Chapter 4

\textbf{\textit{m-} Neighbourly Irregular Graphs}

For \( m \geq 1 \), \( m \)-Neighbourly Irregular graph and \( m \)-Neighbourly Irregular tree are defined in this chapter. It is shown that every graph of order \( n \geq 2 \) is an induced subgraph of \( m \)-Neighbourly Irregular graph, for \( m = 1, 2 \). Also, strict \( m \)-Neighbourly Irregular tree and \( m \)-Neighbourly Irregular strength of a graph are defined and a few properties possessed by 2-Neighbourly Irregular graph and 2-Neighbourly Irregular tree are studied. The 2-Neighbourly Irregular graph of order one to six are listed out. Also, a few results connecting 1-Neighbourly Irregular graph and 2-Neighbourly Irregular graph are proved.

4.1 Introduction

A graph \( G \) is regular if all its vertices have the same degree, otherwise it is irregular[1]. G.S.Bloom, J.K.Kennedy and L.V.Quintas, defined distance \textit{d-irregular} connected graph \( G \) with \( n \geq 2 \) vertices. A graph is distance \textit{d-irregular} if for every two distinct vertices \( u \) and \( v \) of \( G \), the number of vertices at a distance \( d \) from \( u \) is different from the number of vertices at a distance \( d \) from \( v \). But there is no \textit{d}-irregular graph [7].

Irregular graphs that are also connected have been defined and have been referred to as highly irregular graphs [1]. That is, a connected graph \( G \) is highly irregular if every vertex of \( G \) is adjacent only to vertices with distinct degrees. It was shown that for every graph \( H \) of order \( n \geq 2 \), there is a highly irregular graph
G containing H as an induced subgraph.

S.Gnana Prakasam and S.K.Ayyaswamy introduced the concept of Neighbourly Irregular graph in 2004. A connected graph G is said to be a *Neighbourly irregular graph (NI graph)* if no two adjacent vertices of G have the same degree [14].

A connected graph is *Neighbourhood Highly Irregular (NHI)* if each of its vertices adjacent to a vertex has distinct closed neighbourhoods [51]. Every graph of order \( n \geq 2 \) is an induced subgraph of a neighbourhood highly irregular graph of order \( 2n - k \), where \( k \) is the number of pendant vertices of \( G \).

Inspired by these definitions, the \( m \)-Neighbourly Irregular graph and \( m \)-Neighbourly Irregular Tree (abbreviated as \( m \)-NI graph and \( m \)-NI tree) are defined and it is shown that every graph of order \( n \geq 2 \) is an induced subgraph of \( m \)-Neighbourly Irregular graph, for \( m = 1, 2 \).

### 4.2 \( m \)-Neighbourly Irregular Graphs (\( m \)-NI)

For \( m \geq 1 \), \( m \)-Neighbourly Irregular graph is defined and few examples of \( m \)-Neighbourly Irregular graph are presented in this section.

**Definition 4.2.1.** A connected graph \( G \) is said to be \( m \)-Neighbourly Irregular (\( m \)-NI) if no two adjacent vertices of \( G \) have the same number of vertices at a distance \( m \) away from them. Then, a connected graph \( G \) is \( m \)-Neighbourly Irregular \( d_m(u) \neq d_m(v) \), for all \( uv \) in \( E(G) \), where \( d_m(v) \) denotes the number of vertices at a distance \( m \) from \( v \) in \( G \) (\( m \), a positive integer).

**Example 4.2.2.** \( m \)-Neighbourly Irregular Graphs, for \( m = 1, 2 \)

1. Any complete bipartite graph \( K_{l,n} \) is \( m \)-Neighbourly Irregular graph only when \( l \neq n \).
2. \( P_3 \) is \( m \)-Neighbourly Irregular graph.
3. Any complete tripartite graph \( K_{l,n,p} \) is \( m \)-Neighbourly Irregular graph only when \( l \neq n \neq p \).
Example 4.2.3. Graphs shown in Figure 4.1 are $m$-Neighbourly Irregular for $m = 1, 2, 3$.

![Figure 4.1](image)

Example 4.2.4. Graphs given in figure 4.2 are 1-Neighbourly Irregular and 2-Neighbourly Irregular but not 3-Neighbourly Irregular.

![Figure 4.2](image)

The graphs given in figure 4.2 are obtained from star by joining two alternate pendant vertices to one new vertex are $m$-Neighbourly Irregular graphs for $m = 1, 2$. (vertices labeled by their $d_2$-degree)

Example 4.2.5. Herschel graph given in figure 4.3 is 2-Neighbourly Irregular graph, but it is not 1-neighbourly irregular and not 3-Neighbourly irregular. (vertices labeled by their $d_2$-degree)

![Figure 4.3](image)

Remark 4.2.6. Following four graphs are not $m$-Neighbourly Irregular for $m = 1, 2, 3$:

1. $K_{n_1, n_2, \ldots, n_m}$ with at least two $n_i$'s are same.

2. Any path $P_n (n \neq 3)$.

3. Any cycle $C_n (n \geq 3)$. 
4. All complete graphs.

5. Helm graph obtained from a wheel $W_n$ by attaching a pendant edge at each vertex of the $n$-cycle is not $m$-Neighbourly Irregular for $m \geq 1$.

The vertices in the $n$-cycle are adjacent and they have the same $d_m$-degree, for $m \geq 1$. Hence helm graph is not $m$-Neighbourly Irregular graph, for $m \geq 1$ (vertices labeled by their $d_2$-degree).

**Theorem 4.2.7.** Any graph with diameter less than $m$ is not $m$-Neighbourly Irregular graph.

### 4.3 1-Neighbourly Irregular Graphs (1-NI)

A method is suggested in this section to construct 1- Neighbourly Irregular graph containing the given graph as an induced subgraph. Also, minimum number of vertices needed to construct 1- Neighbourly irregular graph containing certain graphs are determined[43].

**Definition 4.3.1.** [14] A connected graph $G$ is 1-Neighbourly Irregular (1-NI) if no two adjacent vertices of $G$ have the same $d_1$-degree.

It is to be noted that Neighbourly Irregular graphs and 1-Neighbourly Irregular graphs are the same.

**Example 4.3.2.** 1-Neighbourly Irregular Graphs.
The following facts are known from literature

**Fact 4.3.3.** [14] The complete bipartite graph $K_{m,n}$ is Neighbourly Irregular if and only if $m \neq n$.

**Fact 4.3.4.** [14] If $v$ is a vertex of maximum degree in a Neighbourly Irregular graph, then at least two of the adjacent vertices of $v$ have the same degree.

**Fact 4.3.5.** [14] If a graph $G$ is Neighbourly Irregular, then no $P_4$ contains internal vertices of the same degree in $G$.

**Fact 4.3.6.** [14] Any graph of order $n$ can be made to be an induced subgraph of a NeighbourlyIrregular graph of order atmost $(n + 1)C_2$.

**Fact 4.3.7.** [14] It is seen that there exists a Neighbourly Irregular graph $K_{n_1, n_2, \ldots, n_m}$ of order $n$, for any +ve integer $n$ and $(n_1, n_2, \ldots, n_m)$ be a partition of $n$ with distinct parts.

**Fact 4.3.8.** [14] Let $G$ be an Neighbourly Irregular graph of order $n$. Then, for any positive integer $m < n$, there exists atmost $m$ vertices of degree $n - m$.

### 4.4 1-Neighbourly Irregular Graphs Containing a Given Graph as an Induced Subgraph

König [20] proved that if $G$ is any graph, whose maximum degree is $r$, then it is possible to add new vertices and to draw new edges joining either two new vertices or a new vertex to an existing point, so that the resulting graph $H$ is a regular graph containing $G$ as an induced subgraph.

**Theorem 4.4.1.** Every graph $G$ is an induced subgraph of 1-Neighbourly irregular graph.

*Proof.* Let $G$ be the given graph.

**Case 1** If $G$ is an $r$-regular graph, then 1-Neighbourly Irregular graph $N$ containing the given graph $G$ can be constructed
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Case 2 If \( G \) is not a regular graph whose maximum degree is \( r \), then \( r \)-regular graph \( H \) whose degree of regularity is the maximum degree of a graph \( G \) can be constructed [20]. Let \( V(H) = \{ h_i : 1 \leq i \leq n \} \). The desired graph \( N_1 \) has the vertex set \( V(N_1) = V(H) \cup V(F) \cup V(T) \), where \( V(F) = \{ f_i : 1 \leq i \leq n - 1 \} \) and \( V(T) = \{ t_i : 1 \leq i \leq n + r - 1 \} \). Let \( E(N_1) = E(H) \cup \{ h_i f_j : 1 \leq i \leq n, 1 \leq j \leq i \} \cup \{ f_i t_j : 1 \leq i \leq n - 1, 1 \leq j \leq n + r - 1 \} \). It is to be noted that the degree of \( h_i \) in \( N_1 = r + i - 1 \), \( 1 \leq i \leq n \) and the degree of \( f_i \) in \( N_1 = n + r + j - 1 \), \( 1 \leq j \leq n - 1 \) and the degree of \( t_i \) in \( N_1 = n - 1 \), \( 1 \leq i \leq n + r - 1 \). Then, the desired graph \( N_1 \) is the 1-Neighbourly Irregular graph of order \( 3n + r - 2 \) containing every \( r \)-regular graph \( H \) of order \( n \) as an induced subgraph. Hence this graph \( N_1 \) is the 1-Neighbourly Irregular graph containing every graph \( G \) as an induced subgraph.

Example 4.4.2. Figure 4.6 illustrates Theorem 4.4.1 for the given graph \( G \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.6.png}
\caption{Figure 4.6}
\end{figure}

Theorem 4.4.3. For \( n \geq 1 \), the minimum order of 1-Neighbourly Irregular graph containing a regular complete bipartite graph \( K_{n,n} \) of order \( 2n \) as an induced subgraph is \( 2n + 1 \).

\textit{Proof.} By attaching one pendant edge to any vertex of \( K_{n,n} \) then the resulting graph is 1-Neighbourly Irregular graph of order \( 2n + 1 \) containing \( K_{n,n} \).

Theorem 4.4.4. For any \( n \geq 3 \), every \( n \)-cycle is an induced subgraph of 1-Neighbourly Irregular graph.

\textit{Proof.} Let \( n \)-cycle be a given graph.

\textbf{Case (i)} \( n \) is even \( (n \geq 4) \). By attaching one pendant edge to alternate vertices of \( n \)-cycle, the resulting graph is 1-Neighbourly Irregular graph of order \( (3n)/2 \).

\textbf{Case (ii)} \( n \) is odd \( (n \geq 3) \). Let \( C_n \) be an odd \( n \)-cycle. \( V(C_n) = \{ v_1, v_2, v_3, v_4, \ldots, v_n \} \).
By attaching two pendant edges to \( v_n \), and attaching one pendant edge to \( v_i \), \( i = 1, 3, 5, 7, \ldots, n - 2 \), the resulting graph is 1-Neighbourly Irregular graph of order \((3n + 3)/2\).

**Example 4.4.5.** Figure 4.7 illustrates Theorem 4.4.4 for \( n = 4, 6 \) and \( n = 5, 7 \).

![Figure 4.7](image)

**Theorem 4.4.6.** For any \( n \geq 3 \), every path \( P_n \) is an induced subgraph of 1-Neighbourly Irregular graph.

*Proof.* Let \( P_n \) be a given graph.

**Case (i)** \( n \) is odd. If \( n = 3 \), then the graph \( P_3 \) is 1-Neighbourly Irregular graph. Let \( P_n(n \geq 5) \) be a path of length \( n-1 \). \( V(P_n) = \{v_1, v_2, v_3, v_4, \ldots, v_n\} \). By attaching one pendant edge to \( v_i, i = 3, 5, 7, \ldots, n - 2 \), the resulting graph is 1-Neighbourly Irregular graph of order \( 3(n-1)/2 \).

**Case (ii)** \( n \) is even. If \( n = 4 \), attach one pendant edge to \( v_3 \). Then the resulting graph is 1-Neighbourly Irregular graph of the smallest order 5. Let \( P_n(n \geq 6) \) be a path of length \( n - 1 \). \( V(P_n) = \{v_1, v_2, v_3, v_4, \ldots, v_n\} \). By attaching one pendant vertex to \( v_i, i = 3, 5, 7, 9, \ldots, n - 1 \), the resulting graph is 1-Neighbourly Irregular graph of order \( (3n - 2)/2 \).

**Example 4.4.7.** Figure 4.8 illustrates Theorem 4.4.6 for \( n = 5, 7 \) and \( n = 6, 8 \).

![Figure 4.8](image)

**Theorem 4.4.8.** For \( n \geq 2 \), the smallest order of 1-Neighbourly Irregular graph containing \( K_n \) as an induced subgraph is \( 2n - 1 \).
Proof. Let $K_n$ be the complete graph of order $n \geq 2$. If $n = 2$, then the graph $P_3$ is 1-Neighbourly Irregular graph containing $K_2$. Let $n \geq 3$ and $V(K_n) = \{v_i : (1 \leq i \leq n)\}$. The desired graph $KN_1$ has the vertex set $V(KN_1) = V(K_n) \cup V(U)$, where $V(U) = \{u_i : (1 \leq i \leq n-1)\}$. Let $E(KN_1) = E(K_n) \cup \{v_iu_j : 1 \leq i \leq n, 1 \leq j < i\}$. Moreover, $d(v_i) = n - 2 + i$, $(1 \leq i \leq n)$ and $d(u_i) = n - i$, $(1 \leq i \leq n - 1)$. Then, the resulting graph $(KN_1)$ is a 1-Neighbourly Irregular graph of order $2n - 1$ which contains $K_n$ as an induced subgraph.

Remark 4.4.9. A graph is called pairlone if it has exactly two vertices of the same degree. It is observed that $d(v_i)$ in $(KN_1)$ is $n + i - 2$, $(1 \leq i \leq n)$, $d(u_i)$ in $(KN_1)$ is $n - i$, $(1 \leq i \leq n - 1)$. The degree sequence is $(1, 2, 3, 4, \ldots, [(2n - 1)/2], [(2n - 1)/2], \ldots, 2n - 2)$. Ebrahim Salehi called this graph a pairlone graph $PL_n$ and proved that for any $n$, there exists a unique pairlone graph of order $n$, which contains at least one vertex of each possible degree[10].

4.5 2-Neighbourly Irregular graph (2-NI)

Definition 4.5.1. A connected graph $G$ is said to be 2-Neighbourly Irregular (2-NI) if no two adjacent vertices of $G$ have the same number of vertices at a distance two away from them. A connected graph $G$ is 2-Neighbourly Irregular if $d_2(u) \neq d_2(v)$, for all $uv$ in $E(G)$, where $d_2(v)$-denotes the number of vertices at a distance two from $v$ in $G$[36].

Example 4.5.2. Graphs shown in Figure 4.9 are 2-Neighbourly Irregular graphs.
**Example 4.5.3.** Gear graph is obtained from the wheel $W_n (n \neq 4)$ by inserting a vertex between every pair of adjacent vertices of the cycle. Graphs in Figure 4.10 are Gear graphs. Gear graph is 1-Neighbourly irregular and 2-Neighbourly Irregular.

Proof. Let the vertex set of Gear graph be $\{v_1, v_2, v_3, \ldots, v_n\} \cup \{v\} \cup \{u_1, u_2, u_3, \ldots, u_n\}$ and edge set $\{vv_i : (1 \leq i \leq n)\} \cup \{v_iu_i : (1 \leq i \leq n)\} \cup \{u_iu_{i+1} : (1 \leq i \leq n - 1)\} \cup \{u_nv_1\}$. It is noted that $d_2(v_i) = n - 1$, $(1 \leq i \leq n)$ and $d_2(u_i) = 3$, $(1 \leq i \leq n)$. Also, $d_2(v) = n$. Hence no two adjacent vertices have the same $d_2$. Gear graph is 2-Neighbourly Irregular, for $n \neq 4$.

**Remark 4.5.4.** When $n = 4$, $d_2(v_i) = 3$, for $(1 \leq i \leq 4)$, $d_2(u_i) = 3$, for $(1 \leq i \leq 4)$ and $d_2(v) = 4$. Hence Gear graph is not 2-Neighbourly Irregular for $n = 4$.

**Theorem 4.5.5.** Any complete $m$ partite graph $K_{n_1,n_2,\ldots,n_m}$ is 2-Neighbourly Irregular if and only if no two $n_i$’s are equal.

Proof. Let $G(V, E)$ be a complete $m$ partite graph with partition $(V_{n_1}, V_{n_2}, \ldots, V_{n_m})$ of $V$. Then every vertex in vertex set $V_{n_i}$ is adjacent to all other vertices in the remaining $m - 1$ partite sets and non-adjacent with vertices in the same set $V_{n_i} (1 \leq i \leq m)$. Hence every vertex in $V_{n_i}$ is at a distance two from $n_i - 1$ vertices, $(1 \leq i \leq m)$. For any vertex $v \in V_{n_i}$, $d_2(v) = n_i - 1$, $(1 \leq i \leq m)$. Let $u, v$ be two adjacent vertices of $G$. Then, $u$ and $v$ are in two different partition sets of $V$. Let $u \in V_{n_i}$ and $v \in V_{n_j}$ where $i \neq j$. $G$ is 2-Neighbourly Irregular if and only if $d_2(u) \neq d_2(v)$, $uv \in E(G)$ if and only if $n_i - 1 \neq n_j - 1$, for $i \neq j$ if and only if $n_i \neq n_j$, for $i \neq j$. Hence $K_{n_1,n_2,\ldots,n_m}$ is 2-Neighbourly Irregular if and only if no two $n_i$’s are equal.

**Example 4.5.6.** Figure 4.11 illustrates Theorem 4.5.5 for $K_{2,3,4}$.
The above graph $K_{2,3,4}$ is $m$-Neighbourly Irregular graph for $m = 1, 2$.

**Theorem 4.5.7.** For any $n \geq 3$, there exists at least one 2-Neighbourly Irregular graph of order $n$.

**Proof.** Every positive integer $n \geq 3$ has a partition $(1, n-1)$. Hence every positive integer $n \geq 3$ has at least one partition with distinct parts. For each partition with distinct parts, there exists a complete partite graph. Then, for any $n \geq 3$, there exists at least one $K_{n_1,n_2,\ldots,n_m}$ such that $n_i's$ are distinct. 

**Remark 4.5.8.** The class of all $K_{n_1,n_2,\ldots,n_m}$ graphs is only a proper subclass of the class of all 2-Neighbourly Irregular graphs. For example, graph in Figure 4.12 is 2-Neighbourly Irregular graph of order 5, which is not in the class of all $K_{n_1,n_2,\ldots,n_m}$ graphs.

**4.6 $S_{m,t}$-graph**

Some other class of 2-Neighbourly Irregular graphs which are not in the class of all $K_{n_1,n_2,\ldots,n_m}$ graphs are discussed in this section.

**Definition 4.6.1.** Let $S_{m,t}$ denote the bipartite graph of order $n$ having distinct partite sets $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_t\}$, where $m < t$, and edge set $E(S_{m,t}) = \cup E_i$, where $E_i = \{u_iv_j : m - i + 1 \leq j \leq t \text{ and } 1 \leq i \leq m\}$. By the
construction of $S_{m,t}, d_2(u_i) = m - 1, (1 \leq i \leq m)$, and $d_2(v_i) = t - 1 (1 \leq i \leq m)$ with $m < t$. Hence $S_{m,t}$ is 2-Neighbourly Irregular graph of order $n$, where $n = m + t$ and $m < t$.

**Example 4.6.2.** Graphs given in Figure 4.13 are 2-Neighbourly Irregular graphs

![Graphs](image)

**Remark 4.6.3.** For each odd $n$, $((n - 1)/2)S_{m,t}$ graphs exist. For each even $n$, $((n - 2)/2)S_{m,t}$ graphs exist.

**Theorem 4.6.4.** For any $n \geq 3$, there exists at least one $S_{m,t}(m < t)$ of order $n = m + t$.

*Proof.* Since every positive integer $n \geq 3$ has at least one partition with distinct parts, the theorem follows.

**Theorem 4.6.5.** If any vertex $v$ in a graph $G$ is adjacent with vertex $u$ of degree $n$ and non adjacent with vertices which are adjacent with $u$, then $v$ is at a distance two from at least $n - 1$ vertices.
**Theorem 4.6.6.** If a graph $G$ is 2-Neighbourly Irregular, and it contains a path $P_4$, then $P_4$ does not contain internal vertices having the same number of vertices at a distance two from them.

**Proof.** Let $G$ be any 2-Neighbourly Irregular graph. Suppose $P_4$ is in $G$, then the internal vertices of $P_4$ being adjacent, there must be different number of vertices at a distance two from them. \qed

**Example 4.6.7.** Figure 4.14 illustrates Theorem 4.6.6

![Figure 4.14](image)

**Remark 4.6.8.** List of connected 2-Neighbourly Irregular graphs of order $n$ up to six.

1. $n = 1, 2$. There is no 2-Neighbourly Irregular graph of order 1 and 2.

2. $n = 3$. Path $P_3$ is the only 2-Neighbourly Irregular graph of order 3.

![Diagram for n=3](image)

3. $n = 4$. Star $K_{1,3}$ is the only 2-Neighbourly Irregular graph of order 4.

![Diagram for n=4](image)

4. $n = 5$. There are five 2-Neighbourly Irregular graph of order 5.

![Diagram for n=5](image)

5. $n = 6$. There are eleven 2-Neighbourly Irregular graph of order 6.
4.7 Construction of 2-Neighbourly Irregular graph Containing a Given Graph as an Induced Subgraph

A 2-Neighbourly Irregular graph containing a given graph as an induced subgraph is constructed by attaching pendant edges and 2-Neighbourly regular strength of $G$ is also defined. Also, minimum number of additional vertices needed to construct 2-Neighbourly Irregular graph containing some particular graphs are determined[36].

**Theorem 4.7.1.** For $n \geq 1$, the minimum order of 2-Neighbourly Irregular graph containing a regular complete bipartite graph $K_{n,n}$ of order $2n$ as an induced subgraph is $2n + 1$.

*Proof.* By attaching one pendant edge to any vertex of $K_{n,n}$ there is a 2-Neighbourly Irregular graph of order $2n + 1$, since, because of this attachment $d_2$ of each vertex in one partition becomes $n$ and $d_2$ of each vertex in another partition set becomes $n - 1$. Hence no two adjacent vertices have the same $d_2$-degree. \(\square\)

**Theorem 4.7.2.** For any $n > 3$, every $n$-cycle is an induced subgraph of 2-Neighbourly Irregular graph of order $2n$.

*Proof.* Let $n$-cycle be the given graph.

Case (i) $n$ is even ($n \geq 4$). By attaching two pendant edges to alternate vertices
of \( n \)-cycle (\( n \) is even), 2-Neighbourly Irregular graph of order \( 2n \) is obtained.

**Case (ii)** \( n \) is odd (\( n \geq 5 \)).

Let \( C_n \) be an odd \( n \)-cycle. \( V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\} \). By attaching three pendant edges to \( v_1 \), and two pendant edges to \( v_i \), for \( i = 3, 5, 7, \ldots, n - 2 \), then the resulting graph will be 2-Neighbourly Irregular graph of order \( 2n \) containing \( C_n \) (n odd) as an induced subgraph.

**Illustration:** Figure 4.16 illustrates Theorem 4.7.2 for \( n = 5, 6, 7, 8 \).

![Figure 4.16](image)

**Theorem 4.7.3.** For any \( n \geq 3 \), every path \( P_n \) is an induced subgraph of 2-Neighbourly Irregular graph (tree) of order \( 2n - 3 \).

**Proof.** Case (i) \( n \) is odd. If \( n = 3 \), \( P_3 \) is 2-Neighbourly Irregular graph. Let \( P_n (n \geq 5) \) be a path of length \( n - 1 \). \( V(P_n) = \{v_1, v_2, v_3, v_4, \ldots, v_n\} \). By attaching one pendant edge to \( v_2 \) and attaching one pendant edge to \( (n - 1)^{th} \) vertex of \( P_n \) and then attaching two pendant edges to \( v_i \), for \( i = 4, 6, 8, \ldots, n - 3 \), 2-Neighbourly Irregular graph (tree) of order \( 2n - 3 \) is obtained.

**Illustration:** Figure 4.17 illustrates Theorem 4.7.3 for \( n = 5, 7, 9 \).

![Figure 4.17](image)

Case (ii) \( n \) is even. If \( n = 4 \), attach one pendant edge to \( v_2 \), then there is a 2-Neighbourly Irregular graph of the smallest order 5. Let \( P_n (n \geq 6) \) be a path of length \( n - 1 \). \( V(P_n) = \{v_1, v_2, v_3, v_4, \ldots, v_n\} \). By attaching one pendant edge to \( v_2 \), and attaching two pendant edges to \( v_i \), (\( 1 \leq i \leq n - 2 \)), there is a 2-Neighbourly Irregular graph of order \( 2n - 3 \).
Illustration: Figure 4.18 illustrates Theorem 4.7.3 for \( n = 4, 6, 8, 10 \).

\[
\begin{array}{cccccc}
2 & 1 & 2 & 1 & 2 & 1 \\
2 & 1 & 5 & 2 & 6 & 2 \\
2 & 3 & 3 & 2 & 3 & 3 \\
2 & 3 & 3 & 3 & 3 & 3 \\
2 & 1 & 5 & 2 & 6 & 2 \\
2 & 6 & 2 & 3 & 1 & 2 \\
2 & 3 & 3 & 3 & 3 & 3 \\
2 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

Figure 4.18

**Theorem 4.7.4.** For \( n \geq 2 \), the smallest order of 2-Neighbourly Irregular graph containing \( K_n \) as an induced subgraph is \( 2n - 1 \).

*Proof.* The graph \( KN_1 \), constructed in Theorem 4.4.8 is a 2-Neighbourly Irregular graph of order \( 2n - 1 \) containing \( K_n \) as an induced subgraph. It is noted that \( d_2(v_i) = n - i \), \( (1 \leq i \leq n) \) and \( d_2(u_i) = n + i - 2 \), \( (1 \leq i \leq n - 1) \). Hence no two adjacent vertices have the same number of vertices at a distance two from them. This graph contains at least one vertex of each possible \( d_2 \)-degree. \( \square \)

Now, \( d_m \)-pairlone graph is defined.

**Definition 4.7.5.** A graph \( G \) is called \( d_m \)-pairlone if it has exactly two vertices of the same \( d_m \)-degree, \( (m, \) a positive integer).

**Remark 4.7.6.** Ebrahim Salehi called the graph \( KN_1 \) as a pairlone graph[10]. The \( d_2 \) sequence of \( KN_1 \) is \( (0, 1, 2, 3, 4, \ldots, [(2n - 1)/2], [(2n - 1)/2] \ldots, (2n - 3)) \). The graph \( KN_1 \) has exactly two vertices of the same \( d_2 \)-degree, and so this graph is called \( d_2 \)-pairlone.

Illustration: Figure 4.19 illustrates Theorem 4.7.4 for \( n = 2, 3, 4, 5 \).
Remark 4.7.7. If one pendant edge is attached to the vertex which has $d_2 = 0$ (or maximum degree vertex) then 2-Neighbourly Irregular graph of even order is obtained. By attaching $n$-pendant edges to the vertex which has $d_2 = 0$ (or maximum degree), then 2-Neighbourly Irregular graph is got, and $d_2$ of each vertex is increased by $n$.

Theorem 4.7.8. There exists a 2-Neighbourly Irregular graph of order $4n - 1$, containing every graph of order $n \geq 2$ as an induced subgraph.

Proof. Let $G$ be any graph of order $n$. Let $V(G) = \{v_i : 1 \leq i \leq n\}$. The desired graph $N_2$ has the vertex set $V(N_2) = V(G) \cup V(T) \cup V(W)$, where $V(T) = \{t_i : 1 \leq i \leq n\}$ and $V(W) = \{w_i : 1 \leq i \leq 2n - 1\}$. Let $E(N_2) = E(G) \cup \{v_it_i : 1 \leq i \leq n\} \cup \{t_iw_j : 1 \leq i \leq n, 1 \leq j \leq n - 1 + i\} \cup \{v_jt_i : v_jv_i \notin E(G), 1 \leq i \leq n, i + 1 \leq j \leq n\}$. Then $d_2(v_i) \in N_2 = 2n - 2 + i$, $(1 \leq i \leq n), d_2(w_i) \in N_2 = 3n - 2, (1 \leq i \leq n)$, $d_2(w_{i+n-1}) \in N_2 = 3n - i - 1, (2 \leq i \leq n)$; $d_2(w_i) \in N_2 \neq d_2(t_i) \in N_2, (1 \leq i \leq n)$, and $d_2(w_{i+n-1}) \in N_2 \neq d_2(t_i) \in N_2, (1 \leq i \leq n)$. Hence the desired graph $N_2$ is the 2-Neighbourly Irregular graph of order $4n - 1$ containing every graph of order $n \geq 2$ as an induced subgraph. Number of edges in the graph $N_2$ is $2n^2$. □

Illustration: Figure 4.20 illustrates Theorem 4.7.8 for $n = 3$.

Remark 4.7.9. It is observed that 2-Neighbourly Irregular graph containing $K_2$ as an induced subgraph is $P_3$ and is of order 3, and 2-Neighbourly Irregular graph containing $K_{2,2}$ as an induced subgraph is of order 5. But in the above construction, 2-Neighbourly Irregular graph containing $K_2$ as an induced subgraph is of order 7, and 2-Neighbourly Irregular graph containing $K_{2,2}$ as an induced subgraph is of order 15. For every graph $G$, the minimum number of additional vertices needed to construct 2-Neighbourly Irregular graph containing $G$, are to be considered, and 2-Neighbourly Regular Strength of $G$ is to be defined.
Definition 4.7.10. Let $G$ be a graph with $n$ vertices. The $m$-Neighbourly Regular Strength ($m$-NRS) of $G$ is the minimum number $k$ denoting the additional vertices needed to construct a $m$-Neighbourly Irregular graph containing $G$ as an induced subgraph of the $m$-Neighbourly Irregular graph.

Definition 4.7.11. Let $G$ be a graph with $n$ vertices. The $2$-Neighbourly Regular Strength ($2$-NRS) of $G$ is the minimum number $k$ denoting the additional vertices needed to construct a $2$-Neighbourly Irregular graph containing $G$ as an induced subgraph of the $2$-Neighbourly Irregular graph ($m$, a positive integer).

Result 4.7.12. $2 - \text{NRS}(C_n) = n$, $2 - \text{NRS}(P_n) = n - 3$, $2 - \text{NRS}(K_{n,n}) = 1$, $2 - \text{NRS}(P_3) = 0$, $2 - \text{NRS}(2-\text{NIgraph}) = 0$, $2 - \text{NRS}(K_n) = (n - 1)$ and $2 - \text{NRS}(N_2) = 0$, where $N_2$ is the graph constructed in theorem 4.7.8.

4.8 Minimal Edge Covering

Minimum number of edges which cover all the vertices of a graph is called minimal edge covering number.

Theorem 4.8.1. The minimal edge covering number of $2$-Neighbourly Irregular graph $N_2$ of order $4n - 1$ containing a given graph of order $n \geq 1$ as an induced subgraph is $3n - 1$.

Proof. Let $w_1, w_2, w_3, w_4, \ldots, w_n, w_{n+1}, \ldots, w_{2n-2}, w_{2n-1}; t_1, t_2, t_3, \ldots, t_n$ and $v_1, v_2, v_3, v_4, \ldots, v_n$ are the vertices of $2$-Neighbourly Irregular graph $N_2$ of order $4n - 1$ (constructed in Theorem 4.7.8) containing any given graph of order $n \geq 1$ as an induced subgraph.

Let $E_1$ be the set of edges $t_1w_1, t_1w_2, t_1w_3, \ldots, t_1w_{n-1}, t_1w_n, t_2w_{n+1}, t_3w_{n+2}, \ldots, t_nw_{2n-1}$. This set $E_1$ covers all the vertices $w_1, w_2, w_3, w_4, \ldots, w_n, w_{n+1}, \ldots, w_{2n-2}, w_{2n-1}$ and $t_1, t_2, t_3, \ldots, t_n$. The remaining vertices $v_1, v_2, v_3, v_4, \ldots, v_n$ are covered by the edges $t_1v_1, t_2v_2, t_3v_3, \ldots, t_{n-1}v_{n-1}, t_nv_n$. Since the given graph is $K_n^c$ then $n$-edges are needed to cover all vertices of the given graph. Hence the minimal edge covering number of $2$-Neighbourly Irregular graph $N_2$ of order $4n - 1$ containing the given graph of order $n$ as an induced subgraph is $3n - 1$. \qed
4.9 Minimal Vertex Covering

Minimum number of vertices which cover all the edges of a graph is called *minimal vertex covering number*.

**Theorem 4.9.1.** The minimal vertex covering number of 2-Neighbourly Irregular graph $N_2$ of order $4n - 1$ containing the given graph of order $n \geq 1$ as an induced subgraph is $2n - 1$.

*Proof.* The vertices $t_1, t_2, t_3, \ldots, t_n$ cover all the edges in the sets $\{v_i t_i : 1 \leq i \leq n\} \cup \{t_i w_j : 1 \leq i \leq n, 1 \leq i \leq n - 1 + j\}$, and the remaining edges are covered by $n - 1$ vertices of $v_1, v_2, v_3, v_4, \ldots, v_n$. Since the given graph is $K_n$, then $n - 1$ vertices are needed to cover all edges of the given graph. Hence the minimal vertex covering number of 2-Neighbourly Irregular graph $N_2$ of order $4n - 1$ containing the given graph of order $n \geq 1$ as an induced subgraph is $2n - 1$. \hfill \square

4.10 $m$-Neighbourly Irregular Trees

For any positive integer $m$, $m$-Neighbourly Irregular tree and strict $m$-Neighbourly Irregular tree are defined in this section and a few properties possessed by 2-neighbourly irregular trees are also included.

**Definition 4.10.1.** A tree $T$ is said to be *$m$-neighbourly irregular tree* if no two adjacent vertices of $T$ have the same $d_m$-degree ($m$, a positive integer).

**Definition 4.10.2.** A tree $T$ is said to be *2-Neighbourly Irregular (2-NI) tree* if no two adjacent vertices of $T$ have the same number of vertices at a distance two away from them.

**Theorem 4.10.3.** Star $K_{1,n}(n \geq 2)$ is $m$-Neighbourly Irregular tree, for $m = 1, 2$.

**Theorem 4.10.4.** Subdivision of $K_{1,n}(n \geq 3)$ is $m$-Neighbourly Irregular tree, for $m = 1, 2$.

*Proof.* Let $K_{1,n}(n \geq 3)$ be a star. Let $u$ be the center vertex of degree $n$. Let $v_1, v_2, \ldots, v_n$ be vertices adjacent to $u$. Subdividing each edge of $K_{1,n}$ one time,
\( Sub(K_{1,n}) \) is got. New \( n \) vertices \( w_1, w_2, w_3, \ldots, w_n \) are got so that each \( w_i \) is adjacent to center vertex \( u \) and adjacent to \( v_i \), \( d_2(u) = n, d_2(v_i) = 1, (1 \leq i \leq n) \), and \( d_2(w_i) = n - 1, (1 \leq i \leq n) \). Hence \( Sub(K_{1,n}, (n \geq 3)) \) is 2-Neighbourly Irregular tree. The size of this graph is \( 2n \). \hfill \square

**Theorem 4.10.5.** If the middle edge of \( B_{n,n}(n \geq 2) \) is subdivided one time, then the resulting graph is \( m \)-Neighbourly Irregular graph, for \( m = 1, 2 \).

**Proof.** Let \( V(K_2) = \{v_1, v_2\} \). Let \( u_i(1 \leq i \leq n) \) be the vertices adjacent to \( v_1 \) and \( w_i(1 \leq i \leq n) \) be the vertices adjacent to \( v_2 \). Subdividing \( v_1v_2 \), a new vertex \( x \) which is adjacent to both \( v_1 \) and \( v_2 \) is got. Then \( d_2(v_1) = 1, d_2(v_2) = 1, d_2(u_i) = n, (1 \leq i \leq n) \) and \( d_2(w_i) = n, (1 \leq i \leq n) \), \( d_2(x) = 2n \). Hence 2-Neighbourly Irregular tree of order \( 2n + 3 \) is got. \hfill \square

**Theorem 4.10.6.** Subdivision of \( B_{n,n}(n \geq 2) \) is 2-Neighbourly Irregular tree of order \( 4n + 3 \).

**Proof.** Let \( V(K_2) = \{v_1, v_2\} \). Let \( u_i(1 \leq i \leq n) \) be the vertices adjacent to \( v_1 \) and \( w_i(1 \leq i \leq n) \), be the vertices adjacent to \( v_2 \). Subdividing each edge of \( B_{n,n} \) one time, \( Sub(B_{n,n}) \) is got. In \( Sub(B_{n,n}) \), for each edge \( v_1u_i(1 \leq i \leq n) \), \( n \) vertices \( x_1, x_2, \ldots, x_n \) are got so that each \( x_i \) is adjacent to \( v_1 \) and \( u_i \), for each edge \( v_2w_i(1 \leq i \leq n) \), \( n \) vertices \( y_1, y_2, \ldots, y_n \) are got so that each \( y_i \) is adjacent to \( v_2 \) and \( w_i \). For the edge \( v_1v_2 \), a new vertex \( x \) which is adjacent to both \( v_1 \) and \( v_2 \) is got. For \( 1 \leq i \leq n, d_2(u_i) = 1, d_2(x_i) = n, d_2(w_i) = 1, d_2(y_i) = n, d_2(v_1) = n + 1, d_2(v_2) = n + 1 \) and \( d_2(x) = 2n \). Hence \( Sub(B_{n,n}) \) is 2-Neighbourly Irregular tree of order \( 4n + 3 \). \hfill \square

**Theorem 4.10.7.** \( B_{n,m}, (n \neq m) \) tree is \( m \)-Neighbourly Irregular, for \( m = 1, 2 \).

**Proof.** Let \( V(K_2) = \{v_1, v_2\} \). Let \( v_i(1 \leq i \leq n) \) be the vertices adjacent to \( v_1 \) and non adjacent with \( v_2 \), and \( u_i(1 \leq i \leq m) \) be the vertices adjacent to \( v_2 \) and non adjacent with \( v_1 \). Then, \( d_2(v_i) = m, d_2(v_i) = n, (1 \leq i \leq n) \) and \( d_2(v_2) = n, d_2(u_i) = m, (1 \leq i \leq m) \). Hence \( B_{n,m} \) is 2-Neighbourly Irregular tree of order \( 2 + m + n \). \hfill \square

**Definition 4.10.8.** An \( m \)-Neighbourly Irregular tree \( T \) is called **strict \( m \)-Neighbourly Irregular tree** if removal of any pendant vertex in \( T \) results in a non \( m \)-Neighbourly Irregular tree \( (m, a \text{ positive integer}) \).
Example 4.10.9. The graphs given in Figure 4.21 are strict 2-Neighbourly Irregular trees.

![Figure 4.21](image)

4.11 Some results related to 1-Neighbourly Irregular graph and 2-Neighbourly Irregular graph

Some results connected with 1-Neighbourly Irregular graphs and 2-Neighbourly Irregular graphs are presented in this section[43].

Example 4.11.1. The graph $K_{n_1,n_2,...,n_m}$ where $n_i$'s are distinct is 1-Neighbourly Irregular and 2-Neighbourly Irregular. But in Figure 4.22, graphs (a) and (b) illustrate '2-Neighbourly Irregular graph that need not be 1-Neighbourly Irregular' and graph (c) illustrates '1-Neighbourly Irregular graph that need not be 2-Neighbourly Irregular". (vertices are labeled by their $d_2$-degrees).

![Figure 4.22](image)

Theorem 4.11.2. Graph $G$ is a 1-Neighbourly Irregular graph of diameter two if and only if $G$ is 2-Neighbourly Irregular graph of diameter two.
Proof. Let $G$ be 1-Neighbourly Irregular graph of diameter two if and only if $d(u) \neq d(v)$, for all $uv \in E(G)$ if and only if $n - 1 - d(u) \neq n - 1 - d(v)$, for all $uv \in E(G)$ if and only if $d_2(u) \neq d_2(v)$, for all $uv \in E(G)$ if and only if $G$ is 2-Neighbourly Irregular graph of diameter two.

Example 4.11.3. Flower graph obtained from a helm by joining each pendant vertex to the central vertex of the helm is not $m$-NI, for $m = 1, 2$ (vertices are labeled by their $d_2$-degrees).

Remark 4.11.4. List of graphs both 1-Neighbourly Irregular and 2-Neighbourly Irregular up to order six

1. If $m = 1, 2$ then the only one $m$-Neighbourly Irregular graph of order three is $P_3$.

2. If $m = 1, 2$ then the is only one $m$-Neighbourly Irregular graph of order four is $K_{1,3}$.

3. If $m = 1, 2$ then there are only four $m$-Neighbourly Irregular graphs of order five.

4. If $m = 1, 2$ then there are only ten $m$-Neighbourly Irregular graphs of order six.
4.12 Graph Products in 2-Neighbourly Irregular Graphs

**Definition 4.12.1.** A graph is 2-Neighbourly Irregular if no two adjacent vertices of $G$ have the same number of vertices at a distance two away from them [36].

**Definition 4.12.2.** The Cartesian product $G_1 \times G_1$ of two graphs $G_1$ and $G_2$ is the simple graph with $V_1 \times V_2$ vertex set, and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G_1 \times G_2$ if and only if either $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $G_2$ or $v_1 = v_2$ and $u_1$ is adjacent to $u_2$ in $G_1[6]$.

**Illustration:**

![Graph Illustration](image)

**Note 1:** Cartesian Product of 2-Neighbourly Irregular graphs need not be 2-Neighbourly Irregular graph.

**Illustration:** $P_3$ is 2-Neighbourly Irregular graph. But Cartesian product of $P_3 \times P_3$ is not 2-Neighbourly Irregular graph.
**Definition 4.12.3.** The *composition* of two simple graphs $G$ and $H$ is the simple graph $G[H]$ denoted with vertex set $V(G) \times V(H)$, in which $(u_1, v_1)$ is adjacent to $(u_2, v_2)$ if and only if either $u_1 u_2 \in E(G)$ (or) $u_1 = u_2$ and $v_1 v_2 \in E(H)$. Also, degree of $(u_1, v_1)$ in $G[H]$ is equal to $d(u_1)|V(H)| + d(v_1)$ [6].

**Note 2:** $P_3$ is 2-Neighbourly Irregular graph. The composition graph $P_3[P_3]$ is 2-Neighbourly Irregular graph.

**Illustration:**

![Figure 4.26](image)

**Note 3:** Composition of two 2-Neighbourly Irregular graphs need not be 2-Neighbourly Irregular graph.

**Illustration:**

![Figure 4.27](image)

$G$ is 2-Neighbourly Irregular graph. But composition graph $G[G]$ is not 2-Neighbourly Irregular graph.

**Definition 4.12.4.** The *join* of graphs $G$ and $H$ is the simple graph denoted by $G + H$ with vertex set $V(G) \cup V(H)$, in which each vertex of $G$ is adjacent to every vertex of $H$. The degree of a vertex $u$ in $G + H$ is equal to $d(u) + |V(H)|$, $u \in V(G)$ and the degree of a vertex $v$ in $G + H$ is equal to $d(v) + |V(G)|$, $v \in V(H)$ [6].
Note 4: Join of two 2-Neighbourly Irregular graph need not be 2-Neighbourly Irregular graph.

Illustration:

Conclusion and Scope: For further investigation, we state the following open problems.

1. The nature of the graph products namely cartesian, composition and join on \( m \)-Neighbourly Irregular graph for \( m \geq 3 \) may be investigated.

2. The nature of the graph products other than cartesian and composition and join on \( m \)-Neighbourly Irregular graphs for \( m \geq 2 \) may be investigated

3. \( m \)-Neighbourly Irregular graph for \( m \geq 3 \) containing given graph of order \( n \geq 2 \), as an induced subgraph may be constructed.