CHAPTER 2

COST OPTIMAL SERIES - PARALLEL SYSTEMS
SUBJECT TO
TWO - MODES OF FAILURE WITH REPAIR PROVISION
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2.1. INTRODUCTION

Redundancy can be used to increase the reliability of a System without any change in the reliability of the individual components. The Series-Parallel configuration is one form of redundancy. In the past, many contributions have been made to the optimisation of Simple and Complex Reliability Systems with the assumption that only random failures of components result in System failure (See [2],[6]). However, in practical situations many types of Systems consist of components that can fail in either of two mutually exclusive ways: Failure to operate when it should be operating (Open) and failure to idle when it should be idle (Short). For example, a network consisting of N relays in parallel has the property that a short-circuit failure of any one relay would cause a System short-circuit failure, and an open-circuit failure of N relays would cause an open-circuit failure of the System. Electronic Diodes, Switches and Transistors also exhibit the open-circuit and short-circuit failure behaviour.

In the present chapter, we deal with a Series-Parallel System, which is subject to two-modes of failure: Open, Short and develop a theoretical procedure leading to optimal number of parallel units \((m^*)\) in a \((N,m)\) Series-Parallel System.

The value of the theoretical results is demonstrated by supporting numerical work.

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2.2. **THE MODEL**

Consider a Series-Parallel System of order \((N,m)\) i.e., a System comprising of \(Nm\) identical units connected in \(N\) Series of Parallel Systems each consisting of \(m\) identical units. Each unit and the System is subject to two-modes of failure on command: Failure to open (Open) and failure to close (Short), and all unit states are mutually independent. The time to failure of a unit is the time when it first fails, either Open or Short. The System is repaired after failure and the corresponding successive repair times \(\{R_i, i=1,2,\ldots\}\) constitute a non-decreasing Geometric Process. The following assumptions are made.

**Assumptions**

(i) The time span is infinite.

(ii) Each unit has the same failure time distribution; the probability of a short failure of a unit, given the failure of the unit, is a constant.

(iii) After a repair the System becomes new in the sense that the same M.T.T.F (Mean Time To Failure) is maintained after a repair completion.

(iv) The successive repair times \(\{R_i, i=1,2,3,\ldots\}\) constitute non-decreasing Geometric Process. (For a sequence of random variables \(\{R_1,R_2,\ldots\}\) and for some \(\alpha > 0\), if \(\alpha^{i-1} R_i, i=1,2,\ldots\) forms a renewal process, then \(\{R_i, i = 1,2,\ldots\}\) is called a Geometric Process (G.P) and \(\alpha\) is the parameter of the Geometric Process. The G.P. is said to be non-increasing, if \(\alpha \geq 1\) and non-decreasing if \(\alpha \leq 1\).) For further details refer [8].

We define the following quantities:

- \(C_1\) acquisition cost of each unit
- \(C_2\) System repair cost per unit time
- \(F(t)\) The failure time distribution of a unit
- \(p\) conditional probability of Short failure, given that a failure has occurred
- \(q\) conditional probability of Open failure, given that a failure has occurred \((p+q=1)\)
- \(R(t)\) Reliability of the System at time \(t\)
- \(\mu_n(m)\) Mean Time To Failure (M.T.T.F) of the System
- \(N,m,n\) a Series-Parallel System of order \((N,m)\) maintained with \(n\) repairs.
- \(\mu_{pN}\) M.T.T.F. of ordinary Parallel system of \(N\) identically and statistically independent units
\[ \mu_{sN} \quad \text{M.T.T.F. of ordinary Series system of } N \text{ identically and statistically independent units.} \]

\[ R_i \quad \text{Random repair time after } i^{th} \text{ system failure, } i=1,2,...,n \text{ and } \mathbb{E}(R_i) = b \]

**2.3. THE SOLUTION**

The assumption (ii) on the model implies

\[ R(t) = \left[ 1 - q^m F^m(t) \right]^N - \left[ 1 - (1-pF(t))^m \right]^N \]

\[ = \sum_{j=1}^{N} \binom{N}{j} (-1)^{j} \left[ q^{mj} F^{mj}(t) - (p+q F'(t))^{mj} \right] \]

\[ = \sum_{j=1}^{N} \binom{N}{j} (-1)^{j+1} q^{mj} \left( 1 - F^{mj}(t) \right) + \sum_{j=1}^{N} \sum_{k=0}^{mj-1} \binom{N}{j} \binom{mj}{k} (-1)^{j+1} q^k p^{(mj-k)} (1-\phi^{(mj-k)}(t)) \]

\[ \mu_{N^{(m)}} = \int_0^\infty R(t) \, dt \]

\[ - \sum_{j=1}^{N} \binom{N}{j} (-1)^{j+1} q^{mj} \mu_p^{mj} + \sum_{j=1}^{N} \sum_{k=0}^{mj-1} \binom{N}{j} \binom{mj}{k} (-1)^{j+1} q^k p^{(mj-k)} \mu_s^{mj-k} \]

(2)

The assumption (iv) on the model leads to

\[ \mathbb{E}(R_i) = \mathbb{E}(\alpha R_2) = \mathbb{E}(\alpha^2 R_3) \ldots = b, \quad (3) \]

Since \( \alpha^{i-1} R_i, \quad i=1,2,\ldots \) forms a renewal process.
Therefore \( E(R_1) = b, \ E(R_2) = b/\alpha, \ E(R_3) = b/\alpha^2 \), ...

\[ E(R_1) \leq E(R_2) \leq E(R_3), \quad \text{for } \alpha \leq 1 \]  

(4)

The expected cost upto \( n \) repairs for the system is given by

\[ \sum_{i=1}^{n} \text{NmC}_i + C_2b\sum_{i=1}^{n} (1/\alpha)^{i-1} \]

(5)

So that the expected cost \( C(N,m,n) \) per unit time (by known renewal theoretic arguments) is given by

\[ \sum_{i=1}^{n} \text{NmC}_i + C_2b\sum_{i=1}^{n} (1/\alpha)^{i-1} \]

\[ C(N,m,n) = \frac{1}{(n+1) \mu_{n}^{(m)}} \]

(6)

We now state and prove the following Lemma.

Lemma 1 : For fixed \( p,q \) and \( N \) the maximum value of \( R(t) \) is attained at

\[ m = \begin{cases} \lfloor m_0 \rfloor + 1 & \text{if } m_0 \text{ is not an integer} \\ m_0 \text{ or } m_0 + 1 & \text{if } m_0 \text{ is an integer} \end{cases} \]

Where

\[ m_0 = N \cdot \frac{\text{Log}(1-pF(t)) - \text{Log} qF(t)}{\text{Log}(1-qF(t)^N) - \text{Log} (1-(1-pF(t))^N)} \]

(7)

and \( m_0 \) denotes the largest integer less than or equal to \( m_0 \).
PROOF : By fixing p, q and N, the Reliability of the System becomes a function of m alone.

So that \( R(t) = R_m(t) \) \hspace{1cm} (8)

Now \( m_0 \) that maximises \( R_m(t) \) must satisfy the following two inequalities.

\[
R_m(t) - R_{m-1}(t) \geq 0 \quad \text{and} \quad R_{m-1}(t) - R_m(t) \leq 0
\]

(9) \hspace{1cm} (10)

The proof of the Lemma follows using (9) and (10)

THEOREM 1 : The \( m^* \) which minimises \( L(N,m,n) \) in (6) satisfies the two inequalities

\[
L(N,m,n) > C_2/C_1 \quad \text{(11)}
\]

\[
L(N,m-1,n) < C_2/C_1 \quad \text{(12)}
\]

Where

\[
L(N,m,n) = \frac{(m+1)N \mu_n^{(m)} - mN \mu_{n(m+1)}}{n \sum_{i=1}^{b} (1/\alpha)^{i-1} [ \mu_n(m+1) - \mu_n(m)]}
\]

(13)

Further \( m^* \) is finite and unique.
**PROOF**: Since \( C(N,m,n) \) is discrete in \( m \), to obtain \( m^* \) which minimises \( C(N,m,n) \), we form the following inequalities

\[
C(N,m+1,n) > C(N,m,n) \quad (14)
\]

\[
C(N,m,n) < C(N,m-1,n) \quad (15)
\]

after substituting (6) in (14), we get

\[
\frac{N(m+1)C_1 + C_2 b \sum_{i=1}^{n} (1/\alpha)^{\nu_i}}{(n+1) \mu_N(m+1)} > \frac{NmC_1 + C_2 b \sum_{i=1}^{n} (1/\alpha)^{\nu_i}}{(n+1) \mu_N(m)}
\]

After some simplification, we obtain

\[
C_1 \left[ (m+1)N\mu_N(m) - Nm\mu_N(m+1) \right] > C_2 \left[ b \sum_{i=1}^{n} (1/\alpha)^{\nu_i} \mu_N(m+1) - b \sum_{i=1}^{n} (1/\alpha)^{\nu_i} \mu_N(m) \right]
\]

\[
\frac{N(m+1) \mu_N(m) - Nm \mu_N(m+1)}{(n+1) \mu_N(m+1)} > \frac{C_2/C_1}{(n+1) \mu_N(m)} \quad (16)
\]

\[
\frac{b \sum_{i=1}^{n} (1/\alpha)^{\nu_i} \left[ \mu_N(m+1) - \mu_N(m) \right]}{(n+1) \mu_N(m+1)}
\]

Where L.H.S. of (16) is noted to be \( L(N,m,n) \). A similar procedure of using (6) in (15) leads to

\[
L(N,m-1,n) < \frac{C_2}{C_1}
\]

The first part of the theorem is thus proved.
To prove that $m^*$ is finite and unique, it is sufficient to show that $L(N,m,n)$ strictly increases with $m$. Then $L(N,m,n)$ crosses $C_2/C_1$ just once. This is shown in the following.

$$L(N,m+1,n) - L(N,m,n)$$

$$= \left[ \sum_{i=1}^{n} b \left( \frac{1}{\alpha} \right)^{-i} \left[ \mu_N(m+2) - \mu_N(m+1) \right] \right] - \left[ \sum_{i=1}^{n} b \left( \frac{1}{\alpha} \right)^{-i} \left[ \mu_N(m+1) - \mu_N(m) \right] \right]$$

After some simplification, we obtain

$$= \left[ \mu_N(m+1) \right] - \left[ \mu_N(m) \right]$$

$$> 0 \text{ by Lemma 1}$$

for $m \leq m_0 + 1$ since $(\mu_N(m+1) - \mu_N(m))$ decreases with $m$ for $m \leq m_0 + 1$.

Thus $L(N,m,n)$ is strictly increasing in $m$ for $m \leq m_0 + 1$ and hence $L(N,m,n)$ crosses the finite value $C_2/C_1$ just once.

This completes the proof of the theorem.
To prove that $m^*$ is finite and unique, it is sufficient to show that $L(N,m,n)$ strictly increases with $m$. Then $L(N,m,n)$ crosses $C_2/C_1$ just once. This is shown in the following.

\[
L(N,m+1,n) - L(N,m,n)
\]

\[
= \left[ \sum_{i=1}^{n} b \frac{1}{\alpha} \left[ \mu_N(m+1) - \mu_N(m+1) \right] \right] \frac{1}{N} - \left[ \sum_{i=1}^{n} b \frac{1}{\alpha} \left[ \mu_N(m) - \mu_N(m) \right] \right] \frac{mn}{N} \]

After some simplification, we obtain

\[
= \left[ \sum_{i=1}^{n} b \frac{1}{\alpha} \left[ \mu_N(m+1) - \mu_N(m+1) \right] \right] \frac{1}{N} - \left[ \sum_{i=1}^{n} b \frac{1}{\alpha} \left[ \mu_N(m) - \mu_N(m) \right] \right] \frac{mn}{N} \]

\[
> 0 \quad \text{by Lemma 1}
\]

for $m \leq m_0+1$ since $\left( \mu_N(m+1) - \mu_N(m) \right)$ decreases with $m$ for $m \leq m_0+1$.

Thus $L(N,m,n)$ is strictly increasing in $m$ for $m \leq m_0+1$ and hence $L(N,m,n)$ crosses the finite value $C_2/C_1$ just once.

This completes the proof of the theorem.
2.4. NUMERICAL RESULTS

For the purpose of illustration, we choose negative exponential failure times, whose distribution is given by $F(t) = 1 - e^{-\lambda t}$. Numerical results are given below. We choose $b = 0.2$, $N = 2$, $\alpha = 0.9$

Using theorem 1 we compute the optimal number of units $m^*$ and resulting cost $C(N,m^*,n)/C_1$, corresponding to different values of $C_2/C_1$, $p, n$ and present them in the following table.

**TABLE 1 C_2/C_1, m^* AND C(N,m^*, n)/C_1, N = 2**

<table>
<thead>
<tr>
<th>$C_2/C_1$</th>
<th>$p=0.01$</th>
<th>$C(N,m^*,n)/C_1$</th>
<th>$m^*$</th>
<th>$C(N,m^*,n)/C_1$</th>
<th>$p=0.02$</th>
<th>$C(N,m^*,n)/C_1$</th>
<th>$m^*$</th>
<th>$C(N,m^*,n)/C_1$</th>
<th>$p=0.03$</th>
<th>$C(N,m^*,n)/C_1$</th>
<th>$m^*$</th>
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2.5. **CONCLUDING REMARKS**

By applying theorem 1 we computed the optimal number of units $m^*$ and the resulting cost corresponding to different values of $C_2/C_1$, $n$, $p$ and presented them in the above Table 1. From this Table 1 we observe that the number of units $m^*$ and the resulting cost increases with increase in $C_2/C_1$ values as well as in $n$. However, as $p$ increases $m^*$ decreases and the resulting cost increases. Therefore, our computation demonstrates that as $p$ increases it is advisable to go in for lower optimal $m^*$ for Series-Parallel Systems.

We now give below the selected list of references.
SELECTED REFERENCES


[5]. RAMI REDDY, C., and VIJAYA BHASKARA RAO, D., *Optimisation of Parallel Systems Subject to Two-modes of Failure with Repair provision*.

