CHAPTER - 2
GLOBAL POSITIONING SYSTEM

2.1 Introduction

The Global Positioning System (GPS) is a satellite-based ranging system with a constellation of 24 satellites in six 12-hour orbits. This navigation system was developed and implemented by the U.S. Department of Defense (DoD) and has been fully deployed since 1993. Figure 2.1 presents the configuration of the GPS constellation. Each satellite continuously broadcasts its own specific pseudo-random noise (PRN) code for identification and ranging purposes.

![Figure 2.1 The Global Positioning System](image)

The constellation comprises 21 satellites with three spares in six orbits. Each orbit is at 55° inclination angle and has a period of 11 hr 57 min. To determine its position, a receiver needs at least four satellites in view to measure at least four pseudoranges to estimate three dimensions of location and user clock error.

The satellite PRN code and navigation information are broadcast at L-band microwave frequencies. The flight time of the GPS PRN signal traveling through space from satellite to receiver provides the basis for the range measurement. Civilian users measure these timing signals, termed pseudoranges (PRs), from the satellites to calculate their positions.
The control station at Colorado Springs is responsible for orbit determination of the satellites and system maintenance. The satellite navigation information includes satellite location and ranging signals. The ranging measurement, pseudorange, is the time-of-flight light of the signals from the satellite to the user. It is converted to range by multiplying the speed of light (C), Equation (2.1). Because pseudorange is a timing signal, it contains not only the true range between the satellite and the receiver but also receiver and satellite clock biases.

Furthermore, as the signal travels through the ionosphere and troposphere, it experiences delays. Finally, as the receiver measures the signal, the local multipath and the receiver thermal noise also introduce errors. Therefore, we can write the PR measurement between the jth satellite and ith receiver as

\[ PR^j_i = \Delta t_{\text{transit}} \cdot C \]

\[ = \rho^j_i + b^j_i - \underbrace{B^j_i}_{\text{receiver clock bias}} + \underbrace{M^j_i}_{\text{multipath error}} + \underbrace{E^j_i}_{\text{measurement noise}} + \underbrace{I^j_i + T^j_i}_{\text{ionospheric and tropospheric delay}} \]

The errors in PR, together with geometry, affect the user positioning accuracy directly [12,14,16].
2.2 GPS Position Solution

To find their positions and time, GPS users must have at least four satellites in view. The least - squares solution is presented below [10, 11, 12, 13, 14]:

[Known]: Satellite position \( X_j, Y_j, Z_j \) in World Geodetic System – 84 (WGS – 84) coordinate, and satellite clocks bias \( B_j \) with respect to GPS time \( j = 1.. N, N \geq 4, \) and \( N \) is the number of satellites in view). All of this information is included in the GPS broadcast messages. Note that satellite broadcast parameters for calculation of \( B_j \) does not include Selective Availability (SA).

[Unknown]: User position \( X_u, Y_u, \) and \( Z_u \) and receiver clock bias \( b_u \), or state vector

\[
\mathbf{R}_u = \begin{bmatrix} X_u & Y_u & Z_u & b_u \end{bmatrix}^T \quad (2.2)
\]

[Measurement]: Pseudoranges (PR) between satellite \( j \) and the user \( u \), i.e.,

\[
PR_u^j = \rho_u^j + b_u - B_j + \epsilon_u^j, \quad \text{or} \quad (2.3)
\]

\[
PR_u^j + B_j = \sqrt{(X_u - X_j)^2 + (Y_u - Y_j)^2 + (Z_u - Z_j)^2} + b_u + \epsilon_u^j \quad (2.4)
\]

Where \( \epsilon_u^j \) contains all the error sources in the PR measurement. For example, it contains the SA, ionospheric and tropospheric delays, multipath and receiver thermal noise in non-differential model. However, in the differential GPS mode, the error, \( \epsilon_u^j \) is greatly reduced by corrections.

[Solution]: Because the number of measurements is greater than the number of states \( N \geq 4 \) and the PR measurements are nonlinear in terms of the state vector \( [X_u, Y_u, Z_u, b_u] \), the GPS positioning is an over determine nonlinear problem. A commonly used technique is to linearize the measurement with respect to an initial guess, \( \hat{\rho}_u \) and \( \hat{b}_u \), and to estimate the solution iteratively using the least squares approach. For the specific problem at hand, the linearized measurements can be expressed in state space form as
\[ z_u^i = \delta PR_u^i = PR_u + B^i - \rho_u - b_u = H^i r_u + V_u^i \] (2.5)

Where

\[ H^i = \begin{bmatrix} \frac{\partial PR_u^i}{\partial X_u} & \frac{\partial PR_u^i}{\partial Y_u} & \frac{\partial PR_u^i}{\partial Z_u} & 1 \end{bmatrix}_{4 \times 1} \] (2.6)

and \( r_u \) is the deviation from best estimate of the previous iteration step. After accumulating over all satellites in view, we have the full matrices,

\[ H = \begin{bmatrix} H & \vdots & \vdots & \vdots \end{bmatrix}_{N \times 4} \quad \text{and} \quad Z = \begin{bmatrix} z_u^1 \\ \vdots \\ \vdots \\ z_u^N \end{bmatrix}_{N \times 4} \] (2.7)

Where \( N \) is the total number of satellites in view.

The over determined least-squares solution is \( \hat{r}_u = H^+ z \), where \( H^+ = (H^T H)^{-1} \)

\( H^T \) is the pseudo-inverse of \( H \), or

\[ \hat{r}_u = (H^T H)^{-1} H^T z \] (2.8)

and the best estimate is found iteratively as \( \hat{R}_{u_k+1} = \hat{R}_{u_k} + \hat{r}_{u_k} \), where \( K \) represents the \( k^{th} \) iteration step until the magnitude of the update, \( \| \hat{r}_k \|_2 \), smaller than a criterion.
Note.

- Because of all the ones at the last column of the observation matrix H, the solutions of receiver clock error is the common part (i.e. the average) of PRs. That is, anything common to all the PR measurements will be absorbed into the receiver clock bias estimate without affecting the accuracy of user’s position.

- From processing experience, the convergent rate of this iteration is fast, usually less than three to five iterations, to reach the convergent criterion \( \| \hat{r}_k \|_2^2 = 1 \) cm.

- A weighted least squares solution of \( \hat{r}_{\text{weighted}} = (H^T WH)^{-1} H^T W z \). From the Best Linear Unbiased Estimation Theory (BLUE), the weighting matrix is \( W = V^{-1} \), where \( V \) is the measurement noise covariance matrix. In this weighted form, the covariance matrix of the estimated state is \( P = (H^T WH)^{-1} \) [8, 9].

- If all the noises in PRs are the same, i.e., \( V = \text{diag} \left[ \sigma_{PR}^2 \right]_{N \times N} \), then the position covariance can be expressed as \( P = \sigma_{PR}^2 (H^T H)^{-1} \). In other words, the variances of the states are the products of the measurements variance and the respective diagonal elements of the matrix \( (H^T H)^{-1} \). These diagonal elements are usually referred to as Dilution of Position (DOPs) and are related to the satellite geometry. From this relationship, we want the DOPs to be as small as possible. This roughly means that the distribution of satellite positions in the sky is as uniform as possible.

- For an underdetermined case \((N < 4)\) the pseudo inverse of \( H \) can be shown to be \( H^+ = H^T (H^T H)^{-1} \). This is also referred to as the minimum norm estimation.

- \( H \) is a \( N \times 4 \) and can consume more computational memory if \( N \) (number of satellites in view) is large. However, for solving navigation position, we need only to form the fixed size \( 4 \times 4 \) matrix \( (H^T WH)_{4 \times 4} \) and the \( 4 \times 1 \) vector \( (H^T Wz)_{4 \times 1} \). To avoid allocating a large array in the program, the following equations can be used for uncorrelated measurement noises where the weighting matrix \( W \) is diagonal:

\[
H^T WH = \sum_{i=1}^{N} (H_i^T W_i H_i)_{4 \times 4} \tag{2.9}
\]
So one must only allocate a 4 X 4 matrix and a 4 X 1 for the accumulations in the above expression.

### 2.3 Velocity Estimation

This section describes a least-squares solution of the user velocity using Doppler measurements [2,3,4,5].

**[Known]**: The satellite 3-D velocity in WGS-84 and satellite clock bias rate \(X_J, Y_J, Z_J\) and \(B_J\). These parameters can be calculated from broadcast navigation information.

**[Unknown]**: The user's velocity and the drift rate of the clock error, that is, the state

\[ \text{Vector } R = [X_u, Y_u, Z_u, b_u]^T \tag{2.11} \]

**[Measurement]**: The range rate \(\rho\) can be calculated from the GPS Doppler measurement:

\[ f_r = f_s (1 - \frac{P}{c}) \text{ then } \rho = \frac{C}{f_s} (f_s - f_r) \tag{2.12} \]

where \(f_s\) is the frequency of the source and \(f_r\) is the received frequency affected by the relative motion between the receiver and the satellite.

**[Solution]**: The range rate between satellite \(j\) and user \(u\), \(\rho'^j_u\), can be written as

\[ \rho'^j_u = 1_{u,los}^j \left( V'^j - V_u \right) + b_u + \varepsilon^j_u \]

\[ = \frac{X'^j - X_u}{\rho'^j_u} \left( X'^j - X_u \right) + \frac{Y'^j - Y_u}{\rho'^j_u} \left( Y'^j - Y_u \right) \]

\[ + \frac{Z'^j - Z_u}{\rho'^j_u} \left( Z'^j - Z_u \right) + b_u + \varepsilon^j_u \tag{2.13} \]

The measurement can be written as

\[ z'^j_u = \rho'^j_u \hat{i}_{u,los}^j \cdot V'^j = \hat{i}_{u,los}^j \cdot V_u + b_u + \varepsilon^j_u \tag{2.14} \]

In the state-space format,
\[ z_u' = H^I . R_u + \varepsilon_u' \]

where

\[ H^I = \begin{bmatrix} X^I - X_u \frac{Y^I - Y_u}{\rho_u^I} \frac{Z^I - Z_u}{\rho_u^I} \end{bmatrix}_{4 \times 1} \]  

Then the weighted least squares solution of velocity is

\[ \hat{\mathbf{v}} = \left( H^T W H \right)^{-1} H^T W z \]  

(2.16)

Where \( H \) and \( Z \) are the accumulated observation matrix and measurement vector.

The weighting function used is

\[ W = \begin{bmatrix} SNR_1^2 & 0 \\ 0 & SNR_N^2 \end{bmatrix} \]  

(2.17)

Note That

- The velocity estimation presented above is linear provided the position is known, therefore no iteration is involved. Furthermore, although the weighting scheme is simple, it has sufficiently de-weighted the low elevation satellites and the test results are found satisfactory.

- A Kalman filter can also be constructed to estimate both user position and velocity simultaneously with PR and Doppler as measurements. However, as in many estimation problems, the filtering process will introduce delays. This delay with the more apparent when the aircraft maneuvers. A third order process model including the acceleration may be used to minimized the latency. However, this third order model significantly increases the complexity of the estimation.

- Satellite velocity is calculated from the differentiation of broadcast satellite orbital elements in ECEF coordinates. To simply calculate velocity from orbital elements without differentiation is NOT appropriate since the orbital plane is not a constant plane under the perturbations.

- For GAGAN users, PRs are corrected for satellite location error, SA and ionospheric delays. The range rate measurements can be corrected from the inferred
PR correction rate, i.e. SA rate and satellite velocity in the user LOS. We also correct the range rates by the change rate of ionospheric and tropospheric delay which are mainly caused by the changing of LOS elevation angles.

2.4 WADGPS Correction Error Analysis

The WADGPS actually estimates the satellite location and sends it to the user.

Therefore, the error in the WADGPS correction for a pseudorange between satellite j and user u can be expressed as [7,8,6,1]

\[
\varepsilon_u = \left( G_u x'_j + V'_u \right) - G_u \hat{x}'_j = G_u \left( x'_j - \hat{x}'_j \right) + V'_u
\]

where \( x'_j = [\Delta r'_j \ B'_j]^T \) is the vector containing true satellite location error \( (\Delta r'_j) \) and clock bias \( (B'_j) \) and \( \hat{x}'_j \) is the wide area estimation of \( X_j \). \( G_u \) is the observation matrix that projects the error or the estimate to the user’s line-of-sight, \( G_u = \begin{bmatrix} I_s^T & 1 \end{bmatrix}_{1 \times 4} \) and \( V'_u \) is the user’s measurement noise.

To calculate the differential correction errors for the user, we need to know the WADGPS corrections first. In this analysis, we used only three reference stations and therefore the minimum-norm estimate of the wide area location error for satellite j can be calculated using the pseudorange residual \( z'_j = G_r x + x + v'_j \) where \( G_r \) is observation matrix, and the state vector \( x'_j = [\Delta r'_j \ B'_j]^T \) (the result justifies this geometric approach). The minimum - norm solution of the state is

\[
\hat{x}'_j = G^+ z = G^+_r \left( G_r G^+_r \right)^{-1}
\]

To find the variance of correction residual, we further assume

1. The user measurement noise and variance are \( E[v'_u] = 0 \) and \( \text{Var}[v'_u] = \sigma^2 \),
2. Each of the wide area reference stations has equal measurement noise and variances, and they are \( E[v_r'] = 0 \) and \( \text{var}[v_r'] = v_r \), and
3. \( E\left[ E_v' \right] = 0 \)

Then the variance of \( \varepsilon_u \) is
\[
\text{Var}[\varepsilon_w] = E[\varepsilon_w^2] = G_u P_{x-\hat{x}} G_u^T + \sigma^2 \quad (2.20)
\]

Where
\[
P_{x-\hat{x}} = E[(\hat{x} - x)(\hat{x} - x)^T]
\quad (2.21)
\]

\[
= E[x'x'] + E[\hat{x}'\hat{x}'] - 2E[x' \hat{x}']
\]

Since
\[
E[\hat{x}'\hat{x}'] = E[x' z \hat{G}^+]
\quad (2.22)
\]

then
\[
P_{x-\hat{x}} = W + E[G^+ z z^T G^+^T] - 2E[x' z G^+]
\quad (2.23)
\]

\[
= W + E[G^+ (G, x' + v') (G, x' + v')^T] - 2E[x' (G, x' + v') G^+]
\]

\[
\]

\[
= (I - G^+ G, ) W (I - G^+ G,) + G^+ V G^+
\]

Substituting into Equation (2.20), the WADGPS variance due to errors in the satellite location becomes
\[
\quad (2.24)
\]