $f$ is a generalized derivation of $R$ associated with $d$ if it is both left and right generalized derivation of $R$.

Throughout this section, $R$ will be a prime ring of char. $\neq 2$, $U$ a nonzero right ideal of $R$ which is semiprime as a ring, $Z$ the center of $R$, $Q_{r}(R)$ the right Martindale ring of quotients, $C$ the extended centroid and $R_{C} = RC$ the central closure.

First we prove the following Lemmas:
**Lemma 3.1.1** : Let $R$ be a prime ring and $U$ a nonzero right ideal of $R$ which is semiprime as a ring. If $aU = 0$ ($Ua = 0$) for all $a \in R$, then $a = 0$.

**Proof** : Since $U \neq \{0\}$, there exist an element $u \in U$ such that $u \neq 0$. Consider that $aRU \subset aU = \{0\}$. Since $u \neq 0$ and $R$ is a prime ring, we have that $a = 0$. □

**Lemma 3.1.2** : Let $f: R \rightarrow R_C$ be an additive map satisfying $f(xy) = xf(y)$, for all $x, y \in R$. Then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.

**Proof** : We extend $f$ from $R$ to $R_C$ such that $\overline{f}(\sum \lambda_i x_i) = \sum \lambda_i f(x_i)$ for all $x_i \in R$ and $\lambda_i \in C$.

Now to show that $\overline{f}$ is well-defined, it is sufficient to prove that $\sum \lambda_i f(x_i) = 0$ implies $\sum \lambda_i f(x_i) = 0$. Let $U$ be a nonzero ideal in $R$ such that $U\lambda_i \subseteq R$ for every $i$.

Let $a \in U$ and we note that factors in the sum $\sum (a\lambda_i)x_i$ lie in $R$. Therefore we have $\overline{f}(\sum \lambda_i x_i) = 0$ implies $a(\sum \lambda_i f(x_i)) = 0$. Since this is true for all $a \in U$, we have $\sum \lambda_i f(x_i) = 0$. By direct computation, we have $\overline{f}(xy) = x \overline{f}(y)$. This proves that $\overline{f}: R_C \rightarrow R_C$ is a right $R_C$ -module map. Hence there exists $q \in Q_r(R_C)$ such that $\overline{f}(x) = qx$, $x \in R_C$. Since $\overline{f}$ is an extension of $f$, this proves the Lemma. □
Lemma 3.1.3: Let $R$ be a prime ring and $U$ a nonzero right ideal of $R$ which is semiprime as a ring. If $d$ is a derivation of $R$ such that $d(U) = 0$, then $d = 0$.

Proof: For all $x \in U, r \in R$, we get

$$0 = d(xr) = xd(r),$$

and so,

$$Ud(R) = 0.$$ 

By Lemma 3.1.1, we obtain that $d = 0$.

Next, we have the following two theorems which are generalization of [8] and [18] respectively.

Theorem 3.1.1: Let $R$ be a prime ring, $U$ a nonzero right ideal of $R$ which is semiprime as a ring and $f$ a left generalized derivation of $R$. If $U$ is noncommutative and $f([x,y]) = 0$, for all $x, y$ in $U$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.

Proof: We replace $y$ by $xy$ in $f([x,y]) = 0$, then

$$0 = f([x,xy]) = f(x[y,y]) = d(x)[x,y] + x f([x,y]),$$

and so, $d(x)[x,y] = 0$ for all $x, y \in U$.

Hence $0 = d(x)[x,y] = d(x)y[x,r] + d(x)[x,y]r$.

Since the second summand is zero, it is clear that $d(x)y[x,r] = 0$, for all $x, y \in U, r \in R$.

By writing $ys, s \in R$, in place of $y$ in the above equation, we get

$$d(x)ys[x,r] = 0$$
Since $R$ is a prime ring, we have $d(x)U = 0$ or $[x, r] = 0$ for all $x \in U, r \in R$.

By Lemma 3.1.1, we get either $d(x) = 0$ or $x \in Z$ for all $x \in U$.

Let $A = \{x \in U \mid d(x) = 0\}$ and $B = \{x \in U \mid x \in Z\}$. Then $A$ and $B$ are two additive subgroups of $(U, +)$ such that $U = A \cup B$. However, a group cannot be the union of proper subgroups. Hence either $U = A$ or $U = B$. If $U = B$, then $U \subseteq Z$, and so $U$ is commutative, which contradicts the hypothesis. So, we must have $d(x) = 0$, for all $x \in U$. By Lemma 3.1.3, we get $d = 0$. Hence, there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$, by Lemma 3.1.2.

**Theorem 3.1.2**: Let $R$ be a prime ring, $U$ a nonzero right ideal of $R$ which is semiprime as a ring and $f$ a left generalized derivation of $R$. If $U$ is noncommutative and $f([x, y]) = \pm [x, y]$, for all $x, y \in U$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x$ in $R$.

**Proof**: We assume that $f([x, y]) = \pm [x, y]$ for all $x, y \in U$.

By replacing $xy$ for $y$ in the above equation, we get $d(x) [x, y] = 0$ for all $x, y \in U$.

By using the same argument as in the proof of Theorem 3.1.1, we get $d = 0$ and so, there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$ by Lemma 3.1.2.

**Corollary 3.1.1**: Let $R$ be prime ring, $U$ a nonzero right ideal of $R$ which is semiprime as a ring and $f$ a left generalized derivation of $R$. If $U$ is noncommutative and $f(xy) = \pm xy$, for all $x, y$ in $U$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$. 

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**Theorem 3.1.3**: Let $R$ be a prime ring, $U$ a nonzero right ideal of $R$ which is semiprime as a ring and $f$ a left generalized derivation of $R$. If $f$ acts as a homomorphism or antihomomorphism on $U$, then there exists $q \in Q_{R}(R)$ such that $f(x) = qx$, for all $x \in R$.

**Proof**: We assume that $f$ acts as a homomorphism on $U$.

Then

$$f(xy) = f(x)f(y) = d(x)y + xf(y), \quad 3.1.1$$

for all $x, y$ in $U$.

We replace $y$ by $yz$, $z \in U$, in the second equality of 3.1.1, we have

$$f(x)f(yz) = f(x)yz + xf(yz) = d(x)yz + xf(y)f(z), \quad 3.1.2$$

since $f$ is a homomorphism. On the other hand, we have

$$f(x)f(yz) = f(xy)f(z) = [d(x)y + xf(y)]f(z). \quad 3.1.3$$

From equations 3.1.2 and 3.1.3 we get

$$d(x)yz = d(x)zf(z).$$

Hence $d(x)y(f(z) - z) = 0$, for all $x, y, z \in U$.

We replace $y$ by $yr$, $r \in R$, in the above equation, then $d(x)yr(f(z) - z) = 0$,

for all $x, y, z \in U, r \in R$.

Since $R$ is a prime ring, we have either $f$ is the identity map on $U$, or $d(U) = 0$.

We suppose that $f(y) = y$, for all $x \in U$. Then
\[ xy = f(xy) \]
\[ = d(x)y + xf(y) \]
\[ xy = d(x)y + xy. \]

and so, \( d(x)y = 0 \), for all \( x, y \in U \).

Hence, we conclude that \( d = 0 \) by Lemma 3.1.1. Thus, there exists \( q \in Q_r(R_C) \) such that \( f(x) = qx \), for all \( x \in R \) by Lemma 3.1.2.

Now we assume that \( f \) acts as an antihomomorphisms on \( U \). Then
\[ \begin{align*}
  f(xy) & = f(y)f(x) = d(x)y + xf(y), \quad 3.1.4 \\
  & \text{for all } x, y \in U.
\end{align*} \]

By replacing \( xy \) for \( y \) in equation 3.1.4 we get
\[ f(xy)f(x) = d(x)xy + xf(xy). \]

Hence \( d(x)yf(x) + xf(y)f(x) = d(x)xy + xf(y)f(x) \), so
\[ d(x)yf(x) = d(x)xy, \quad 3.1.5 \]

for all \( x, y \in U \).

We replace \( y \) by \( yr, r \in R \) in equation 3.1.5 to get
\[ d(x)yrf(x) = d(x)xyr = d(x)yf(x)r. \]

That is,
\[ d(x)y[f(x),r] = 0, \quad 3.1.6 \]

for all \( x, y \in U, r \in R \).

Again by writing \( y \) as \( xs, s \in R \) we have either \( [f(x),r] = 0 \) or \( d(x)U = 0 \), for all \( x \in U, r \in R \). According to Brauer’s Trick and Lemma 3.1.1 we conclude that \( f(U) \subset Z \) or \( d(U) = 0 \). In the second case, the proof is complete. The first case gives that \( f \) acts as a homomorphism on \( U \). Thus, there exists \( q \in Q_r(R_C) \) such that \( f(x) = qx \), for all \( x \in R \).  \( \square \)
Theorem 3.1.4: Let $R$ be a prime ring with char. $\neq 2$, $U$ a nonzero right ideal of $R$ which is semiprime as a ring and $f$ a left generalized derivation of $R$. If $U$ is noncommutative and $[x,f(x)] = 0$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.

Proof: By linearizing $[x,f(x)] = 0$, we get
\[
d(x) [x,y] + [x,d(x)]y = 0,
\]
for all $x, y \in R$.

We replace $y$ by $zy$ in equation 3.1.7 and using this equation, we obtain that
\[
d(x)z [x,y] = 0,
\]
for all $x, y, z \in U$.

By replacing $z$ by $zr$, $r \in R$ in equation 3.1.8, we get $[x,y] = 0$ or $d(x)U = 0$ for all $x, y \in R$. By Lemma 3.1.1, we have either $[x,y] = 0$ or $d(x) = 0$, for all $x \in U$. By a standard argument one of these must be held for all $x \in U$.

The first result cannot hold, since $U$ is noncommutative, so the second possibility gives $d(U) = 0$, and hence $d = 0$. Therefore, the proof is completed by using Lemma 3.1.2. □

Theorem 3.1.5: Let $R$ be a prime ring with char. $\neq 2$, $U$ a nonzero right ideal of $R$ which is semiprime as a ring and $f$ a left generalized derivation of $R$. If $U$ is noncommutative, $d(Z) \neq 0$ and $[f(x), f(y)] = [x,y]$, for all $x, y \in U$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$. 

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**Proof**: By taking $xy$ instead of $y$ in the hypothesis, then we get

$$[x,xy] = [f(x), f(xy)].$$

$$x[x,y] = [f(x), d(x)y + xf(y)]$$

$$= d(x) [f(x), y] + [f(x), d(x)]y + x[f(x), y] + [f(x), xy]$$

and so,

$$[f(x), x] f(y) + d(x) [f(x), y] + [f(x), d(x)]y = 0, \quad 3.1.9$$

for all $x, y \in U$.

We replace $y$ by $cy = yc$, where $c \in Z$, and using 3.1.9, we obtain

$$[f(x), x] d(c)y = 0 \text{ for all } x, y \in U.$$

Since $0 \neq d(c) \in Z$ and $U$ is a nonzero right ideal of $R$, we have $[f(x), x] = 0$,

for all $x \in U$. The proof is now completed by using Theorem 3.1.4. □

**Theorem 3.1.6**: Let $R$ be a prime ring with char. $\neq 2$, $U$ a nonzero right ideal of $R$ which is semiprime as a ring and $f$ a left generalized derivation of ring $R$. If $U$ is noncommutative and $f(U) \subseteq Z$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$, for all $x \in R$.

**Proof**: For all $r \in R$, we have $[f(yr), y] = 0$.

So, $[f(yx), y] = 0$, for all $x, y \in U$.

$$\Rightarrow \quad [d(y)x + yf(x), y] = 0$$

$$\Rightarrow \quad d(y) [x,y] + [d(y), y]x = 0.$$

By expanding this equation, we conclude that

$$d(x)xy = yd(x)x, \quad 3.1.10$$

for all $x, y \in U$. 

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We replace $x$ by $zx$ in equation 3.1.10 and using this equality, we get

$$d(y) zxy = yd(y) zx = d(y) zyx.$$  

That is, $d(y)z [x,y] = 0$, for all $x, y, z \in U$.

By taking $zr, r \in R$ in place of $z$ in the above equation and using the fact that $R$ is prime, we conclude that $d(y) = 0$ or $[x,y] = 0$ for all $x, y \in U$. By the standard argument, we have either $d = 0$ or $U$ is commutative. Since $U$ is noncommutative, the proof is complete. □

**Theorem 3.1.7** : Let $R$ be a prime ring with char. $\neq 2$, $U$ a nonzero right ideal of $R$ which is semiprime as a ring and $f$ a left generalized derivation of $R$ and $a \in R$. If $U$ is noncommutative, $d(Z) \neq 0$ and $[a, f(x)] \in Z$ for all $x \in U$, then $a \in Z$.

**Proof** : Since $d(Z) \neq 0$, there exists $c \in Z$ such that $d(c) \neq 0$. Also since $d$ is a derivation, it is clear that $d(c) \in Z$. By replacing $x$ by $xc = cx$ in the hypothesis, we have

$$Z \ni [a, f(cx)] = [a, d(c)x + cf(x)]$$

$$= c[a, f(x)] + d(c) [a,x]$$

Since, the first term lies in $Z$, we get $d(c) [a,x] \in Z$, for all $x \in U$. Thus, we obtain that $[a,x] \in Z$ for all $x \in U$, and so

$$[r, [a,x]] = 0, \quad 3.1.11$$

for all $x \in U, r \in R$.

By taking $x^2$ instead of $x$ and using equation 3.1.11 we have

$$0 = [r, [a,x]x + x[a,x]] = 2[r, [a,x]x] = 0, \text{ for all } x \in U, r \in R.$$
Since char. \( \neq 2 \) and \([a,x] \in Z\), we obtain \([a,x][x,r] = 0\), for all \( x \in U, r \in R \) and so \([a,x] = 0 \) or \([x,r] = 0\), for all \( x \in U, r \in R \).

Let \( A = \{x \in U/ [a,x] = 0\} \) and \( B = \{x \in U/ x \in Z\} \).

Then \( A \) and \( B \) are two additive subgroups of \( (U, +) \) such that \( U = A \cup B \). By Brauer's Trick, we have either \( U = A \) or \( U = B \). Since \( U \) is noncommutative, we have \( U = A \). Hence \([a,U] = 0\) and so \( a \in Z \). \( \square \)

Now we also have the following

**Corollary 3.1.2**: Let \( R \) be a prime ring with char. \( \neq 2 \), \( U \) a nonzero right ideal of \( R \) which is semiprime as a ring and \( f \) a left generalized derivation of \( R \). If \( U \) is noncommutative, \( d(Z) \neq 0 \) and \([f(U), f(U)] \subseteq Z\), then there exists \( q \in Q_{r}(R_{C}) \) such that \( f(x) = qx \), for all \( x \in R \).

### 3.2 Centralizing generalized derivations on semiprime rings

Bell and Martindale [10] studied centralizing mappings of semiprime rings and proved that if \( d \) is a nonzero derivation of a prime ring \( R \) such that \([d(x),x] = 0\) for all \( x \) in a nonzero left ideal of \( R \), then \( R \) is commutative. Bell and Daif [7] investigated commutativity in prime and semiprime rings admitting a derivation or an endomorphism which is strong commutativity preserving on a nonzero right ideal. Ali and Shah [2] extended some results of Bell and Martindale [10] for generalized derivations.
In this section, we prove that if $d$ is a nonzero derivation of a prime ring $R$ and $f$ is a generalized derivation, then $f$ is a strong commutativity preserving. Using this, we prove that $R$ is commutative.

We know that an additive map $f: R \rightarrow R$ is a generalized derivation on $R$ if $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$, where $d$ is a derivation on $R$. A mapping $f$ is commuting on a left ideal $U$ of $R$ if $[f(x), x] = 0$ for all $x \in U$ and $f$ is centralizing if $[f(x), x] \in Z$ for all $x \in U$. A mapping $f: R \rightarrow R$ is called strong commutativity preserving if $[f(x), f(y)] = [x, y]$ for all $x, y \in R$.

Throughout this section, $R$ will denote a semiprime ring and $Z$ its center. We use the following fact in this section.

**Remark 3.2.1**: For a nonzero element $a \in Z$, if $ab \in Z$, then $b \in Z$.

To prove the main results we require the following Lemmas:

**Lemma 3.2.1**: If $f$ is an additive mapping from $R$ to $R$ such that $f$ is centralizing on a left ideal $U$ of $R$, then $f(x) \in Z$, for all $x \in U \cap Z$.

**Proof**: Since $f$ is centralizing on $U$, we have $[f(x+y), x+y] \in Z$, for all $x, y \in U$. This implies that

$$[f(x), y] + [f(y), x] \in Z.$$  \hspace{1cm} 3.2.1

Now if $x \in Z$, then from equation 3.2.1, $[f(x), y] \in Z$. We replace $y$ by $f(x)y$, then $f(x)[f(x), y] \in Z$. If $[f(x), y] = 0$, then $f(x) \in C_R(U)$, the centralizer of $U$ in $R$ and hence $[10]$ belongs to $Z$. But on the other hand, if $[f(x), y] \neq 0$, it again follows, from Remark 3.2.1 that $f(x) \in Z$. \hfill $\square$
Lemma 3.2.2: Let $R$ be a semiprime ring and $U$ a nonzero ideal of $R$. If $z$ in $R$ centralizes the set $[U,U]$, then $z$ centralizes $U$.

Proof: Let $z$ centralizes $[U,U]$. Then for all $x, y$ in $U$, we have $z[x,xy] = [x,xy]z$, which can be rewritten as $zx[x,y] = x[x,y]z$. Hence $[z,x] [x,y] = 0$ for all $x, y$ in $U$. If we replace $y$ by $yz$, then $[z,x]U [z,x] = \{0\}$. Since $U$ is an ideal, it follows that, $[z,x] U R [z,x]U = \{0\} = U[z,x]RU [z,x]$, so that $[z,x]U = U[z,x] = \{0\}$. Thus $[[z,x],x] = 0$ for all $x$ in $U$ and by [11], $z$ centralizes $U$. □

Now we prove the following results:

Theorem 3.2.1: Let $d: R \rightarrow R$ be a nonzero derivation of a prime ring $R$ and $f$ be a generalized derivation on a nonzero left ideal $U$ of $R$. If $f$ acts as a homomorphisms on $U$, then $f$ is strong commutativity preserving on $U$.

Proof: We assume that $f$ acts as a homomorphism on $U$ and $f$ be a generalized derivation on $U$. Then

$$f(xy) = f(x)f(y) = f(x)y + xd(y), \tag{3.2.1}$$

for all $x, y$ in $U$.

We replace $x$ by $xz, z \in U$, the second equality of 3.2.1 we have

$$f(xz)f(y) = f(xz)y + xzd(y) = f(x)f(z)y + xzd(y), \tag{3.2.2}$$

since $f$ is a homomorphism. On the other hand, we have

$$f(xz)f(y) = f(x)f(z)f(y) = f(x) f(zy) = f(x) \left[f(z)y + zd(y)\right] = f(x)f(z)y + f(x) zd(y). \tag{3.2.3}$$

From equations 3.2.2 and 3.2.3 we get $f(x) zd(y) = xzd(y)$. 63
We replace $x$ by $[x,y]$ in equation 3.2.4, then

$$\langle f([x,y]) - [x,y] \rangle zd(y) = 0.$$ 3.2.4

By replacing $z$ by $rz$, $r \in R$ in the above equation, then

$$\langle f([x,y]) - [x,y] \rangle rz d(y) = 0.$$ 

By the primeness of $R$, we have either $f([x,y]) - [x,y] = 0$ or $Ud(y) = 0 \Rightarrow d(y) = 0.$

Since $d \neq 0$, then $f([x,y]) - [x,y] = 0.$

$$\Rightarrow \quad [f, f] = [x,y].$$

Hence $f$ is a strong commutativity preserving on $U. \quad \square$

**Theorem 3.2.2**: Let $R$ be a semiprime ring and $f$ be a generalized derivation on a nonzero left ideal $U$ of $R.$ If $f$ acts as a homomorphism on $U$, then $f$ is commuting on $U.$

**Proof**: From the above Theorem 3.2.1, $f$ is strong commutativity preserving. Then for all $x, y \in U,$ we have $[x, yx] = [f(x), f(xy)].$ This implies that $[x,y]x = [x,y] f(x)$ and so $[x,y] (f(x) - x) = 0.$

From $[x,xy] = [f(x), f(xy)]$ we can similarly show $(f(x) - x) [x,y] = 0.$

For all $x, y \in U,$ we replace $y$ by $ry$, then $[x,ry](f(x)-x) = 0.$

This implies that $[x,r] U (f(x)-x) = 0$ and so $[x,r]RU (f(x)-x) = 0.$
Since $R$ is semiprime, it must contain a family $w = \{P_{\alpha} | \alpha \in \Lambda\}$ of prime ideals such that $\cap P_{\alpha} = 0$. If $P$ is a member of $w$ and $x \in U$, then from the last equation, we have $U(f(x) - x) \subseteq P$ or $[x,R] \subseteq P$. Suppose $\exists y \in U$ such that $[y,R] \not\subseteq P$. This implies that $U(f(y) - y) \subseteq P$.

Let $z$ be any element of $U$ such that $[y+z,R] \subseteq P$. This means that $[z,R] \not\subseteq P$ and hence $U(f(z) - z) \subseteq P$. On the other hand if $[y+z,R] \subseteq P$, then $U(f(y+z) - (y+z)) \subseteq P$. This implies that $U(f(z) - z) \subseteq P$. Thus we conclude that $U(f(z) - z) \subseteq P$ for all $z \in U$ and hence $[U,U] (f(z)-z) \subseteq P$ for all $z \in U$.

Since $P$ is arbitrary and $\cap P_{\alpha} = 0$, we have $[U,U] (f(z)-z) = \{0\}$ for all $z \in U$.

Similarly, we can show that $(f(z)-z) [U,U] = \{0\}$.

This implies that $(f(z)-z) \in CR [U,U]$ for all $z \in U$.

By Lemma 3.2.2 of [18], we have $(f(x)-x) \in CR (U)$, for all $z \in U$. Thus we have $[f(x)-x,x] = 0$ for all $x \in U$. This implies that $[f(x),x] = 0$ for all $x \in U$.

Hence $f$ is commuting on $U$.

**Theorem 3.2.3**: Let $R$ be a semiprime ring and $d: R \to R$ be a nonzero derivation. If $f$ is a generalized derivation on a left ideal $U$ of $R$, then $R$ is commutative.

**Proof**: Since $f$ is commuting on $U$ by the above Theorem 3.2.2. Then we have $[f(x),x]=0$ for all $x \in U$. We replace $x$ by $x+y$, then $[f(x),y]+[f(y),x]=0$.

Now by substituting $y = yx$, then we get $y[f(x),x] + [f(x),y]x + [f(y),x]x + [yd(x),x] = 0$. By simplifying we arrive at $[yd(x),x] = 0$. We replace $y$ by $ry$, then $[r,x]U d(x) = 0$ for all $x \in U$ and $r \in R$. Since $R$ is semiprime ring, it
must contain a family \( w = \{ P_\alpha / \alpha \in \Lambda \} \) of prime ideals such that \( \bigcap P_\alpha = 0 \).

If \( P \) is a member of \( w \) and \( x \in U \), then from the last equation \([R,x] \subseteq P \) or \( d(x) \subseteq P \). Since \( d \) is nonzero on \( R \), then by [35], \( d \) is nonzero on \( U \). Suppose \( d(x) \subseteq P \), for some \( x \in U \), then \([R,x] \subseteq P \). Suppose \( z \in U \) is such that \( z \notin Z \), then \( d(z) \subseteq P \) and \( x+z \notin Z \). This implies \( d(x+z) \subseteq P \) and so \( d(x) \subseteq P \), a contradiction. This implies \( z \in Z \) for all \( z \in U \). Thus \( U \) is commutative and hence [35], \( R \) is commutative.

**Theorem 3.2.4**: Let \( U \) be a left ideal of a semiprime \( R \) such that \( U \cap Z \neq 0 \).

Let \( d \) be a nonzero derivation and \( f \) be a generalized derivation on \( R \) such that \( f \) is centralizing on \( U \). Then \( R \) is commutative.

**Proof**: We assume that \( Z \neq 0 \) because \( f \) is commuting on \( U \) and there is nothing left to prove.

Since \( f \) is centralizing on \( U \), we have \([f(x+y), x+y] \in Z \) for all \( x, y \in U \). This implies that

\[
[f(x),y] + [f(y),x] \in Z, \tag{3.2.5}
\]

for all \( x, y \in U \).

We replace \( x \) by \( zy \) in 3.2.5, then \([f(z),y]y + z[d(y),y] + z[f(y),y] \in Z \).

Now by Lemma 3.2.1, \( f(z) \in Z \) and therefore \( z[d(y),y] + z[f(y),y] \in Z \). But \( f \) is centralizing on \( U \), we have \( z[f(y),y] \in Z \) and consequently \( z[d(y),y] \in Z \).

Since \( z \) is nonzero, it follows from Remark 3.2.1 that \([d(y),y] \in Z \). This
implies $d$ is centralizing on $U$ and hence by [10], we conclude that $R$ is commutative.

\[ \square \]

3.3 Generalized derivations of prime near rings

Golbasi [22] extended some well-known results concerning derivations of prime rings to generalized derivations of prime near rings.

In this section, we consider a prime near ring $N$ of char. $\neq 2$ with center $Z$ and two generalized derivations $(f,d)$ and $(g,h)$ of $N$. We prove that if $f([x,y]) = 0$ or $f([x,y]) = \pm [x,y]$ or $f([x,y]) = \pm xy$ or $f(xy) = \pm xy \in Z$ or $f(xy) + yx \in Z$ or $f(x)e Z$, then $N$ is a commutative ring. Also we prove that if $af(x) = 0$ for $a \neq 0 \in N$, then $a \in Z$ and hence $d(a) \in Z$.

We know that a left (right) near ring is a set $N$ with two operations $+$ and $\cdot$ such that $(N, +)$ is a group and $(N, \cdot)$ is a semigroup satisfying the left distributive law : $x(y+z) = xy + xz$ (right distributive law : $(x+y)z = xz + yz$) for all $x, y, z \in N$. Also a near ring $N$ is prime if $xNy = 0$ implies $x = 0$ or $y = 0$. An additive map $f$ from a ring $N$ to $N$ is a generalized derivation if there exists a derivation $d$ of $N$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in N$.

Throughout this section $N$ will be a prime near ring, and $Z$ its center. The symbol $[x,y]$ will represent the commutator $xy - yx$ for all $x, y$ in $N$,
while the symbol \((x,y)\) will denote the additive - group commutator 
\(x+y-x-y\).

To prove the main results we will make use of the following Lemmas:

**Lemma 3.3.1**: Let \(d\) be an arbitrary derivation on a near ring \(N\). Then \(N\) satisfies the following partial distributive law \((ad(b) + d(a)b)c = ad(b)c + d(a)bc\) and \((d(a)b + ad(b))c = d(a)bc + ad(b)c\) for all \(a, b, c \in N\).

**Proof**: We note that \(d((ab)c) = abd(c) + (ad(b) + d(a)b)c\), and \(d(a(bc)) = ad(bc) + d(a)bc = a(bd(c) + d(b)c) + d(a)bc = abd(c) + ad(b)c + d(a)bc\).

By equating these two expressions for \(d(abc)\) now yields the partial distributive law. □

**Lemma 3.3.2**: Let \(N\) be a 3-prime near ring

(i) if \(z \in Z\{0\}\), then \(z\) is not a zero divisor in \(N\).

(ii) If \(Z\{0\}\) contains an element \(z\) for which \(z+z \in Z\), then \((N, +)\) is abelian.

(iii) Let \(d\) be a nonzero derivation on \(N\). Then \(xd(N) = \{0\}\) implies \(x=0\) and \(d(N)x = \{0\}\) implies \(x = 0\).

(iv) If \(N\) is a char. \(\neq 2\) and \(d\) is a derivation on \(N\) such that \(d^2 = 0\), then \(d = 0\).

**Proof**: (i) If \(z \in Z\{0\}\) and \(zx = 0\), then \(zNx = \{0\}\). Hence \(x = 0\).

(ii) Let \(z \in Z\{0\}\) be an element such that \(z+z \in Z\), and let \(x, y \in N\). Since \(z+z\) is distributive, we get \((x+y)\ (z+z) = x\ (z+z) + y(z+z) = xz + xz + \)
\[ yz + yz = z(x+x+y+y). \] On the other hand, \((x+y)(z+z) = (x+y)z + (x+y)z = z(x+y+x+y).\]

Thus, \(x+x+y+y = x+y+x+y\) and therefore \(x+y = y+x\)

(iii) Let \(xd(N) = \{0\}\), and let \(r,s\) be arbitrary elements of \(N\).

Then \(0 = xd(rs) = xrd(s) + xd(r)s = xrd(s).\)

Thus, \(xNd(N) = \{0\}\) and since \(d(N)x = \{0\}, x = 0.\) A similar argument works if \(d(N)x=\{0\}\). Since Lemma 3.3.1 provides enough distributivity to carry it through.

(iv) For arbitrary \(x, y \in N\), we have

\[ 0 = d^2(xy) = d(xd(y) + d(xy)) = xd^2(y) + d(x)d(y) + d(x)d(y) + d^2(x)y = 2d(x)d(y). \]

Since \(N\) is char. \(\neq 2, d(x)d(N) = 0\) for each \(x \in N\), and (iii) yield \(d = 0.\)

Lemma 3.3.3: Let \(d\) be a derivation on \(N\), and suppose \(u \in N\) is not a left zero divisor. If \([u,d(u)] = 0\), then \((x,u)\) is a constant for every \(x \in N\).

Proof: From \(u(u+x) = u^2 + ux\), we obtain \(ud(u+x) + d(u)(u+x) = ud(u) + d(u)u + ud(x) + d(ux)\), which reduces to \(ud(x) + d(u)u = d(u)u + ud(x)\).

Since \(d(u)u = ud(u), this equation is expressible as u(d(x) + d(u) - d(x) - d(u)) = 0 = u d((x,u)).\) Thus \(d((x,y)) = 0.\)

Lemma 3.3.4: (i) Let \(f\) be a right generalized derivation of a near ring \(N\) associated with \(d\). Then \(f(xy) = xd(y) + f(x)y\) for all \(x, y \in N.\)

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(ii) Let $f$ be a left generalized derivation of a near ring $N$ associated with $d$. Then $f(xy) = xf(y) + d(x)y$ for all $x, y \in N$.

**Proof:** (i) We suppose that $f(xy) = f(x)y + xd(y)$, for all $x, y \in N$. From $x(y+y) = xy + xy$ and $N$ satisfies left distributive law,

$$f(x(y+y)) = f(x)(y+y) + xd(y+y) = f(x)y + f(x)y + xd(y) + xd(y).$$

and

$$f(xy + xy) = f(xy) + f(xy) = f(x)y + f(x)y + xd(y) + xd(y).$$

By comparing these two equalities, we have $f(x)y + xd(y) = xd(y) + f(x)y$. Hence $f(xy) = xd(y) + f(x)y$.

(ii) We suppose that $f(xy) = d(x)y + xf(y)$, for all $x, y \in N$. From $(x+x)y = xy + xy$ and $N$ satisfies right distributive law,

$$f((x+x)y) = d(x+x)y + (x+x)f(y) = d(x)y + d(x)y + xf(y) + xf(y)$$

and

$$f(xy+xy) = f(xy) + f(xy) = d(x)y + xf(y) + d(x)y + xf(y).$$

By comparing these two equalities, we have $d(x)y + xf(y) = xf(y) + d(x)y$. Hence $f(xy) = xf(y) + d(x)y$.

**Lemma 3.3.5:** (i) Let $f$ be a right generalized derivation of the near ring $N$ associated with $d$. Then $(f(x)y + xd(y))z = f(x)yz + xd(y)z$ for all $x, y \in N$.

(ii) Let $f$ be a left generalized derivation of the near ring $N$ associated with $d$. Then $(d(x)y + xf(y))z = d(x)yz + x f(y)z$ for all $x, y \in N$. 

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Proof: From the calculation for \( f((xy)z) = f(x(yz)) \) and by Lemma 3.3.4, we can induce the result. □

Lemma 3.3.6: Let \( N \) be a prime near ring, \( f \) a nonzero generalized derivation of \( N \) associated with the nonzero derivation \( d \), and \( a \in N \).

(i) If \( af(N) = 0 \), then \( a = 0 \).

(ii) If \( f(N) a = 0 \), then \( a = 0 \).

Proof: We suppose that \( af(N) = \{0\} \) for \( a \in N \). Then for all \( b, r \in N \),

\[
0 = af(rb) = a\{f(r)b + rd(b)\} = af(r)b + ard(b) = ard(b).
\]

Hence \( aNd(b) = \{0\} \) for all \( b \in N \).

Since by hypothesis, \( d \) is a nonzero derivation on \( N \), then we must have \( a = 0 \).

(ii) We suppose that \( f(N) a = 0 \), for all \( a \in N \). Then for all \( b, r \in N \),

\[
0 = f(br)a = \{d(b)r + bf(r)\} a = d(b)ra + bf(r)a = d(b)ra.
\]

Hence \( d(b)Na = \{0\} \) for \( b \in N \). Since \( d \) is a nonzero derivation on \( N \), we have \( a = 0 \). □

Lemma 3.3.7: Let \( f \) be a generalized derivation of \( N \) associated with the nonzero derivation \( d \). If \( N \) is a near ring of char. \( \neq 2 \) and \( \hat{f}^2 = 0 \), then \( f = 0 \).

Proof: For all \( x, y \in N \), we have \( \hat{f}^2(xy) = 0 \). Then

\[
0 = \hat{f}^2(xy) = f(f(xy)) = f(f(x)y + xd(y))
\]

\[
= \hat{f}(x)y + 2f(x)d(y) + xd^2(y).
\]

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In particular, \( f(x)c + 2f(x)d(c) + xd^2(c) = 0 \) for all \( x \in N, c \in Z \). Since the first summand is zero, we have \( 2f(x)d(c) + xd^2(c) = 0 \), for all \( x \in N, c \in Z \).

By taking \( f(x) \) instead of \( x \) in the above equation, then we obtain

\[
2f(x)d(c) + f(x)d^2(c) = 0, \text{ for all } x \in N, c \in Z.
\]

Since \( f(x)d(c) = 0 \), and so \( f(x)d^2(c) = 0 \) for all \( x \in N, c \in Z \), we conclude \( f(x)d^2(c) = 0 \) for all \( x \in N, c \in Z \). Since \( N \) is prime, we get \( d^2(Z) = 0 \) or \( f(N) = 0 \). Also since \( d \) is a nonzero derivation of \( N \), then \( f = 0 \). \( \square \)

**Lemma 3.3.8**: Let \( d \) be a nonzero derivation of a prime near ring \( N \) and \( a \in N \) such that \( ad(x) = 0 \) (or \( d(x)a = 0 \)) for all \( a \in N \). Then \( a = 0 \).

**Proof**: We suppose that \( ad(x) = 0 \) for all \( x \in N \). By replacing \( x \) by \( xy \), we have that \( a \cdot d(xy) = 0 = ad(x)y + axd(y) \). Then \( axd(y) = 0 \) for all \( x, y \in N \).

Therefore \( aNd(y) = 0 \). By primeness of \( N \), we have either \( a = 0 \) or \( d = 0 \).

Since \( d \) is a nonzero derivation on \( N \), then \( a = 0 \). \( \square \)

Now we prove the following results:

**Theorem 3.3.1**: Let \((f,d)\) be a generalized derivation of \( N \). If \( f([x,y]) = 0 \) for all \( x, y \in N \), then \( N \) is a commutative ring.

**Proof**: We assume that \( f([x,y]) = 0 \) for all \( x, y \in N \). By substituting \( xy \) instead of \( y \), we obtain \( f([x,xy]) = f(x[xy]) = d(x) [xy] + x f([xy]) = 0 \).

Since the second term is zero, it is clear that

\[
d(x)xy = d(x)yx, \tag{3.3.1}
\]

for all \( x, y \in N \).
We replace $y$ by $yz$ in equation 3.3.1 and using this equation, we get

$$d(x) N[x,z] = 0 \text{ for all } x, y \in N.$$ 

Hence either $x \in Z$ or $d(x) = 0$. Let $K = \{x \in N \mid x \in Z\}$ and $L = \{x \in N \mid d(x) = 0\}$. Then $K$ and $L$ are two additive subgroups of $(N, +) = K \cup L$.

However, a group cannot be the union of proper subgroups, hence either $N = K$ or $N = L$. Since $d \neq 0$, we conclude that $N$ is a commutative ring. \(\square\)

**Theorem 3.3.2**: Let $(f,d)$ be a generalized derivation of $N$. If $f([x,y]) = \pm [x,y]$ for all $x, y \in N$, then $N$ is a commutative ring.

**Proof**: We assume that $f([x,y]) = \pm [x,y]$ for all $x, y \in N$. By replacing $y$ by $xy$ in the hypothesis, we have $f([x,xy]) = \pm (x^2y - xyx) = \pm x[x,y]$.

On the other hand,

$$f([x,xy]) = f(x[x,y]) = d(x) [x,y] + xf([x,y]) = d(x) [x,y] + x(\pm [x,y]).$$

If follows from the two expressions for $f([x,xy])$ that $d(x)xy = d(x)yx$ for $x, y \in N$. By using the same argument as in the proof of Theorem 3.3.1, we get that $N$ is a commutative ring. \(\square\)

**Theorem 3.3.3**: Let $(f,d)$ be a generalized derivation of $N$. If $f([x,y]) = \pm xy$, for all $x, y \in N$, then $N$ is a commutative ring.

**Proof**: We assume that $f([x,y]) = \pm xy$ for all $x, y \in N$. By replacing $y$ by $xy$ in the hypothesis, we have $f([x,xy]) = \pm x(xy)$.

On the other hand,

$$f([x,xy]) = f(x[x,y]) = d(x) [x,y] + xf([x,y]) = d(x) [x,y] + x(\pm xy).$$

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If follows from the two expressions for \( f([x,xy]) \) that
\[
d(x)xy = d(x)yx, \quad 3.3.2
\]
for all \( x,y \in N \).

By using the same argument as in the proof of Theorem 3.1.1, we get that \( N \) is a commutative ring. \( \square \)

**Theorem 3.3.4:** Let \((f,d)\) be a generalized derivation of \(N\). If \( f(xy) - xy \in Z \) (or) \( f(xy) + xy \in Z \) for all \( x, y \in N \), then \( N \) is a commutative ring.

**Proof:** We assume \( f(xy) - xy \in Z \) which implies that
\[
d(x)y + x f(y) - xy \in Z, \quad 3.3.3
\]
for all \( x, y \in N \).

By replacing \( x \) by \( zx \) in equation 3.3.3 to get \( d(zx)y + zxf(y) - zxy \in Z \), for all \( x,y,z \in N \). And hence \([d(z)xy + z(d(x)y + xf(y) - xy), z] = 0\), which reduces to
\[
[d(z)xy, z] = 0, \quad 3.3.4
\]
for all \( x, y \in N \).

Now let us rewrite equation 3.3.4 as the following
\[
d(z) x[y,z] + d(z) [x,z]y + [d(z),z]xy = 0, \quad 3.3.5
\]
for all \( x, y, z \in N \).

In equation 3.3.5, we replace \( y \) by \( yw \) and using 3.3.5 to get \( d(z)xy [w,z] = 0 \), for all \( x, y, z, w \in N \). By putting \( xy = r, r \in R \) such that the equation reduces to \( d(z)r [w,z] = 0 \), which implies that \( d(z)R[w,z] = 0 \). By primeness of \( N \), we have either \( d(z) = 0 \) or \( [w,z] = 0 \). Since by hypothesis \( d \) is nonzero then

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$[w,z] = 0$. Hence $N$ is a commutative ring. Similarly, the case $f(xy) + xy \in Z$ implies the commutativity of $R$ is clear because we can replace $f$ by $-f$ in the above case, as required.

**Theorem 3.3.5:** Let $(f,d)$ be a generalized derivation of $N$. If $f(xy) - yx \in Z$ (or) $f(xy) + yx \in Z$ for all $x, y \in N$, then $N$ is a commutative ring.

**Proof:** We have $f(xy) - yx \in Z$, which implies that

$$[d(x)y + xf(y) - yx, r] = 0,$$

for all $x, y \in N$ and $r \in R$.

Now equation 3.3.6 can be rewrite as the following

$$d(x)[y, r] + [d(x), r]y + x[f(y), r] + [x, r]f(y) = y[x, r] + [y, r]x,$$

for all $x, y \in N$ and $r \in R$.

We replace $x$ by $rx$ in equation 3.3.7 to get $r(d(x)[y, r] + [d(x), r]y + x[f(y), r] + [x, r]f(y)) + [x, r]f(y)) + d(r)x[y, r] + d(r)[x, r]y + [d(r), r]xy = y r[x, r] + [y, r]rx$, for all $x, y \in N$ and $r \in R$.

By combining equations 3.3.7 with above equation to get

$$[y,x] [x, r] + [(y, r), r] x + d(r)x[y, r] + d(r)[x, r]y + [d(r), r]xy = 0,$$

for all $x, y \in N$ and $r \in R$.

We replace $x$ by $xy$ in equation 3.3.8 and using 3.3.8 to get

$$d(r)xy [y, r] + [y, r] x[y, r] = 0,$$

for all $x, y \in N$ and $r \in R$.

If we replace $r$ by $r + y$ in equation 3.3.9 and using 3.3.9 to get

$$d(y)xy [x, r] = 0,$$

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for all \( y \in N \) and \( r \in R \).

Put \( xy = z \) in 3.3.10 then

\[
d(x) z[x,r] = 0, 3.3.11
\]

for all \( x, y, z \in N \) and \( r \in R \).

By primeness of \( N \), either \( d(x) = 0 \) or \([x,r] = 0\).

If \( d(x) = 0 \), then by hypothesis \( d \) is nonzero. Therefore \([x,r] = 0\) and hence \( N \) is a commutative ring.

Similarly the case \( f(xy) +yx \in Z \) can be proved if we replace \( f \) by \(-f\) in the above case, as required. □

**Theorem 3.3.6**: Let \((f,d)\) be a nonzero generalized derivation of \( N \). If \( f \) acts as a homomorphism on \( N \), then \( f \) is the identity map.

**Proof**: We assume that \( f \) acts as a homomorphism on \( N \). Then

\[
f(xy) = f(x)f(y) = d(x)f(y) + xf(y), 3.3.12
\]

for all \( x, y \in N \).

By substituting \( yz \) for \( y \) in equation 3.3.12 we arrive at

\[
f(x)f(yz) = d(x)yz + xf(yz).
\]

Since \((f,d)\) be a generalized derivation and \( f \) acts as a homomorphism on \( N \), we deduce that

\[
f(xy)f(z) = d(x)yz + xd(y)z + xyf(z).
\]

By Lemma 3.3.5 (ii), we get

\[
d(x)yf(z) + xf(y)f(z) = d(x)yz + xd(y)z + xyf(z)
\]

and so
\[d(x)yf(z) + xf(yz) = d(x)yz + xd(y)z + xyf(z)\].

That is,
\[d(x)yf(z) + xd(y)z + xyf(z) = d(x)yz + xd(y)z + xyf(z)\].

Hence, we deduce that \(d(x)y (f(z) - z) = 0\) for all \(x, y, z \in N\).

Since \(N\) is prime and \(d \neq 0\), we have \(f(z) = z\) for all \(z \in N\). Thus, \(f\) is the identity map. \(\square\)

**Theorem 3.3.7**: Let \((f,d)\) be a nonzero generalized derivation of \(N\). If \(f\) acts as an antihomomorphism on \(N\), then \(f\) is the identity map.

**Proof**: By the hypothesis, we have
\[f(xy) = f(y)f(x) = d(x)y + xf(y),\]  \(\text{3.3.13.}\)

for all \(x, y \in N\).

We replace \(y\) by \(xy\) in the last equation, we obtain
\[f(xy)f(x) = d(x)xy + xf(xy)\].

Since \((f,d)\) is a generalized derivation and \(f\) acts as an antihomomorphism on \(N\), we get
\[(d(x)y + xf(y))f(x) = d(x)xy + xf(y)f(x)\].

By Lemma 3.3.5 (ii), we conclude that
\[d(x)yf(x) + xf(y)f(x) = d(x)xy + xf(y)f(x),\]

and so
\[d(x)yf(x) = d(x)xy,\] for all \(x, y \in N\).

We replace \(y\) by \(yz\) and using this equation, we have
\[d(x) N [f(x),z] = 0\] for all \(x, z \in N\).
Hence we obtain the following alternatives: \( d(x) = 0 \) or \( f(x) \in N \), for all \( x \in N \).

By a standard argument, one of these must hold for all \( x \in N \). Since \( d \neq 0 \), the second possibility gives that \( N \) is commutative ring by [23], and so we deduce that \( f \) is the identity map by Theorem 3.3.6. \( \Box \)

**Theorem 3.3.8**: Let \( (f,d) \) be a generalized derivation of \( N \) such that \( d(Z) \neq 0 \), and \( a \neq 0 \in N \). If \( af(x) = 0 \) for all \( x \in N \), then \( a \in Z \).

**Proof**: Since \( d(Z) \neq 0 \), there exists \( c \in Z \) such that \( d(c) \neq 0 \). Furthermore, as \( d \) is a derivation, it is clear that \( d(c) \in Z \). We replace \( x \) by \( cx \) in the hypothesis and using Lemma 3.3.5 (ii), we have \( af(cx) \in Z \).

\[ \Rightarrow ad(c)x + acf(x) \in Z. \]

So, \( [ad(c)x,r] + [caf(x),r] = 0 \) which implies that

\[ ad(c)[x,r] + [a,r] d(c)x = 0. \]  \( \text{3.3.14} \)

We replace \( x \) by \( xa \) and using the above equation. Then

\[ ad(c) [xa,r] + [a,r] d(c) xa = 0. \]

\[ \Rightarrow ad(c)x[a,r] = 0 \]

\[ \Rightarrow ad(c) N [a,r] = 0, \text{ for all } x \in N. \]

By the primeness of \( N \), either \( ad(c) = 0 \) or \( [a,r] = 0 \).

If \( ad(c) = 0 \) then by Lemma 3.3.8, we have \( a = 0 \).

But by hypothesis \( a \neq 0 \). Therefore we must have \( [a,r] = 0 \), which implies \( a \in Z \). \( \Box \)
**Theorem 3.3.9** : Let $(f,d)$ be a generalized derivation of $N$ and $a \neq 0 \in N$. If $af(x) \in Z$ for all $x \in N$, then $d(a) \in Z$.

**Proof** : If $a = 0$, then there is nothing to prove. Hence, we assume that $a \neq 0$.

By substituting $ax$ for $x$ in the hypothesis, we have $af(ax) \in Z$.

$$\Rightarrow \quad ad(a)x + aaf(x) \in Z.$$ 

So, $$[ad(a)x + aaf(x), r] = 0$$

By using $af(x) \in Z$, we have

$$a \ d(a) \ [x,r] + a[d(a),r]x + [a,r] \ d(a)x = 0.$$ 

We replace $x$ by $xa$ and using the above equation, then $ad(a) \ [xa,r] + a[d(a),r]xa + [a,r]d(a)xa = 0$, which implies $ad(a)x \ [a,r] = 0$.

$$\Rightarrow \quad ad(a) \ N \ [a,r] = 0, \text{ for all } x \in N.$$ 

By primeness of $N$, either $ad(a) = 0$ or $[a,r] = 0$.

If $ad(a) = 0$, then $a = 0$ by Lemma 3.3.8. Since $a \neq 0$ by the hypothesis.

Hence $[a,r] = 0$ implies $a \in Z$.

If $0 \neq a \in Z$, then $(N, +)$ is abelian by Lemma 3.3.2 (ii).

Thus $f(xya) = faxy)$

$$f(x)ya + xd(y)a + xyd(a) = d(a)xy + af(x)y + axd(y).$$ 

and so,

$$[d(a),z] = 0, \text{ for all } x \in N.$$ 

Hence $d(a) \in Z$. \[\Box\]

**Theorem 3.3.10** : Let $(f,d)$ be a generalized derivation of $N$. If $N$ is a near ring of char. $\neq 2$ such that $f^2(N) \subset Z$, then $N$ is a commutative ring.
**Proof**: Suppose that \( f^*(N) \subset Z \). Then we get
\[
 f^*(xy) = f^*(x) + 2f(x) \ d(y) \quad \text{for all } x, y \in N.
\]
In particular, \( f^*(x)c + 2f(x)d(c) + xd^2(c) \in Z \), for all \( x \in N, c \in Z \). Since the first summand is an element of \( Z \), we have
\[
 2f(x)d(c) + xd^2(c) \in Z, \quad 3.3.15
\]
for all \( x \in N, c \in Z \).

By taking \( f(x) \) instead of \( x \) in equation 3.3.15, we obtain that \( 2f^*(x) \ d(c) + f(x)d^2(c) \in Z \), for all \( x \in N, c \in Z \).

Since \( d(c) \in Z \), \( f^*(x) \in Z \), and so \( f^*(x)d(c) \in Z \), for all \( x \in N, c \in Z \), we conclude \( f(x) \ d^2(c) \in Z \), for all \( x \in N, c \in Z \).

Since \( N \) is prime, we get \( d^2(Z) = 0 \) or \( f(N) \subseteq Z \).

If \( f(N) \subseteq Z \) then \( N \) is a commutative ring by [23].

Hence, we assume \( d^2(Z) = 0 \). By 3.1.15, we get \( 2f(x) \ d(c) \in Z \), for all \( x \in N, c \in Z \).

Since \( N \) is a near ring of char. \( \neq 2 \) and \( d(c) \in Z \), we obtain that either \( f(N) \subseteq Z \) or \( d(Z) = 0 \). If \( f(N) \subseteq Z \), then we are already done. So, we may assume that \( d(Z) = 0 \). Then \( f(cx) = f(xc) \).

\[
 f(c)x + cd(x) = f(x)c + xd(c),
\]
and so
\[
 f(c)x + cd(x) = f(x)c, \quad 3.3.16
\]
for all \( x \in N, c \in Z \).
Now replacing $x$ by $f(x)$ \textit{3.3.16}, and using the fact that $\hat{f}(N) \subseteq Z$, we get

$$f(c) f(x) + cd(f(x)) = \hat{f}(x)c,$$

for all $x \in N$, $c \in Z$.

That is,

$$f(c) f(x) + cd(f(x)) \in Z,$$  \textit{3.3.17}

for all $x \in N$, $c \in Z$.

Again by taking $f(x)$ instead of $x$ in this equation, then we obtain

$$f(c) \hat{f}(x) + cd(\hat{f}(x)) \in Z,$$

for all $x \in N$, $c \in Z$.

The second term is equal to zero because $d(Z) = 0$. Hence we have

$$f(c) \hat{f}(x) \in Z,$$

for all $x \in N$, $c \in Z$.

Since $\hat{f}(N) \subseteq Z$ by the hypothesis, we get either $\hat{f}(N) = 0$ or $f(Z) \subseteq Z$. If $\hat{f}(N) = 0$, then the theorem holds by Lemma 3.1.7. If $f(Z) \subseteq Z$, then

$$f(x f(c)) = f(f(c) x)$$

for all $x \in N$, $c \in Z$, and so $d(x) f(c) = f(c) f(x)$, for all $x \in N$, $c \in Z$.

By using $f(c) \in Z$, we have

$$f(c) (d(x) - f(x)) = 0,$$

for all $x \in N$, $c \in Z$.

Since $f(Z) \subseteq Z$, we have either $f(Z) = 0$ or $d = f$. If $d = f$, then $f$ is a derivation of $N$ and so $N$ is a commutative ring by Lemma 3.3.7.

Now we assume that $f(Z) = 0$. From equation \textit{3.3.16}, we have

$$c(d(x) - f(x)) = 0,$$

for all $x \in N$, $c \in Z$.

Since $c \in Z$, we have either $d = f$ or $Z = 0$. Clearly $d = f$ implies the theorem holds. If $Z = 0$, then $\hat{f}(N) = 0$ by the hypothesis, and so $N$ is a commutative ring by Lemma 3.3.2 (iv). Hence, the proof is completed. \hfill $\Box$
Corollary 3.3.1: Let $N$ be a near ring of char. $\neq 2$, $(f,d)$ a nonzero generalized derivation of $N$. If $[f(N), f(N)] = 0$, then $N$ is a commutative ring.

Theorem 3.3.11: Let $(f,d)$ and $(g,h)$ be two generalized derivations of $N$. If $(fg, dh)$ acts as a generalized derivation on $N$, then $(N, +)$ is abelian.

Proof: By calculating $fg(xy)$ in two different ways, then $f(x)h(y) + g(x)d(y) = 0$ for all $x, y \in N$.

We suppose $y + z$ for $y$, thereby obtaining $f(x)h(y) + f(x)h(z) + g(x)d(y) + g(x)d(z) = 0$.

By using the hypothesis, we get $f(x)h(y,z) = 0$, for all $x, y, z \in N$.

It follows by Lemma 3.3.6 (ii) that $h(y,z) = 0$, for all $y, z \in N$. For any $w \in N$, we have $h(wy, wz) = h(w(y,z)) = h(w)(y,z) + w h(y,z) = 0$, and so, $h(w)(y,z) = 0$, for all $w, y, z \in N$.

By Lemma 3.3.2 (ii) implies that $(N, +)$ is abelian. □

Theorem 3.3.12: Let $N$ be a near ring of char.$\neq 2$, $(f,d)$ and $(g,h)$ be two generalized derivations on $N$. If $(fg, dh)$ acts as a generalized derivation on $N$, then $f=0$ or $g=0$.

Proof: If $h = 0$ or $d = 0$, then the proof of the theorem is obvious. So, we may assume that $h \neq 0$ and $d \neq 0$. Therefore, we know that $(N, +)$ is abelian by Theorem 3.3.11. By calculating $fg(xy)$ in two different ways, then we obtain $g(x)d(y) + f(x)h(y) = 0$ for all $x, y \in N$. 

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We replace $x$ by $uv$ in the above equation and using the hypothesis, then

$$f(uv) h(y) + g(uv) d(y) = uf(v) h(y) + d(u)v h(y) + u g(v) d(y) + h(u)$$

$vd(y) = 0$, and so

$$d(u)v h(y) = - h(u)v d(y), \quad 3.3.18$$

for all $u, v, y \in N$.

By taking $yt$ instead of $y$ in the above relation, we obtain

$$d(u)v h(y)t + d(u)v y h(t) = -h(u)v d(y)t - h(u)v y d(t).$$

That is,

$$d(u)v y h(t) = -h(u)v y d(t), \quad 3.3.19$$

for all $u, v, y, t \in N$.

We replace $y$ by $h(y)$ in equation 3.3.19 and using this relation we obtain

$$h(u) N(d(y) h(t) - h(y) d(t)) = 0, \text{ for all } u, y, t \in N.$$  

Since $N$ is a prime near ring and $h \neq 0$, we obtain that

$$d(y) h(t) = h(y) d(t), \quad 3.3.20$$

for all $y, t \in N$.

Now again taking $uv$ instead of $x$ in the initial hypothesis, we get

$$f(u)v h(y) + ud(v) h(y) + g(u)v d(y) + uh(v) d(y) = 0.$$  

By using equation 3.3.20 implies that $f(u)v h(y) + 2uh(v) d(y) + g(u)v d(y) = 0, \text{ for all } u, v, y \in N$. We take $h(v)$ instead of $v$ in this equation, then

$$f(u) h(v) h(y) + 2uh^2(v) d(y) + g(u) h(v) d(y) = 0.$$  

By the hypothesis and equation 3.3.20, we have

$$0 = -g(u)d(v) h(y) + 2uh^2(v) d(y) + g(u) h(v) d(y).$$

$$= -g(u) h(v) d(y) + 2uh^2(v) d(y) + g(u) h(v) d(y),$$
and so

\[ 2u h^{2}(v) d(y) = 0, \text{ for all } u, v, y \in N. \]

Since \( N \) is a prime near ring of char.\( \neq 2 \), we obtain that \( h^{2}(N) d(N) = 0 \). By Lemma 3.3.2 (iii) and (iv) gives that \( h = 0 \). This contradicts by our assumption. Thus the proof is completed.