1.1 DEFINITIONS

PROBABILITY SPACE

Let $\Omega$ be a non-empty set, $\mathcal{F}$ be a $\sigma$ field of subsets of $\Omega$ and $P$ be a measure defined on $\mathcal{F}$ satisfying $P(\Omega) = 1$. Then the triplet $(\Omega, \mathcal{F}, P)$ is called a probability space and $P$ is a probability measure.

DISTRIBUTION FUNCTION

By a distribution function $F$, we mean a real valued function $F$ defined on $(-\infty, \infty)$, which is non-decreasing, right continuous and which satisfies $F(-\infty) = 0$ and $F(\infty) = 1$. If $X$ is a random variable then its distribution function $F$ is defined by $F_X(x) = P(X \leq x)$.

CONVERGENCE IN DISTRIBUTIONS

A sequence of random variables $\{X_n, n \geq 1\}$ with corresponding distribution functions $\{F_n\}$ converges weakly to a random variable $X$ with the distribution function $F$, if, $F_n(x) \xrightarrow{w} F(x)$, as $n \to \infty$, at all continuity points of $F$. Such a convergence is expressed through $X_n \xrightarrow{L} X$ or $F_n \xrightarrow{w} F$. 
CONVERGENCE IN PROBABILITY

A sequence of random variables \( \{X_n\} \) is said to converge in probability to a random variable \( X \), if for every \( \varepsilon > 0 \).

\[
\lim_{n \to \infty} P \left( |X_n - X| > \varepsilon \right) = 0.
\]
This is denoted by \( X_n \overset{P}{\longrightarrow} X \).

ALMOST SURE CONVERGENCE

A sequence \( \{X_n\} \) of random variables is said to converge almost surely to a random variable \( X \), if and only if.

\[
P \left( \lim_{n \to \infty} X_n = X \right) = 1.
\]
We denote this by \( X_n \overset{a.s.}{\longrightarrow} X \) (i.e., \( X_n \overset{P}{\longrightarrow} X \) almost surely).

INFINITELY DIVISIBLE DISTRIBUTIONS

A Distribution function \( F \) is called infinitely divisible if and only if for every integer \( n \), it can be written as a convolution of non-degenerate distribution functions \( F_1, F_2, \ldots, F_n \). The Characteristic function \( f(t) \) of an infinitely divisible distribution function can be represented by the Levy canonical form (see Lukacs [ ], page 118).

\[
\log f(t) = \gamma t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{\infty} \left\{ \exp(itu) - 1 - \frac{itu}{1+u^2} \right\} dH(u).
\]
where the spectral function \( H(u) \) and the constants \( \gamma \) and \( \sigma^2 \) satisfy the following conditions.
(i) \( H(u) \) and \( H(-u) \) are non-decreasing in the interval \( 0 < u < \infty \).

(ii) The integrals \( \int_0^u u^2 \, dH(u) \) and \( \int_0^\infty u^2 \, dH(u) \) are finite for every \( u > 0 \).

(iii) \( H(-\infty) = H(\infty) = 0 \)

(iv) \( \gamma \) is any real constant.

(v) \( \sigma^2 \) is a non-negative constant.

**STABLE DISTRIBUTIONS**

A distribution function \( F \) is said to be Stable if and only if, for every \( b_1 > 0, b_2 > 0, a_1 \) real, there exists \( b > 0 \) and a real such that the relation

\[
F\left(\frac{x-a_1}{b_1}\right) * F\left(\frac{x-a_2}{b_2}\right) = F\left(\frac{x-a}{b}\right)
\]

holds. The characteristic function \( f(t) \) of a Stable distribution has the following representation,

\[
\log f(t) = \begin{cases} 
  i\nu t - c|t|^\alpha \left[ i + i\beta \frac{t}{|t|} \tan \left( \frac{n\alpha}{2} \right) \right], & \text{if } \alpha \neq 1, \\
  i\nu t - c|t|^\alpha \left[ i + i\beta \frac{t}{|t|} \left( \frac{2}{\pi} \log |t| \right) \right], & \text{if } \alpha = 1
\end{cases}
\]

where \( \alpha, \beta, \gamma \) and \( C \) are real constants with \( C \geq 0 \), \( 0 < \alpha \leq 2 \). Here \( \alpha \) is called the characteristic exponent. The parameters \( \gamma \) and \( C \) merely determine location and scale and hence without loss of generality, we assume that \( \gamma = 0 \) and \( C = 1 \).

A Stable random variable is positive valued (negative valued) when ever \( 0 < \alpha < 1 \), and \( \beta = -1, (\beta = 1) \) in the characteristic function representation. A Stable random variable with \( \alpha = 2 \) is a normal random variable.
Let \( \{X^i\} \) be a sequence of independent and identically distributed random variables with a common distribution function \( F \). Let \( S_n^\infty \sum_{k=1}^{\infty} X_k, n \geq 1 \). Set \( Z_n = \frac{S_n}{B_n} - A_n \), where \( \{A_n\} \) and \( \{B_n\} \) are sequence of real constants with \( B_n > 0 \). If the sequence of normalized sums, i.e., \( \{Z_n\} \), converges in law to some random variable with distribution function, say \( G \), then \( F \) is said to belong to the domain of attraction of \( G \) and it is well known that the limit law is always a Stable law (see[ ], page 162).

**DOMAIN OF NORMAL ATTRACTION**

A distribution function \( F \) (or a r.v. \( X \) with distribution function \( F \)) belongs to the domain of normal attraction of a Stable law with characteristic with \( B_n = a n^{1/\alpha} \), where \( a \) is some positive constant.
1.2 NOTATIONS

The following notations may be noted as they appear frequently in the text.

r.v (s) random variable (s).
d.f(s) distribution function (s).
ch.fn(s) characteristic function (s).
i.i.d. independent identically distributed.
R Set of real numbers.
I Set of integers.
Lim. Sup Limit supremum.
Lim. Inf Limit infimum.
a.s. almost surely.
L convergence in law.
P convergence in probability.
a.s. convergence in probability.
DA Domain of attraction.
DNA Domain of normal attraction.
N (integer), t, c, m With or without a suffix; positive constants, not necessarily the same at each occurrence.
1.3. SOME KNOWN RESULTS

THEOREM 1.3.1 [Beyde, C. C., [9], Theorem, page 1576]

Suppose that \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. r.v.s which do not belong to the domain of partial attraction of the normal distribution, i.e., those for which

\[
\liminf_{u \to \infty} \frac{u^2 \mathbb{P}\left( |X| > u \right)}{\int \mathbb{E}^2 \mathbb{d}F(x) / |x| \leq u} > 0,
\]

where \( F(x) = \mathbb{P}(X \leq x) \). Let \( \sum_{k=1}^{n} X_k \) and \( (Y_n) \) is a monotone sequence of positive numbers such that \( Y_n \to \infty \) as \( n \to \infty \) and \( Y_n^{-1} \mathbb{E}_n \to 0 \). For any \( \alpha > 1 \), let,

\[
\liminf_{n \to \infty} \frac{\mathbb{P}\left( |X| > Y_n \right)}{\mathbb{P}\left( |X| > Y_n \right)} > 0.
\]

Then

\[
0 < \liminf_{n \to \infty} \frac{\mathbb{P}\left( |S_n| > x_n \right)}{\mathbb{P}\left( |X| > x_n \right)} \leq \limsup_{n \to \infty} \frac{\mathbb{P}\left( |S_n| > x_n \right)}{\mathbb{P}\left( |X| > x_n \right)} < \infty.
\]

THEOREM 1.3.2 [Borel-Cantelli Lemma (B.C. Lemma), see Stout, W [9], page]

Let \( \{A_n\} \) be a sequence of events in probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

(i) If \( \sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \) then \( \mathbb{P}(A_n \ i.o) = 0 \).

(ii) If \( \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \) and if \( A_1, A_2, \ldots \) are mutually independent events then \( \mathbb{P}(A_n \ i.o) = 1 \).
THEOREM 1.3.3. [Chover, J, [3] page 441, Theorem ]

Let \( \{X_n, n \geq 1\} \) be mutually independent r.v.s, identically distributed according to the symmetric Stable distribution with exponent \( \alpha \), \( 0 < \alpha < 2 \). Then,

\[
\limsup_{n \to \infty} \frac{n^{1/\alpha} S_n}{(\log n)^{-1}} = e^{1/\alpha} \text{ a.s.}
\]

THEOREM 1.3.4 [Allan Gut, [1], Theorem 2.1, Page 30 ].

Let \( \{n_k\}_{k \geq 1} \) be strictly increasing subsequence of the positive integers such that,

\[
\liminf_{k \to \infty} \frac{n_k}{n_{k+1}} > 0.
\]

Let \( \{X_n, n \geq 1\} \) be i.i.d. r.v.s. and \( S_n = \sum_{k=1}^{n} X_k \). Suppose that \( E(X_1) = 0 \), \( E(X_1^2) = \sigma^2 < \infty \). Then,

\[
\limsup_{k \to \infty} \frac{S_{n_k}}{\sqrt{n_k \log \log n_k}} > 0. \text{ Then } E(X_1^2) = \infty \text{ and } E(X_1) = 0.
\]

THEOREM 1.3.5 [ Allan Gut, [1], Theorem 2.2, page 31 ]

Let \( \{n_k\} \) be strictly increasing subsequence of the positive integers such that, \( \limsup_{k \to \infty} \frac{n_k}{n_{k+1}} < 1 \) and let \( \{X_n, n \geq 1\} \) be i.i.d. r.v.s.

Set \( S_n = \sum_{k=1}^{n} X_k \) and suppose that \( E(X_1) = 0 \) and \( E(X_1^2) = \sigma^2 < \infty \). Then,
\[ \limsup \frac{S_{n_k}}{\sqrt{n_k \log \log n_k}} = \sigma \varepsilon^* \quad (\sigma \varepsilon^*) \text{ a.s.,} \]

where \( \varepsilon^* = \inf \left\{ \varepsilon > 0 : \sum_{k=3}^{\infty} \left( \log n_k \right)^{-\varepsilon/2} < \infty \right\} \). In particular, if \( \varepsilon^* = 0 \),

\[ \frac{S_{n_k}}{\sqrt{n_k \log \log n_k}} \to 0 \text{ a.s. as } k \to \infty. \]

For the converse, suppose that \( \varepsilon^* > 0 \). If

\[ P \left\{ \limsup_{k \to \infty} \frac{|S_{n_k}|}{\sqrt{n_k \log \log n_k}} < \infty \right\} > 0. \]

Then \( E(X_1^2) < \infty \) and \( E(X_1) = 0 \).