Chapter 3

$(r, 2, m(r - 1))$ - Regular Graphs

For any $n \geq 5$, $(n \neq 6, 8)$ and any $r > 1$, there exists an $(r, 2, r(r - 1))$-regular graph on $n \times 2^{r-2}$ vertices with girth five and the lower bound of $(r, 2, r(r - 1))$-regular graph is discussed in this chapter. It is proved that for any $r \geq 2$, there is an $(r, 2, (r-1)(r-1))$-regular graph on $4 \times 2^{r-2}$ vertices. Also, it is proved that for any $r \geq n$, there is an $(r, 2, (r-n)(r-1))$-regular graph on $(n+1) \times 2^{r-n}$ vertices. Also, the existence of $(r, 2, m(r - 1))$-regular graphs is proved.

3.1 Introduction

In this chapter only $(r, 2, k)$-regular graphs are studied. If $G$ is an $(r, m, k)$-regular graph, then $0 \leq k \leq r(r - 1)^{m-1}[29]$. This implies that if $G$ is an $(r, 2, k)$-regular graph. Then $0 \leq k \leq r(r - 1)[29].$

Example 3.1.1. Some $(r, 2, k)$-regular graphs, $(0 \leq k \leq r(r - 1))$.

(i) Let $r = 0, (k = 0)$.

\[ \bullet \]

$(0, 2, 0)$-regular

(ii) Let $r = 1, (k = 0)$.

\[ \bullet \]

$(1, 2, 0)$-regular.
(iii) Let \( r = 2 \), \((k \text{ lies between 0 and 2})\).

\[
\begin{align*}
(2,2,0)\text{-regular} & \quad (2,2,1)\text{-regular} & \quad (2,2,2)\text{-regular} \\
\end{align*}
\]

(iv) Let \( r = 3 \), \((k \text{ lies between 0 and 6})\).

\[
\begin{align*}
(3,2,0)\text{-regular} & \quad (3,2,2)\text{-regular} & \quad (3,2,3)\text{-regular} \\
(3,2,4)\text{-regular} & \quad (3,2,5)\text{-regular} & \quad (3,2,6)\text{-regular} \\
\end{align*}
\]

There is no \((3,2,1)\)-regular graph. A question arises now. "Is there any \((r,2,1)\)-regular graph for \( r \) odd?". The answer is "NO". For any odd \( r \geq 3 \), there need not exist \((r,2,1)\)-regular graph. For, let \( G \) be an odd regular graph. Suppose \( G \) is \((2,1)\)-regular graph, then \( G \) is of type \( P_4 \) or \( UP_2 \). Since \( G \) is a regular graph, \( G \) must be of type \( UP_2 \). Then \( G \) is a regular graph of even degree, which is a contradiction. So, \( G \) is not \((2,1)\)-regular. Hence there is no \((2,1)\)-regular graph for \( r = 3, 5, 7, 9, \ldots \).

A question arises now. "Is it possible to construct the \((r,2,k)\)-regular graphs for all values of \( k \) lies between 0 and \( r(r-1) \), for any \( r \)?"

With this motivation, the following graphs are constructed.

1. \((r,2, r(r-1))\)-regular graph for \( r \geq 2[33] \).
2. \((r,2, (r-1)(r-1))\)-regular graph \( r \geq 2[34] \).
3. \((r,2, (r-n)(r-1))\)-regular graph for any \( r \geq n[47] \).

Also, the existence of \((r,2, m(r-1))\)-regular graph is proved[42].
3.2 \((r, 2, r(r - 1))\)-Regular Graphs

**Definition 3.2.1.** A graph \(G\) is \((r, 2, r(r - 1))\)-regular if each vertex in the graph \(G\) is at a distance one away from exactly \(r\) number of vertices and each vertex in the graph \(G\) is at a distance two away from exactly \(r(r - 1)\) number of vertices.

**Example 3.2.2.** (i) Any cycle \(C_n\) \((n \geq 5)\) is \((2, 2, 2(2 - 1))\)-regular graph.

(ii) \((3, 2, 3(3 - 1))\)-regular graphs are shown in Figure 3.2.

![Figure 3.2](image)

3.3 \((r, 2, r(r - 1))\)-Regular Graphs Related with Girth

**Theorem 3.3.1.** For any \(r > 1\), a graph \(G\) is \((r, 2, r(r - 1))\)-regular if and only if \(G\) is \(r\)-regular with girth at least five.

**Proof.** For any \(r > 1\), let \(G\) be a \((r, 2, r(r - 1))\)-regular graph. Then \(d(v) = r\) and \(d_2(v) = r(r - 1)\), for all \(v \in V(G)\). Fix \(v \in V(G)\). Let \(N(v) = \{v_1, v_2, v_3, \ldots, v_r\}\). For \(i \neq j\), \(N(v_i) \cap N(v_j) = \{v\}\) and each \(N(v_i)\) contains \(r\) elements. Also, no two \(v_j\) are adjacent. Then \(G(N(v))\) has no edges. Therefore, \(G\) is an \(r\)-regular graph which does not contain triangles and four cycles and hence \(G\) is an \(r\)-regular graph having girth at least five. \(\Box\)

Now constructing \((r, 2, r(r - 1))\)-regular graph is equivalent to constructing \(r\)-regular graph with girth at least five. With this characterization, \((r, 2, r(r - 1))\)-regular graphs can be easily constructed.
3.4 Construction of \((r, 2, r(r - 1))\)-Regular graphs

Consider the Petersen graph \(P(n, 2)\) which has the vertex set \(\{x_i^{(1)}, x_i^{(2)} : 0 \leq i \leq n - 1\}\) and the edge set \(\{x_i^{(1)}x_{i+1}^{(2)}, x_i^{(2)}x_{i+1}^{(1)} : 0 \leq i \leq n - 1\}\), where subscripts are taken modulo \(n\).

**Theorem 3.4.1.** For any \(n \geq 5\), \((n \neq 6, 8)\) and any \(r > 1\), there exists an \((r, 2, r(r - 1))\)-regular graph on \(n \times 2^{r-2}\) vertices with girth five.

**Proof.** Let this result be proved by induction on \(r\). If \(r = 2\), then any \(n\)-cycle \((n \geq 5)\) is the required graph. Let \(G\) be a Petersen graph \(P(n, 2)\) with the vertex set \(V(G) = \{x_i^{(1)}, x_i^{(2)} : 0 \leq i \leq n - 1\}\) and edge set \(E(G) = \{x_i^{(1)}x_{i+1}^{(2)}, x_i^{(2)}x_{i+1}^{(1)} : 0 \leq i \leq n - 1\}\) (subscripts are taken modulo \(n\)). If \(n = 6\), then \(G\) is a graph with girth three, and if \(n = 8\), \(G\) is a graph with girth four as shown in Figure 3.3.

![Figure 3.3](image)

For \(n = 6, 8\) there is no graph with girth five. For any \(n \neq 6, 8\), \(0 \leq i \leq n - 1\),

\[
N_2(x_i^{(1)}) = \{x_{i+n-4}^{(1)}, x_{i+4}^{(1)}, x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+n-2}^{(2)}, x_{i+n-1}^{(2)}\}
\]

and \(d_2(x_i^{(1)}) = 6 = 3(3 - 1)\).

\[
N_2(x_i^{(2)}) = \{x_{i+2}^{(2)}, x_{i+n-2}^{(2)}, x_{i+1}^{(1)}, x_{i+2}^{(1)}, x_{i+n-2}^{(1)}, x_{i+n-1}^{(1)}\}
\]

and \(d_2(x_i^{(2)}) = 6 = 3(3 - 1)\). Hence \(G\) is \((3, 2, 3(3 - 1))\)-regular graph on \(n \times 2^{3-2}\) vertices with girth 5.

**Step 1** Take another copy of \(G\) as \(G'\). Let \(V(G') = \{x_i^{(3)}, x_i^{(4)} : 0 \leq i \leq n - 1\}\) and \(E(G') = \{x_i^{(3)}x_{i+1}^{(4)}, x_i^{(4)}x_{i+1}^{(3)} : 0 \leq i \leq n - 1\}\) (subscripts are taken modulo \(n\)). The desired graph \(G_1\) has the vertex set \(V(G_1) = V(G) \cup V(G')\) and edge set \(E(G_1) = E(G) \cup E(G') \cup \{x_i^{(1)}x_{i+1}^{(4)}, x_i^{(2)}x_i^{(3)} : 0 \leq i \leq n - 1\}\). Now the resulting
graph $G_1$ is 4-regular graph having $n \times 2^{4-2}$ vertices with girth five.

Now, consider the edges $x_i^{(1)} x_{i+1}^{(4)}$, $(0 \leq i \leq n - 1)$.

$N(x_i^{(1)}) = \{x_{i+2}^{(1)}, x_{i+n-2}^{(1)}, x_i^{(2)}\}$ in $G$ and $|N(x_i^{(1)})| = 3$ in $G$, $(0 \leq i \leq n - 1)$

$N(N(x_i^{(1)})) = \{x_{i+2}^{(4)}, x_{i+n-2}^{(4)}, x_i^{(3)}\}$ in $G'$ and $|N(N(x_i^{(1)}))| = 3$ in $G'$, $(0 \leq i \leq n - 1)$

$N(x_{i+1}^{(4)}) = \{x_{i+2}^{(4)}, x_{i+3}^{(3)}\}$ in $G'$ and $|N(x_{i+1}^{(4)})| = 3$ in $G'$, $(0 \leq i \leq n - 1)$

$N(N(x_{i+1}^{(4)})) = \{x_{i+n-1}^{(1)}, x_{i+1}^{(1)}, x_{i+1}^{(2)}\}$ in $G$ and $|N(N(x_{i+1}^{(4)}))| = 3$ in $G$, $(0 \leq i \leq n - 1)$.

$d_2$ of each vertex in $C^{(1)}$, where $C^{(1)}$ is the cycle induced by the vertices $\{x_i^{(1)}\}, (0 \leq i \leq n - 1)$ in $G_1$.

$d_2(x_i^{(1)})$ in $G_1 = d_2(x_{i+1}^{(1)})$ in $G + |N(x_{i+1}^{(4)})| \in G' + |N(N(x_{i+1}^{(4)}))| \in G' = 6 + 3 + 3 = 12 = 4(4 - 1), (0 \leq i \leq n - 1)$.

$d_2$ of each vertex in $C^{(4)}$, where $C^{(4)}$ is the cycle induced by the vertices $\{x_i^{(4)}\}, (0 \leq i \leq n - 1)$ in $G_1$.

$d_2(x_i^{(4)})$ in $G_1 = (d_2(x_{i+1}^{(4)}))$ in $G' + |N(x_i^{(1)}))| in G + |N(N(x_{i+1}^{(4)}))| in G = 6 + 3 + 3 = 12 = 4(4 - 1), (0 \leq i \leq n - 1)$.

Next, consider the edges $x_i^{(2)} x_i^{(3)}$, $(0 \leq i \leq n - 1)$.

$N(x_i^{(2)}) = \{x_{i+2}^{(2)}, x_{i+n-1}^{(2)}, x_i^{(1)}\}$ in $G$ and $|N(x_i^{(2)})) = 3$ in $G$, $(0 \leq i \leq n - 1)$

$N(N(x_i^{(2)})) = \{x_{i+2}^{(3)}, x_{i+n-1}^{(3)}, x_i^{(4)}\}$ in $G' and |N(N(x_i^{(2)})) = 3 in G', (0 \leq i \leq n - 1)$

$N(x_i^{(3)}) = \{x_{i+2}^{(3)}, x_{i+n-2}^{(3)}, x_i^{(4)}\}$ in $G'$ and $|N(x_i^{(3)})| = 3 in G', (0 \leq i \leq n - 1)$

$N(N(x_i^{(3)})) = \{x_{i+n-2}^{(2)}, x_{i+1}^{(2)}, x_i^{(1)}\}$ in $G$ and $|N(N(x_i^{(3)})) = 3 in G, (0 \leq i \leq n - 1)$.

$d_2$ of each vertex in $C^{(2)}$, where $C^{(2)}$ is the cycle induced by the vertices $\{x_i^{(2)}\}, (0 \leq i \leq n - 1)$ in $G_1$.

$d_2(x_i^{(2)})$ in $G_1 = d_2(x_{i+1}^{(2)})$ in $G + |N(x_i^{(3)})) in G' + |N(N(x_{i+1}^{(2)}))| in G' = 6 + 3 + 3 = 12 = 4(4 - 1), (0 \leq i \leq n - 1)$.

$d_2$ of each vertex in $C^{(3)}$, where $C^{(3)}$ is the cycle induced by the vertices $\{x_i^{(3)}\}, (0 \leq i \leq n - 1)$ in $G_1$.

$d_2(x_i^{(3)})$ in $G_1 = d_2(x_{i+1}^{(3)})$ in $G' + |N(x_i^{(2)})) in G + |N(N(x_{i+1}^{(3)})) in G = 6 + 3 + 3 = 12 = 4(4 - 1), (0 \leq i \leq n - 1)$. In $G_1$, for $(1 \leq t \leq 4), d_2(x_i^{(t)}) = 4(4 - 1), (0 \leq i \leq n - 1)$. Hence $G_1$ is $(4, 2, 4(4 - 1))$-regular on $n \times 2^{4-2}$ vertices with the vertex set $V(G_1) = \{x_i^{(t)} | (1 \leq t \leq 2^{4-2}), (0 \leq i \leq n - 1)\}$ and edge set $E(G_1) = E(G) \cup E(G') \cup \{x_i^{(1)} x_{i+1}^{(4)}, x_i^{(2)} x_i^{(3)} : 0 \leq i \leq n - 1\}$ of girth 5. Then the result is true for $r = 4$.

**Step 2:** Take another copy of $G_1$ as $G'_1$ with the vertex set $V(G'_1) = \{x_i^{(t)} :
\[ (2^{4-2} + 1 \leq t \leq 2^{4-1})0 \leq i \leq n - 1 \] and each \( x_i^{(t)} \) \((1 \leq t \leq 2^{4-2})\), corresponds to \( x_i^{(t)} \) \((1 \leq t \leq 2^{4-1})\), for \(0 \leq i \leq n - 1\). The desired graph \( G_2 \) has the vertex set \( V(G_2) = V(G_1) \cup V(G'_1) \) and edge set \( E(G_2) = E(G_1) \cup E(G'_1) \cup \{ x_i^{(1)}, x_{i+1}, x_i^{(8)}, x_{i+1}^{(7)}, x_i^{(6)}, x_{i+1}^{(6)}, x_i^{(5)} \} \). Hence the resulting graph \( G_2 \) is 5-regular graph having \( n \times 2^{5-2} \) vertices with girth 5.

consider the edges \( x_i^{(1)}, x_{i+1}^{(8)} \), \((0 \leq i \leq n - 1)\)

\[ N(x_i^{(1)}) = \{ x_{i+2}, x_{i+n-1}, x_i^{(2)}, x_i^{(4)} \} \text{ in } G_1 \text{ and } |N(x_i^{(1)})| = 4 \text{ in } G_1. \]

\[ N(N(x_i^{(1)})) = \{ x_{i+4}, x_{i+n-2}, x_i^{(8)}, x_i^{(7)}, x_i^{(5)} \} \text{ in } G'_1 \text{ and } |N(N(x_i^{(1)}))| = 4 \text{ in } G'_1. \]

\[ N(x_i^{(8)}) = \{ x_{i+1}, x_i^{(8)}, x_{i+1}^{(7)}, x_i^{(5)} \} \text{ in } G'_1 \text{ and } |N(x_i^{(8)})| = 4 \text{ in } G'_1. \]

\[ N(N(x_i^{(8)})) = \{ x_{i+1}, x_{i+n-1}, x_i^{(8)}, x_i^{(4)} \} \text{ in } G_1 \text{ and } |N(N(x_i^{(8)}))| = 4 \text{ in } G_1. \]

\(d_2 \) of each vertex in \( \text{C}(1)\), where \( \text{C}(1) \) is the cycle induced by the vertices \( \{x_i^{(1)}\} \), \((0 \leq i \leq n - 1)\) in \( G_2\).

\[ d_2(x_i^{(1)}) \text{ in } G_2 = d_2(x_i^{(1)}) \text{ in } G_1 + |N(x_i^{(8)})| \text{ in } G'_1 + |N(N(x_i^{(1)}))| \text{ in } G'_1 = 12 + 4 + 4 = 20 = 5(5 - 1), \text{ (0 \leq i \leq n - 1).} \]

\(d_2 \) of each vertex in \( \text{C}(8)\), where \( \text{C}(8) \) is the cycle induced by the vertices \( \{x_i^{(8)}\} \), \((0 \leq i \leq n - 1)\) in \( G_2\).

\[ d_2(x_i^{(8)}) \text{ in } G_2 = (d_2(x_i^{(8)}) \text{ in } G'_1 + |N(x_i^{(1)}))| \text{ in } G_1 + |(N(N(x_i^{(8)}))| \text{ in } G_1 = 12 + 4 + 4 = 20 = 5(5 - 1), \text{ (0 \leq i \leq n - 1).} \]

Next, consider the edges \( x_i^{(2)}, x_i^{(7)} \), \((0 \leq i \leq n - 1)\).

\[ N(x_i^{(2)}) = \{ x_{i+2}, x_{i+n-1}, x_i^{(2)}, x_i^{(3)} \} \text{ in } G_1 \text{ and } |N(x_i^{(2)})| = 4 \text{ in } G_1. \]

\[ N(N(x_i^{(2)})) = \{ x_{i+4}, x_{i+n-2}, x_i^{(7)}, x_i^{(6)} \} \text{ in } G'_1 \text{ and } |N(N(x_i^{(2)}))| = 4 \text{ in } G'_1. \]

\[ N(x_i^{(7)}) = \{ x_{i+2}, x_{i+n-2}, x_i^{(7)}, x_i^{(6)} \} \text{ in } G'_1 \text{ and } |N(x_i^{(7)})| = 4 \text{ in } G'_1. \]

\[ N(N(x_i^{(7)})) = \{ x_{i+2}, x_{i+n-1}, x_i^{(3)} \} \text{ in } G_1 \text{ and } |N(N(x_i^{(7)}))| = 4 \text{ in } G_1. \]

\(d_2 \) of each vertex in \( \text{C}(2)\), where \( \text{C}(2) \) is the cycle induced by the vertices \( \{x_i^{(2)}\} \), \((0 \leq i \leq n - 1)\) in \( G_2\).

\[ d_2(x_i^{(2)}) \text{ in } G_2 = d_2(x_i^{(2)}) \text{ in } G_1 + |N(x_i^{(7)})| \text{ in } G'_1 + |N(N(x_i^{(2)}))| \text{ in } G'_1 = 12 + 4 + 4 = 20 = 5(5 - 1), \text{ (0 \leq i \leq n - 1).} \]

\(d_2 \) of each vertex in \( \text{C}(7)\), where \( \text{C}(7) \) is the cycle induced by the vertices \( \{x_i^{(7)}\} \), \((0 \leq i \leq n - 1)\) in \( G_2\).

\[ d_2(x_i^{(7)}) \text{ in } G_2 = d_2(x_i^{(7)}) \text{ in } G'_1 + |N(x_i^{(2)}))| \text{ in } G_1 + |(N(N(x_i^{(7)}))| \text{ in } G_1 = 12 + 4 + 4 = 20 = 5(5 - 1), \text{ (0 \leq i \leq n - 1).} \]

Next, consider the edges \( x_i^{(3)}, x_i^{(6)} \), \((0 \leq i \leq n - 1)\).
\[ N(x_i^{(3)}) = \{x_{i+2}^{(3)}, x_{i+n-2}^{(3)}, x_i^{(4)}, x_i^{(2)}\} \text{ in } G_1 \text{ and } |N(x_i^{(3)})| = 4 \text{ in } G_1. \]

\[ N(N(x_i^{(3)})) = \{x_{i+n-2}^{(6)}, x_{i+n-1}^{(6)}, x_i^{(5)}, x_i^{(7)}\} \text{ in } G'_1 \text{ and } |N(N(x_i^{(3)}))| = 4 \text{ in } G'_1. \]

\[ N(x_{i+1}^{(6)}) = \{x_{i+2}^{(6)}, x_{i+1}^{(5)}, x_i^{(7)}\} \text{ in } G'_1 \text{ and } |N(x_{i+1}^{(6)})| = 4 \text{ in } G'_1. \]

\[ N(N(x_{i+1}^{(6)})) = \{x_{i+n-1}^{(3)}, x_{i+1}^{(3)}, x_{i+1}^{(4)}, x_i^{(2)}\} \text{ in } G_1 \text{ and } |N(N(x_{i+1}^{(6)}))| = 4 \text{ in } G_1. \]

\[ d_2 \text{ of each vertex in } C(3), \text{ where } C(3) \text{ is the cycle induced by the vertices } \{x_i^{(3)}\}, (0 \leq i \leq n-1) \text{ in } G_2. \]

\[ d_2(x_i^{(3)}) \text{ in } G_2 = d_2(x_i^{(3)}) \text{ in } G_1 + |N(x_i^{(6)})| \text{ in } G'_1 + |N(N(x_i^{(3)}))| \text{ in } G'_1. \text{ = } 12 + 4 + 4 = 20 = 5(5 - 1), (0 \leq i \leq n - 1). \]

\[ d_2 \text{ of each vertex in } C(6), \text{ where } C(6) \text{ is the cycle induced by the vertices } \{x_i^{(6)}\}, (0 \leq i \leq n - 1) \text{ in } G_2. \]

\[ d_2(x_{i+1}^{(6)}) \text{ in } G_2 = d_2(x_{i+1}^{(6)}) \text{ in } G'_1 + |N(x_i^{(3)})| \text{ in } G_1 + |N(N(x_{i+1}^{(6)}))| \text{ in } G_1 = 12 + 4 + 4 = 20 = 5(5 - 1), (0 \leq i \leq n - 1). \]

Next, consider the edges \[ x_i^{(4)}, x_i^{(5)}, (0 \leq i \leq n - 1). \]

\[ N(x_i^{(4)}) = \{x_i^{(4)}, x_{i+n-1}^{(4)}, x_i^{(3)}, x_i^{(1)}\} \text{ in } G_1 \text{ and } |N(x_i^{(4)})| = 4 \text{ in } G_1. \]

\[ N(N(x_i^{(4)})) = \{x_{i+1}^{(5)}, x_{i+n-1}^{(5)}, x_{i+1}^{(6)}, x_i^{(8)}\} \text{ in } G'_1 \text{ and } |N(N(x_i^{(4)}))| = 4 \text{ in } G'_1. \]

\[ N(x_i^{(5)}) = \{x_{i+2}^{(5)}, x_{i+n-2}^{(5)}, x_i^{(6)}, x_i^{(8)}\} \text{ in } G'_1 \text{ and } |N(x_i^{(5)})| = 4 \text{ in } G'_1. \]

\[ N(N(x_i^{(5)})) = \{x_{i+n-1}^{(4)}, x_{i+n-2}^{(4)}, x_i^{(3)}, x_i^{(1)}\} \text{ in } G_1 \text{ and } |N(N(x_i^{(5)}))| = 4 \text{ in } G_1. \]

\[ d_2 \text{ of each vertex in } C(4), \text{ where } C(4) \text{ is the cycle induced by the vertices } \{x_i^{(4)}\}, (0 \leq i \leq n - 1) \text{ in } G_2. \]

\[ d_2(x_i^{(4)}) \text{ in } G_2 = d_2(x_i^{(4)}) \text{ in } G_1 + |N(x_i^{(5)})| \text{ in } G'_1 + |N(N(x_i^{(4)}))| \text{ in } G'_1 = 12 + 4 + 4 = 20 = 5(5 - 1), (0 \leq i \leq n - 1). \]

\[ d_2 \text{ of each vertex in } C(5), \text{ where } C(5) \text{ is the cycle induced by the vertices } \{x_i^{(5)}\}, (0 \leq i \leq n - 1) \text{ in } G_2. \]

\[ d_2(x_i^{(5)}) \text{ in } G_2 = d_2(x_i^{(5)}) \text{ in } G'_1 + |N(x_i^{(4)})| \text{ in } G_1 + |N(N(x_i^{(5)}))| \text{ in } G_1 = 12 + 4 + 4 = 20 = 5(5 - 1), \text{ for } (0 \leq i \leq n - 1). \]

In \( G_2 \), for \( 1 \leq t \leq 8 \), \( d_2(x_i^{(t)}) = 5(5 - 1), (0 \leq i \leq n - 1). \)

Hence \( G_2 \) is \( (5, 2, 5(5 - 1)) \)-regular graph on \( n \times 2^{5-2} \) vertices with the vertex set \( V(G_2) = \{x_i^{(t)} : 1 \leq t \leq 2^{5-2}, 0 \leq i \leq n - 1\} \) and edge set \( E(G_2) = E(G_1) \cup E(G'_1) \cup \{x_i^{(1)}, x_{i+1}^{(8)}, x_i^{(2)}, x_i^{(7)}, x_i^{(3)}, x_{i+1}^{(6)}, x_i^{(4)}, x_i^{(5)} : 0 \leq i \leq n - 1\} \) is of girth five. Then the result is true for \( r = 5 \).

Let us assume that this result is true for \( r = m+3 \). Then, there exists \( (m+3, 2, (m+3)(m+2)) \)-regular graph on \( n \times 2^{m+1} \) vertices with the vertex set \( V(G_m) = \{x_i^{(t)} : 1 \leq t \leq 2^{m+1}\} \).
$t \leq 2^{m+1}, 0 \leq i \leq n-1 \}$ and edge set $E(G_m) = E(G_{m-1}) \cup E(G'_{m-1}) \bigcup_{t=1}^{2m} \{x_i^{(t)}x_{i+t(mod2)}^{2m+1-t+1}: 0 \leq i \leq n-1 \}$ of girth 5. Then $d_2(x_i^{(t)}) = (m+3)(m+2)$ and $d(x_i^{(t)}) = m+3$, $(1 \leq t \leq 2^{m+1}), (0 \leq i \leq n-1)$. Take another copy of $G_m$ as $G'_m$ with the vertex set $V(G'_m) = \{x_i^{(t)}: 2^{m+1}+1 \leq t \leq 2^{m+2}, 0 \leq i \leq n-1 \}$ and each $x_i^{(t)}, (2^{m+1}+1 \leq t \leq 2^{m+2})$, corresponds to $x_i^{(t)}, (1 \leq t \leq 2^{m+1}), (0 \leq i \leq n-1)$. The desired graph $G_{m+1}$ has the vertex set $V(G_{m+1}) = V(G_m) \cup V(G'_m)$ and edge set $E(G_{m+1}) = E(G_m) \cup E(G'_m) \bigcup_{t=1}^{2^{m+1}} \{x_i^{(t)}x_{i+t(mod2)}^{2m+2-t+1}: 0 \leq i \leq n-1 \}$. Now the resulting graph $G_{m+1}$ is $(m+4)$-regular graph having $n \times 2^{m+2}$ vertices with girth 5.

**Consider the edges** $\bigcup_{t=1}^{2^{m+1}} \{x_i^{(t)}x_{i+t(mod2)}^{2m+2-t+1}: 0 \leq i \leq n-1 \}$.

$d_2$ of each vertex in $C^{(t)}(1 \leq t \leq 2^{m+1})$, where $C^{(t)}$ is the cycle induced by the vertices $\{x_i^{(t)}: (0 \leq i \leq n-1) \}$ in $G_{m+1}$.

$d_2(x_i^{(t)})$ in $G_{m+1} = d_2(x_i^{(t)})$ in $G_m + |N(x_i^{(t)}x_{i+t(mod2)}^{2m+2-t+1})| \in G'_m + |N(N(x_i^{(t)}))| \in G'_m = (m+3)(m+2) + (m+3) + (m+3), (0 \leq i \leq n-1) = (m+3)(m+2+1+1) = (m+3)(m+4)$.

$d_2$ of each vertex in $C^{(2^{m+2}-t+1)}(2^{m+1}+1 \leq t \leq 2^{m+2})$, where $C^{(2^{m+2}-t+1)}$ is the cycle induced by the vertices $\{x_i^{(2^{m+2}-t+1)}: (0 \leq i \leq n-1) \}$ in $G_{m+1}$.

$d_2(x_i^{2m+2-t+1})$ in $G_{m+1} = d_2(x_i^{2m+2-t+1})$ in $G'_m + |N(x_i^{(t)})| \in G'_m + |N(N(x_i^{2m+2-t+1}))| \in G'_m = (m+3)(m+2) + (m+3) + (m+3), (1 \leq i \leq n-1) = (m+3)(m+4)$.

In $G_{m+1}, (1 \leq t \leq 2^{m+2}), d_2(x_i^{(t)}) = (m+3)(m+4), (0 \leq i \leq n-1)$. Then, there exists $(m+4, 2, (m+3)(m+4))$-regular graph on $n \times 2^{m+2}$ vertices with the vertex set $V(G_m) = \{x_i^{(t)}: 1 \leq t \leq 2^{m+2}, 0 \leq i \leq n-1 \}$ and edge set $E(G_{m+1}) = E(G_m) \cup E(G'_m) \bigcup_{t=1}^{2^{m+1}} \{x_i^{(t)}x_{i+t(mod2)}^{2m+2-t+1}: 0 \leq i \leq n-1 \}$ of girth 5. Then $(1 \leq t \leq 2^{m+2}), d_2(x_i^{(t)}) = (m+3)(m+4), (0 \leq i \leq n-1)$ and $d(x_i^{(t)}) = m+4$. If the result is true for $r = m+3$, then it is true for $r = m+4$. Hence the result is true for all $r > 2$.

Then, for any $n \geq 5 (n \neq 6, 8)$ and for any $r > 1$, there is an $(r, 2, r(r-1))$-regular graph on $n \times 2^{r-2}$ vertices with girth 5. \[\square\]
Corollary 3.4.2. For any \( r > 1 \), there is an \((r, 2, r(r - 1))\)-regular graph on \( 5 \times 2^{r - 2} \) vertices.

Example 3.4.3. Graphs given in Figure 3.4 illustrate the Corollary 3.4.2, for \( r = 2, 3, 4 \).

Figure 3.4

Corollary 3.4.4. For any \( r > 1 \), there is an \((r, 2, r(r - 1))\)-regular graph on \( 7 \times 2^{r - 2} \) vertices with girth 5.

Corollary 3.4.5. For any \( r > 1 \), there is an \((r, 2, r(r - 1))\)-regular graph on \( 9 \times 2^{r - 2} \) vertices with girth 5.

Corollary 3.4.6. For any \( r > 1 \), there is an \((r, 2, r(r - 1))\)-regular graph on \( 10 \times 2^{r - 2} \) vertices with girth 5.

It is observed that if the value of \( n \) increases, then order of graph increases.

3.5 Construction of \((r, 2, r(r - 1))\)-Regular Graphs of Girth Six

The Petersen graph \( P(n, 3) \) has vertex set \( \{ x_{1}^{(1)}, x_{i}^{(2)} : 0 \leq i \leq n - 1 \} \) and edge set \( \{ x_{i}^{(2)} x_{i+1}^{(2)} : 0 \leq i \leq n - 1 \} \cup \{ x_{i}^{(1)} x_{i}^{(2)} : 0 \leq i \leq n - 1 \} \cup \{ x_{i}^{(1)} x_{i+3}^{(1)} : 0 \leq i \leq n - 1 \} \) (where subscripts are taken modulo \( n \)).

Theorem 3.5.1. For any \( n \geq 7, n \neq 9, 12, 15 \) and \( r > 1 \), there is an \((r, 2, r(r - 1))\)-regular graph on \( n \times 2^{r - 2} \) vertices with girth six.
Proof. Take $P(n, 3)$ instead of taking $P(n, 2)$ in Theorem 3.4.1. The result follows.

\[\square\]

**Remark 3.5.2.** If $n = 15$, then there is an $(r, 2, r(r-1))$-regular graph of order $n \times 2^{r-2}$ with girth 5. If $n = 9, 12$, then there are graphs with girth 3, 4 respectively as shown in Figure 3.5.

![Figure 3.5](image)

**3.6 Construction of $(r, 2, r(r-1))$-Regular Graphs of Girth Seven**

Consider the Petersen graph $P(n, 4)$, which has the vertex set $\{x_{i}^{(1)}, x_{i}^{(2)} : 0 \leq i \leq n-1\}$ and edge set $\{x_{i}^{(2)} x_{i+1}^{(2)} : 0 \leq i \leq n-1\} \cup \{x_{i}^{(1)} x_{i}^{(2)} : 0 \leq i \leq n-1\} \cup \{x_{i}^{(1)} x_{i+4}^{(1)} : 0 \leq i \leq n-1\}$ (where subscripts are taken modulo $n$).

**Theorem 3.6.1.** For any $n \geq 9$, $n \neq 12, 16, 20, 24$ and any $r > 1$, there is an $(r, 2, r(r-1))$-regular graph on $n \times 2^{r-2}$ vertices with girth seven.

Proof. Take $P(n, 4)$ instead of taking $P(n, 2)$ in Theorem 3.4.1. The result follows.

\[\square\]

**Remark 3.6.2.** If $n = 20, 24$, then there is an $(r, 2, r(r-1))$-regular graph of order $n \times 2^{r-2}$ with girth 5, 6 respectively. If $n = 12, 16$ then there are graphs with girth 3, 4 respectively as shown in Figure 3.6.
Consider the generalized Petersen graph $P(n, k)$, for $n \geq 5$ and $(2 \leq k < n/2)$, having vertex set $V(G) = \{x_i^{(1)}, x_i^{(2)} : 0 \leq i \leq n-1\}$ and edge set $\{x_i^{(2)} x_{i+1}^{(2)} : 0 \leq i \leq n-1\} \cup \{x_i^{(1)} x_{i+k}^{(1)} : 0 \leq i \leq n-1\}$, where subscripts are taken modulo $n$.

**Theorem 3.6.3.** For any $n \geq 5$ and $(2 \leq k < n/2)$, $n \not= (2 + t)k$, $1 \leq t \leq 2$ and for any $r \geq 2$, there is an $(r, 2, r(r-1))$-regular graph on $n \times 2^{r-2}$ vertices with girth $k + 3$. In particular, when $n = (2 + t)k$, $(2 < t \leq k)$ one can get an $(r, 2, r(r-1))$-regular graph of order $n \times 2^{r-2}$ with girth $(2 + t)$.

**Proof.** The generalized Petersen graph $P(n, k)$ is taken, when $n \geq 5$ and $(1 \leq k < n/2)$ instead of taking $P(n, 2)$ in Theorem 3.4.1.

**Remark 3.6.4.** If $n = (2 + t)k$, $t = 1, 2$ then one can have graphs with girth 3, 4 respectively. Then, there is no $(r, 2, r(r-1))$-regular graph for $n = (2 + t)k$, for $t = 1, 2$ by this construction.

### 3.7 Lower Bound of $(r, 2, r(r-1))$-Regular Graphs

If $G$ is observed as $(r, 2, r(r-1))$-regular graph, then $|V(G)| \geq 1 + r(r-1) + r = r^2 + 1$. The order of $(r, 2, r(r-1))$-regular graph is at least $r^2 + 1$. Let us look at $(r, 2, r(r-1))$-regular graph of order precisely $r^2 + 1[41]$.

**Theorem 3.7.1.** A graph $G$ is an $(r, 2, r(r-1))$-regular graph of order $r^2 + 1$ if and only if $diam(G) = 2$.

**Proof.** Let $G$ is an $(r, 2, r(r-1))$-regular graph of order $r^2 + 1$ if and only if Order of $(G) = r + r(r-1) + 1$. Therefore, by theorem 2.4.5 $G$ is an $(r, 2, r(r-1))$-regular if and only if $diam(G) = 2$. \qed
3.8 \((r, 2, r(r - 1))\)-Regular Graphs of order \(r^2 + 1\)

In this section, three \((r, 2, r(r - 1))\)-regular graphs with exactly \(r^2 + 1\) vertices are listed.

**List of \((r, 2, r(r - 1))\)-regular graphs of order \(r^2 + 1\):**

(i) For \(r = 2\), Pentagon \(C_5\) is a unique graph with diameter 2 and girth 5 of order 5.

(ii) For \(r = 3\), Petersen graph is a unique graph with diameter 2 and girth 5 of order 10.

(iii) For \(r = 7\), Hoffman’s singleton graph is a unique graph with diameter 2 and girth 5 of order 50. It is a \((7, 2, 7(7 - 1))\)-regular graph of order \(7^2 + 1\).

**Remark 3.8.1.** Hoffman’s singleton [29] proved that there exists a regular graph of degree \(r\) with \(r^2 + 1\) vertices and diameter two only for \(r = 2, 3, 7\) and possibly 57. The existence of a graph with \(r = 57\) has not yet been established.

3.9 Construction of Hoffman Singleton Graph due to Neil Robertson

**Construction of Hoffman Singleton Graph:** Take five pentagons \(P_h, (0 \leq h \leq 4)\) and five pentagrams \(Q_i, (0 \leq i \leq 4)\) so that vertex \(j\) of \(P_h\) is adjacent to vertices \(j - 1, j + 1\) of \(P_h\) and vertex \(j\) of \(Q_i\) is adjacent to vertices \(j - 2, j + 2\) of \(Q_i\). Now, join vertex \(j\) of \(P_h\) to vertex \(hi + j\) of \(Q_i\) (all indices \mod 5). The resulting graph is Hoffman singleton graph[11].

Using the method of construction of Hoffman Singleton Graph by Neil Robertson[26], \(r\)-regular graphs of girth 5 can be constructed for \(r = 3, 4, 5, 6\).

For \(r = 3, 4, 5, 6\), take \((r - 2)\) pentagons \(P_h, (0 \leq h \leq r - 3)\) and \((r - 2)\) pentagrams \(Q_i, (0 \leq i \leq r - 3)\) so that vertex \(j\) of \(P_h\) is adjacent to vertices \(j - 1, j + 1\) of \(P_h\) and vertex \(j\) of \(Q_i\) is adjacent to vertices \(j - 2, j + 2\) of \(Q_i\). Now, join vertex \(j\) of \(P_h\) to vertex \(hi + j\) of \(Q_i\) (all indices \mod 5).
1. If \( r = 3 \), then Petersen graph \((a)\) of Figure (3.7) is obtained which is \((3, 2, 3(3 - 1))\)-regular graph on 10 vertices with girth 5.

2. If \( r = 4 \), then \((b)\) of Figure (3.7) is obtained which is \((4, 2, 4(4 - 1))\) regular graph on 20 vertices with girth 5. Figure 3.7. illustrates Construction of Hoffman Singleton Graph for \( n = 5 \) and \( r = 3, 4 \).

3. If \( r = 5 \), then there is a \((5, 2, 5(5 - 1))\)- regular graph on 30 vertices with girth 5.

4. If \( r = 6 \), then there is a \((6, 2, 6(6 - 1))\)- regular graph on 40 vertices with girth 5.

**Remark 3.9.1.** For \( r = 8 \), (construction due to Neil Robertson of Hoffman Singleton graph [29, 26], take six pentagons \( P_h \) and six pentagrams \( Q_i \) so that vertex \( j \) of \( P_h \) is adjacent to vertices \( j - 1, j + 1 \) of \( P_h \) and vertex \( j \) of \( Q_i \) is adjacent to vertices \( j - 2, j + 2 \) of \( Q_i \). Now, join vertex \( j \) of \( P_h \) to vertex \( hi + j \) of \( Q_i \) (all indices mod 5). Joining the vertex 0 of \( P_0 \) to vertex 0 of \( Q_6 \) and joining the vertex 0 of \( P_6 \) to 0 of \( Q_0 \), a graph with four cycle is obtained. Hence there is no \((8, 2, 8(8 - 1))\)-regular graph using this construction.

If \( n = 5 \), degree \( r \) varies from 3 to 7 only. For any \( r(3 \leq r \leq 7) \), the vertex labeled \( j 0 \leq j \leq 4 \) residing in a pentagon labeled \( P_h \), \( (0 \leq h \leq r - 2) \) and pentagrams labeled \( Q_i \), \( (0 \leq i \leq r - 2) \) so that vertex \( j \) of \( P_h \) is adjacent to vertices
j − 1, j + 1 of \( P_h \) and vertex \( j \) of \( Q_i \) is adjacent to vertices \( j − 2, j + 2 \) of \( Q_i \). Now, join vertex \( j \) of \( P_h \) to vertex \( hi + j \) of \( Q_i \) (all indices mod 5).

For any \( r(3 \leq r \leq 7) \), \((r, 2, r(r − 1))\)-regular graph having 10\((r − 2)\) vertices and 5\(nr(r − 2)\) edges with girth 5 is obtained.

**Theorem 3.9.2.** If \( n \geq 5(n \neq 6, 8) \) is an integer such that \( 3 \leq r \leq n + 2 \), then there is an \((r, 2, r(r − 1))\)-regular graph of order \( 2nr(r − 2) \) and \( nr(r − 2) \) edges.

*Proof.* By the construction in 3.9, there is a graph with girth 3 if \( n = 6 \) and a graph with girth 4 if \( n = 8 \). Let \( n \geq 5 \), \((n \neq 6, 8)\) for \( 3 \leq r \leq n + 2 \), take the vertex labeled \( i \), \((0 \leq i \leq n − 1)\) residing in a labeled \( P_h \), \((0 \leq h \leq r − 2)\) and labeled \( Q_i \), \((0 \leq i \leq r − 2)\) so that vertex \( j \) of \( P_h \) is adjacent to vertices \( j − 1, j + 1 \) of \( P_h \) and vertex \( j \) of \( Q_i \) is adjacent to vertices \( j − 2, j + 2 \) of \( Q_i \). Now, join the vertex \( j \) of \( P_h \) to vertex \( hi + j \) of \( Q_i \) (all indices mod \( n \)). This is an \( r \)-regular graph with \( 2nr(r − 2) \) vertices and \( nr(r − 2) \) edges with girth 5. \(\square\)

**Theorem 3.9.3.** If \( n \geq 7 \), \((n \neq 9, 12, 15)\) is an integer such that \( 3 \leq r \leq n + 2 \), then there is an \((r, 2, r(r − 1))\)-regular graph of order \( 2nr(r − 2) \) vertices and \( nr(r − 2) \) edges with girth 6.

*Proof.* By the construction in 3.9, there is a graph of girth 3 if \( n = 9 \) and there is a graph of girth 4 if \( n = 12 \). If \( n = 15 \), then there is an \((r, 2, r(r − 1))\)-regular graph of order \( 30(r − 2) \) and \( 15r(r − 2) \) edges with girth 5.

Let \( n \geq 7 \), \((n \neq 9, 12, 15)\) be an integer. For any \( r \), \(3 \leq r \leq n + 2\), take the vertex labeled \( i \) \((0 \leq i \leq n − 1)\) residing in a labeled \( P_h \) \((0 \leq h \leq r − 2)\) and labeled \( Q_i \), \((0 \leq i \leq r − 2)\) so that vertex \( j \) of \( P_h \) is adjacent to vertices \( j − 1, j + 1 \) of \( P_h \) and vertex \( j \) of \( Q_i \) is adjacent to vertices \( j − 3, j + 3 \) of \( Q_i \). Now join vertex \( j \) of \( P_h \) to vertex \( hi + j \) of \( Q_i \) (all indices mod \( n \)). The resulting graph is an \((r, 2, r(r − 1))\)-regular of order \( 2nr(r − 2) \) and \( nr(r − 2) \) edges with girth 6. \(\square\)

**Theorem 3.9.4.** If \( n \geq 9 \) \((n \neq 12, 16, 20, 24)\) is an integer such that \( 3 \leq r \leq n + 2 \), then there is an \((r, 2, r(r − 1))\)-regular graph of order \( 2nr(r − 2) \) and \( nr(r − 2) \) edges with girth 7.

*Proof.* If \( n = 12 \), then by construction in 3.9, there is a graph with girth 3 and \( n = 16 \) there is a graph with girth 4. If \( n = 20, 24 \), there is an \((r, 2, r(r − 1))\)-regular
graph of order $2n(r - 2)$ and $nr(r - 2)$ edges with girth 5 and girth 6 respectively. Let $n \geq 9$, $(n \neq 12, 16, 20, 24)$ be an integer. For $3 \leq r \leq n + 2$, take the vertex labeled $i$ $(0 \leq i \leq n - 1)$ residing in a labeled $P_h$, $(0 \leq h \leq r - 2)$ and labeled $Q_i$, $(0 \leq i \leq r - 2)$ so that vertex $j$ of $P_h$ is adjacent to vertices $j - 1, j + 1$ of $P_h$ and vertex $j$ of $Q_i$ is adjacent to vertices $j - 4, j + 4$ of $Q_i$. Now, join vertex $j$ of $P_h$ to vertex $hi + j$ of $Q_i$ (all indices mod $n$). The resulting graph is an $(r, 2, m(r - 1))$-regular graph of order $2n(r - 2)$ and $nr(r - 2)$ edges with girth $7$. If $n \geq 9 (n \neq 12, 16, 20, 24)$ is an integer such that $3 \leq r \leq n + 2$, then there is an $(r, 2, m(r - 1))$-regular graph of order $2n(r - 2)$ and $nr(r - 2)$ edges with girth $7$.

From Theorem 3.9.2, Theorem 3.9.3 and Theorem 3.9.4, the following generalization can be formed.

**Theorem 3.9.5.** If $n \geq 5$ and $(2 \leq k < n/2)$ and $n \neq (2 + t)k, 1 \leq t \leq 2$, then for any $r \geq 2$, there is an $(r, 2, m(r - 1))$-regular graph on $2n(r - 2)$ vertices and $nr(r - 2)$ edges with girth $k + 3$. In particular, if $n = (2 + t)k, (3 \leq t \leq k)$, there is an $(r, 2, m(r - 1))$-regular graph of order $2n(r - 2)$ and $nr(r - 2)$ edges with girth $(2 + t)$.

**Remark 3.9.6.** If $n = (2 + t)k, (t = 1, 2)$, then there is a graph with girth 3, 4 respectively. Hence there is no $(r, 2, m(r - 1))$-regular graph for $n = (2 + t)k$, for $t = 1, 2$.

### 3.10 $(r, 2, (r - 1)(r - 1))$-Regular Graphs

A method is given in this section to construct an $(r, 2, (r - 1)(r - 1))$-regular graph with $4 \times 2^{r-2}$ vertices, for any $r > 1$[34].

**Definition 3.10.1.** A graph $G$ is $(r, 2, (r - 1)(r - 1))$-regular if every vertex has degree $r$ and every vertex in the graph $G$ is at a distance two from exactly $(r - 1)(r - 1)$ number of vertices.

**Example 3.10.2.**
1. $C_4$ is a $(2, 2, (2 - 1)(2 - 1))$-regular graph.
2. Graphs shown in Figure 3.8 are $(3, 2, (3 - 1)(3 - 1))$-regular graph.
Theorem 3.10.3. For $r > 1$, if $G$ is an $(r, 2, (r - 1)(r - 1))$-regular graph, then $G$ has girth four.

Proof. Let $G$ be an $(r, 2, (r - 1)(r - 1))$-regular graph and $v$ be any vertex of $G$. Let $N(v) = \{v_1, v_2, v_3, \ldots, v_r\}$. Then $\deg(v) = r$ and $\deg(v) = (r - 1)(r - 1)$, for all $v$ in $G$. Then, at least one vertex of $N_2(v)$ is adjacent with more than one vertex of $N(v)$ and $G(N(v))$ has no edges. Hence $G$ does not contain triangles, and $G$ contains four cycles so that $G$ has girth four.

Theorem 3.10.4. For any $r \geq 1$, there is an $(r, 2, (r - 1)(r - 1))$-regular graph on $4 \times 2^{r-2}$ vertices.

Proof. If $r = 1$, then $K_2$ is the required graph. If $r = 2$, then $C_4$ is the required graph. This result can be proved by induction on $r$. Let $G$ be a graph with vertex set $V(G) = \{x_1^{(1)}, x_1^{(2)} : (0 \leq i \leq 3)\}$ and edge set $E(G) = \{x_0^{(1)}x_2^{(1)}, x_0^{(1)}x_3^{(1)}, x_1^{(1)}x_2^{(1)}, x_1^{(2)}x_3^{(2)}\} \cup \{x_i^{(1)}x_i^{(2)}, x_i^{(2)}x_{i+1} : 0 \leq i \leq 3\}$ (subscripts are taken modulo 4). Figure 3.9 represents 3-regular graph $G$.

\[ N_2(x_1^{(1)}) = \{x_1^{(1)}, x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\} \text{ and } \deg(x_1^{(1)}) = 4 = (3 - 1)(3 - 1). \]

\[ N_2(x_1^{(2)}) = \{x_2^{(1)}, x_1^{(1)}, x_3^{(1)}, x_4^{(1)}\} \text{ and } \deg(x_1^{(2)}) = 4 = (3 - 1)(3 - 1). \]

Hence $G$ is a $(3, 2, (3 - 1)(3 - 1))$-regular graph on $4 \times 2^{3-2} = 8$ vertices.

Step 1 Take another copy of $G$ as $G'$. $V(G') = \{x_i^{(3)}, x_i^{(4)} : 0 \leq i \leq 3\}$ and $E(G') = \{x_0^{(3)}x_2^{(3)}, x_0^{(3)}x_3^{(3)}, x_1^{(3)}x_2^{(3)}, x_1^{(3)}x_3^{(3)}\} \cup \{x_i^{(3)}x_i^{(4)}, x_i^{(4)}x_{i+1} : 0 \leq i \leq 3\}$. (subscripts are
taken modulo 4). The desired graph $G_1$ has the vertex set $V(G_1) = V(G) \cup V(G')$ and edge set $E(G_1) = E(G) \cup E(G') \cup \{x_i^{(1)}, x_{i+1}^{(1)}, x_i^{(2)}, x_{i+1}^{(2)} : 0 \leq i \leq 3\}$ (subscripts are taken modulo 4). Now, the resulting graph $G_1$ is a regular graph having $4 \times 2^{4-2} = 16$ vertices. Figure 3.10 represents the graph $G_1$.

Next, consider the edges $x_i^{(1)}, x_{i+1}^{(1)}, (0 \leq i \leq 3)$.

$$N(x_i^{(1)}) = \{x_{i+3}^{(1)}, x_i^{(2)}\} \text{ in } G \text{ and } |N(x_i^{(1)})| = 3 \text{ in } G, \ (0 \leq i \leq 3).$$

$$N(N(x_i^{(1)})) = \{x_i^{(4)}, x_{i+3}^{(4)}, x_i^{(3)}\} \text{ in } G' \text{ and } |N(x_i^{(1)})| = 3 \text{ in } G', \ (0 \leq i \leq 3).$$

$$N(x_{i+1}^{(4)}) = \{x_{i+1+3}^{(4)}, x_{i+1}^{(4)}, x_{i+1}^{(3)}\} \text{ in } G' \text{ and } |N(x_{i+1}^{(4)})| = 3 \text{ in } G', \ (0 \leq i \leq 3).$$

$$N(N(x_{i+1}^{(4)})) = \{x_{i+1+3}^{(4)}, x_i^{(4)}, x_i^{(2)}\} \text{ in } G \text{ and } |N(N(x_{i+1}^{(4)}))| = 3 \text{ in } G, \ (0 \leq i \leq 3)$$

To find $d_2$ of each vertex in $C^{(1)}$, where $C^{(1)}$ is the cycle induced by the vertices $\{x_i^{(1)} : 0 \leq i \leq 3\}$ in $G_1$.

$$N_2(x_i^{(1)}) \text{ in } G_1 = N_2(x_i^{(1)}) \text{ in } G \cup N(x_{i+1}^{(4)}) \text{ in } G' \cup N(N(x_i^{(1)})) \text{ in } G'$$

$$= \{x_{i+1}^{(1)}, x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}\} \text{ in } G \cup \{x_i^{(4)}, x_{i+2}^{(4)}, x_{i+1}^{(3)}\} \text{ in } G'$$

$$\cup \{x_{i+3}^{(4)}, x_{i+1}^{(4)}, x_i^{(3)}\} \text{ in } G'$$

$$= \{x_{i+1}^{(1)}, x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}\} \cup \{x_i^{(4)}, x_{i+2}^{(4)}, x_{i+1}^{(3)}, x_i^{(3)}\} \cup \{x_{i+3}^{(4)}, x_{i+1}^{(4)}, x_i^{(3)}\} \cup \{x_{i+1}^{(1)}, x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}\} \text{ in } G'$$

Note that $x_i^{(4)}$ is the common element in $N(x_{i+1}^{(4)}) \text{ in } G'$ and $N(N(x_i^{(1)})) \text{ in } G'$.

$$d_2(x_i^{(1)}) \text{ in } G_1 = d_2(x_i^{(1)}) \text{ in } G + |N(x_{i+1}^{(4)})| \text{ in } G' + |N(N(x_i^{(1)}))| \text{ in } G' - 1.$$
To find $d_2$ of each vertex in $C^{(4)}$, where $C^{(4)}$ is the cycle induced by the vertices \{x^{(4)}_i : 0 \leq i \leq 3\} in $G_1$.

$$N_2(x^{(4)}_{i+1}) \text{ in } G_1 = N_2(x^{(4)}_{i+1}) \text{ in } G' \cup N(x^{(1)}_i) \text{ in } G \cup N(N(x^{(4)}_{i+1})) \text{ in } G$$
$$= \{x^{(4)}_{i+3}, x^{(3)}_{i+3}, x^{(3)}_{i+2}, x^{(3)}_i\} \cup \{x^{(1)}_{i+2}, x^{(1)}_{i+3}; x^{(2)}_i\} \text{ in } G$$
$$\cup \{x^{(1)}_{i+3}; x^{(1)}_{i+1}; x^{(2)}_i\} \text{ in } G$$
$$= \{x^{(4)}_{i+3}, x^{(3)}_{i+3}, x^{(3)}_{i+2}, x^{(3)}_i\} \cup \{x^{(1)}_{i+2}, x^{(1)}_{i+3}, x^{(2)}_i, x^{(1)}_{i+1}, x^{(1)}_{i+1}\} \text{ in } G.$$ Note that $x^{(1)}_{i+3}$ is the common element in $N(x^{(1)}_i)$ in $G$ and $N(N(x^{(4)}_{i+1}))$ in $G$.

$$d_2(x^{(4)}_{i+1}) \text{ in } G_1 = d_2(x^{(4)}_{i+1}) \text{ in } G' + (|N(x^{(1)}_i)| \text{ in } G + |N(N(x^{(4)}_{i+1}))| \text{ in } G - 1$$
$$= 6 + (3 + 3) - 1 = 9 = (4 - 1)(4 - 1), (0 \leq i \leq 3).$$

Next, consider the edges \(x^{(2)}_i x^{(3)}_i, (0 \leq i \leq 3)\).

$$N(x^{(2)}_i) = \{x^{(2)}_{i+1}, x^{(2)}_{i+3}, x^{(1)}_i\} \text{ in } G \text{ and } |N(x^{(2)}_i)| = 3 \text{ in } G, \ (0 \leq i \leq 3).$$
$$N(N(x^{(2)}_i)) = \{x^{(3)}_{i+1}, x^{(3)}_{i+3}, x^{(4)}_i\} \text{ in } G' \text{ and } |N(N(x^{(2)}_i))| = 3 \text{ in } G', \ (0 \leq i \leq 3)$$
$$N(x^{(3)}_i) = \{x^{(3)}_{i+2}, x^{(3)}_{i+3}, x^{(4)}_i\} \text{ in } G' \text{ and } |N(x^{(3)}_i)| = 3 \text{ in } G', \ (0 \leq i \leq 3)$$
$$N(N(x^{(3)}_i)) = \{x^{(2)}_{i+2}, x^{(2)}_{i+3}, x^{(1)}_i\} \text{ in } G \text{ and } |N(N(x^{(3)}_i))| = 3 \text{ in } G, \ (0 \leq i \leq 3).$$

To find $d_2$ of each vertex in $C^{(2)}$, where $C^{(2)}$ is the cycle induced by the vertices \{x^{(2)}_i : 0 \leq i \leq 3\} in $G_1$.

$$N_2(x^{(2)}_i) \text{ in } G_1 = N_2(x^{(2)}_i) \text{ in } G' \cup N(x^{(3)}_i) \text{ in } G' \cup N(N(x^{(2)}_i)) \text{ in } G'$$
$$= \{x^{(1)}_{i+1}, x^{(1)}_{i+2}, x^{(2)}_{i+2}, x^{(1)}_{i+3}\} \text{ in } G \cup \{x^{(3)}_{i+2}, x^{(3)}_{i+3}, x^{(4)}_i\} \text{ in } G'$$
$$\cup \{x^{(3)}_{i+1}, x^{(3)}_{i+3}, x^{(4)}_i\} \text{ in } G'$$
$$= \{x^{(1)}_{i+1}, x^{(1)}_{i+2}, x^{(2)}_{i+2}, x^{(1)}_{i+3}\} \text{ in } G \cup \{x^{(3)}_{i+2}, x^{(3)}_{i+3}, x^{(4)}_i, x^{(3)}_{i+1}, x^{(4)}_i\} \text{ in } G'.$$ Note that $x^{(3)}_{i+3}$ is the common element in $N(x^{(3)}_i)$ in $G'$ and $N(N(x^{(2)}_i))$ in $G'$.

$$d_2(x^{(2)}_i) \text{ in } G_1 = d_2(x^{(2)}_i) \text{ in } G' + (|N(x^{(3)}_i)| \text{ in } G' + |N(N(x^{(2)}_i))| \text{ in } G') - 1 = 4 + (3 + 3) - 1 = 9 = (4 - 1)(4 - 1), (0 \leq i \leq 3).$$

To find $d_2$ of each vertex in $C^{(3)}$, where $C^{(3)}$ is the cycle induced by the vertices
\( \{x_i^{(3)} : 0 \leq i \leq 3\} \) in \( G_1 \).

\[
N_2(x_i^{(3)}) \text{ in } G_1 = N_2(x_i^{(3)}) \text{ in } G' \cup N(x_i^{(2)}) \text{ in } G \cup N(N(x_i^{(3)})) \text{ in } G \\
= \{x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}\} \text{ in } G' \cup \{x_{i+2}, x_{i+3}, x_{i+4}\} \text{ in } G \\
= \{x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}\} \text{ in } G' \cup \{x_{i+2}, x_{i+3}, x_{i+4}\} \text{ in } G.
\]

Note that \( x_{i+3}^{(2)} \) is the common element in \( N(x_i^{(2)}) \text{ in } G \) and \( N(N(x_i^{(3)})) \text{ in } G \).

\[
d_2(x_i^{(3)}) \text{ in } G_1 = d_2(x_i^{(3)}) \text{ in } G' + |N(x_i^{(2)})| \text{ in } G + |N(N(x_i^{(3)}))| \text{ in } G - 1. \\
= 4 + (3 + 3) - 1 = 9 = (4 - 1)(4 - 1), (0 \leq i \leq 3).
\]

In \( G_1 \), \( d_2(x_i^{(t)}) = (4 - 1)(4 - 1), (0 \leq i \leq 3), \) for \( 1 \leq t \leq 4 \). Hence \( G_1 \) is a \( (4, 2, (4 - 1)(4 - 1)) \)-regular graph on \( 4 \times 2^{4-2} = 16 \) vertices with the vertex set \( V(G_1) = \{x_i^{(t)} : 1 \leq t \leq 2^{4-2}, (0 \leq i \leq 3)\} \) and \( E(G_1) = E(G) \cup E(G') \cup \{x_i^{(1)} x_{i+1}, x_i^{(2)} x_i^{(3)} : 0 \leq i \leq 3\} \). Hence the result is true for \( r = 4 \).

**Step 2** Take another copy of \( G_1 \) as \( G'_1 \) with the vertex set \( V(G'_1) = \{x_i^{(t)} : (2^{4-2} + 1 \leq t \leq 2^{4-1}), (0 \leq i \leq 3)\} \) and each \( x_i^{(t)}(2^{4-2} + 1 \leq t \leq 2^{4-1}), \) corresponds to \( x_i^{(t)}(1 \leq t \leq 2^{4-2}) \) and \( (0 \leq i \leq 3) \). The desired graph \( G_2 \) has the vertex set \( V(G_2) = V(G_1) \cup V(G'_1) \) and edge set \( E(G_2) = E(G_1) \cup E(G'_1) \cup \{x_i^{(1)} x_{i+1}, x_i^{(2)} x_i^{(3)}, x_i^{(3)} x_{i+1}, x_i^{(4)} x_i^{(5)}\} \). Now, the resulting graph \( G_2 \) is 5 regular graph having \( 4 \times 2^{5-2} = 32 \) vertices. Figure 3.11 represents the graph \( G_2 \).
Consider the edges $x_i^{(1)} x_{i+1}^{(8)}$, $(0 \leq i \leq 3)$

$$N(x_i^{(1)}) = \{x_{i+2}^{(1)}, x_{i+3}^{(1)}, x_i^{(2)}, x_{i+1}^{(4)}\} \text{ in } G_1 \text{ and } |N(x_i^{(1)})| = 4 \text{ in } G_1$$

$$N(N(x_i^{(1)})) = \{x_{i+3}^{(8)}, x_i^{(8)}, x_i^{(7)}, x_{i+1}^{(5)}\} \text{ in } G_1' \text{ and } |N(N(x_i^{(1)}))| = 4, \text{ in } G_1'$$

$$N(x_{i+1}^{(8)}) = \{x_i^{(8)}, x_{i+2}^{(8)}, x_i^{(7)}, x_{i+1}^{(5)}\} \text{ in } G_1' \text{ and } |N(x_{i+1}^{(8)})| = 4, \text{ in } G_1'$$

$$N(N(x_{i+1}^{(8)})) = \{x_{i+1}^{(1)}, x_{i+3}^{(1)}, x_{i+1}^{(2)}, x_i^{(4)}\} \text{ in } G_1 \text{ and } |N(N(x_{i+1}^{(8)}))| = 4, \text{ in } G_1.$$

To find $d_2$ of each vertex in $C^{(1)}$, where $C^{(1)}$ is the cycle induced by the vertices $\{x_i^{(1)} : 0 \leq i \leq 3\}$ in $G_2$.

$$N_2(x_i^{(1)}) \text{ in } G_2 = N_2(x_i^{(1)}) \text{ in } G_1 \cup N(x_{i+1}^{(8)}) \text{ in } G_1' \cup N(N(x_i^{(1)})) \text{ in } G_1'$$

$$= N_2(x_i^{(1)}) \text{ in } G_1 \cup \{x_{i+2}^{(8)}, x_i^{(7)}, x_{i+1}^{(5)}\} \text{ in } G_1'$$

$$= N_2(x_i^{(1)}) \text{ in } G_1 \cup \{x_{i+2}^{(8)}, x_i^{(7)}, x_{i+1}^{(5)}\} \text{ in } G_1'$$

Note that $x_i^{(8)}$ is the common element in $N(x_{i+1}^{(8)}) \text{ in } G_1'$ and $N(N(x_i^{(1)})) \text{ in } G_1'$.

$$d_2(x_i^{(1)}) \text{ in } G_1 = d_2(x_i^{(1)}) \text{ in } G_1 + |N(x_{i+1}^{(8)})| \text{ in } G_1' + |N(N(x_i^{(1)}))| \text{ in } G_1' - 1.$$

$$= 9 + (4 + 4) - 1 = 16 = (5 - 1)(5 - 1), (0 \leq i \leq 3).$$

To find $d_2$ of each vertex in $C^{(8)}$, where $C^{(8)}$ is the cycle induced by the vertices $\{x_i^{(8)} : 0 \leq i \leq 3\}$ in $G_2$.

$$N_2(x_{i+1}^{(8)}) \text{ in } G_2 = N_2(x_{i+1}^{(8)}) \text{ in } G_1' \cup N(x_i^{(1)}) \text{ in } G_1 \cup N(N(x_{i+1}^{(8)})) \text{ in } G_1.$$}

$$= N_2(x_{i+1}^{(8)}) \text{ in } G_1' \cup \{x_{i+2}^{(1)}, x_i^{(1)}, x_i^{(2)}, x_{i+1}^{(4)}\} \text{ in } G_1$$

$$= N_2(x_{i+1}^{(8)}) \text{ in } G_1' \cup \{x_{i+2}^{(1)}, x_i^{(1)}, x_i^{(2)}, x_{i+1}^{(4)}\} \text{ in } G_1$$

Note that $x_{i+3}^{(1)}$ is the common element in $N(x_i^{(1)}) \text{ in } G_1$ and $N(N(x_{i+1}^{(8)})) \text{ in } G_1$.

$$d_2(x_{i+1}^{(8)}) \text{ in } G_2 = d_2(x_{i+1}^{(8)}) \text{ in } G_1' + |N(x_i^{(1)})| \text{ in } G_1 + |N(N(x_{i+1}^{(8)}))| \text{ in } G_1 - 1$$

$$= 9 + 4 + 4 - 1 = 9 + 8 - 1 = (5 - 1)(5 - 1), (0 \leq i \leq 3).$$
Next, consider the edges $x^{(2)}_i, x^{(7)}_i$, $(0 \leq i \leq 3)$.

$$N(x^{(2)}_i) = \{x^{(2)}_{i+1}, x^{(2)}_{i+3}, x^{(1)}_i, x^{(3)}_i\} \text{ in } G_1 \text{ and } |N(x^{(2)}_i)| = 4 \text{ in } G_1.$$  

$$N(N(x^{(2)}_i)) = \{x^{(7)}_{i+1}, x^{(7)}_{i+3}, x^{(8)}_{i+1}, x^{(6)}_{i+1}\} \text{ in } G'_1 \text{ and } |N(N(x^{(2)}_i))| = 4 \text{ in } G'_1.$$  

$$N(x^{(7)}_i) = \{x^{(7)}_{i+2}, x^{(7)}_{i+3}, x^{(8)}_i, x^{(6)}_i\} \text{ in } G'_1 \text{ and } |N(x^{(7)}_i)| = 4 \text{ in } G'_1.$$  

$$N(N(x^{(7)}_i)) = \{x^{(2)}_{i+2}, x^{(2)}_{i+3}, x^{(1)}_{i+3}, x^{(3)}_{i+3}\} \text{ in } G_1 \text{ and } |N(N(x^{(7)}_i))| = 4 \text{ in } G_1.$$  

To find $d_2$ of each vertex in $C^{(2)}$, where $C^{(2)}$ is the cycle induced by the vertices $\{x^{(2)}_i : 0 \leq i \leq 3\}$ in $G_2$.

$$N_2(x^{(2)}_i) \text{ in } G_2 = N_2(x^{(2)}_i) \text{ in } G_1 \cup N(x^{(7)}_i) \text{ in } G'_1 \cup N(N(x^{(2)}_i)) \text{ in } G'_1$$  

$$= N_2(x^{(2)}_i) \text{ in } G_1 \cup \{x^{(7)}_{i+3}, x^{(7)}_{i+2}, x^{(6)}_i, x^{(8)}_i\} \text{ in } G'_1$$  

$$\cup \{x^{(8)}_{i+1}, x^{(6)}_{i+1}, x^{(7)}_{i+1}, x^{(7)}_i\} \text{ in } G'_1$$  

$$= N_2(x^{(2)}_i) \text{ in } G_1 \cup \{x^{(7)}_{i+3}, x^{(7)}_{i+2}, x^{(6)}_i, x^{(8)}_i, x^{(8)}_i, x^{(7)}_{i+1}, x^{(7)}_{i+1}, x^{(7)}_i\} \text{ in } G'_1.$$  

Note that $x^{(7)}_{i+3}$ is the common element in $N(x^{(7)}_i)$ in $G_1$ and $N(N(x^{(2)}_i))$ in $G'_1$.  

$$d_2(x^{(2)}_i) \text{ in } G_2 = d_2(x^{(2)}_i) \text{ in } G_1 + (|N(x^{(7)}_i)| \text{ in } G'_1 + |N(N(x^{(2)}_i))| \text{ in } G'_1) - 1.$$  

$$= 9 + (4 + 4) - 1 = 16 = (5 - 1)(5 - 1), \ (0 \leq i \leq 3).$$  

To find $d_2$ of each vertex in $C^{(7)}$, where $C^{(7)}$ is the cycle induced by the vertices $\{x^{(7)}_i : 0 \leq i \leq 3\}$ in $G_2$.

$$N_2(x^{(7)}_i) \text{ in } G_2 = N_2(x^{(7)}_i) \text{ in } G_1 \cup N(x^{(2)}_i) \text{ in } G_1 \cup N(N(x^{(7)}_i)) \text{ in } G_1$$  

$$= N_2(x^{(7)}_i) \text{ in } G'_1 \cup \{x^{(2)}_{i+1}, x^{(2)}_{i+3}, x^{(3)}_i, x^{(1)}_i\} \text{ in } G_1$$  

$$\cup \{x^{(2)}_{i+3}, x^{(3)}_{i+3}, x^{(3)}_{i+2}\} \text{ in } G_1.$$  

$$= N_2(x^{(7)}_i) \text{ in } G'_1 \cup \{x^{(2)}_{i+1}, x^{(2)}_{i+3}, x^{(3)}_i, x^{(3)}_i, x^{(1)}_{i+3}, x^{(1)}_{i+3}, x^{(3)}_{i+2}\} \text{ in } G_1.$$  

Note that $x^{(2)}_{i+3}$ is the common element in $N(x^{(2)}_i)$ in $G_1$ and $N(N(x^{(7)}_i))$ in $G_1$.  

$$d_2(x^{(7)}_i) \text{ in } G_2 = (d_2(x^{(7)}_i) \text{ in } G'_1 + (|N(x^{(2)}_i)| \text{ in } G_1 + |N(N(x^{(7)}_i))| \text{ in } G_1) - 1$$  

$$d_2(x^{(7)}_i) \text{ in } G_2 = 9 + 4 + 4 - 1 = 9 + 8 - 1 = (5 - 1)(5 - 1), \ (0 \leq i \leq 3).$$
Next, consider the edges \( x_i^{(3)} x_{i+1}^{(6)} \), \( 0 \leq i \leq 3 \).

\[
N(x_i^{(3)}) = \{x_i^{(3)}, x_{i+3}^{(3)}, x_i^{(4)}, x_i^{(2)}\} \text{ in } G_1 \text{ and } |N(x_i^{(3)})| = 4 \text{ in } G_1.
\]

\[
N(N(x_i^{(3)})) = \{x_{i+3}^{(6)}, x_i^{(6)}, x_i^{(5)}, x_i^{(7)}\} \text{ in } G'_1 \text{ and } |N(N(x_i^{(3)}))| = 4 \text{ in } G'_1.
\]

\[
N(x_{i+1}^{(6)}) = \{x_i^{(6)}, x_i^{(6)}, x_{i+1}^{(6)}, x_{i+1}^{(7)}\} \text{ in } G'_1 \text{ and } |N(x_{i+1}^{(6)})| = 4 \text{ in } G'_1.
\]

\[
N(N(x_{i+1}^{(6)})) = \{x_i^{(3)}, x_{i+1}^{(3)}, x_i^{(4)}, x_{i+1}^{(2)}\} \text{ in } G_1 \text{ and } |N(N(x_{i+1}^{(6)}))| = 4 \text{ in } G_1.
\]

To find \( d_2 \) of each vertex in \( C^{(3)} \), where \( C^{(3)} \) is the cycle induced by the vertices \( \{x_i^{(3)} : 0 \leq i \leq 3\} \) in \( G_2 \).

\[
N_2(x_i^{(3)}) \text{ in } G_2 = N_2(x_i^{(3)}) \text{ in } G_1 \cup N(x_{i+1}^{(6)}) \text{ in } G'_1 \cup N(N(x_i^{(3)})) \text{ in } G'_1
\]

\[
= N_2(x_i^{(3)}) \text{ in } G_1 \cup \{x_{i+2}^{(6)}, x_{i+1}^{(6)}, x_{i+1}^{(7)}\} \text{ in } G'_1
\]

\[
\quad \cup \{x_{i+3}^{(6)}, x_i^{(5)}, x_i^{(7)}\} \text{ in } G'_1
\]

\[
= N_2(x_i^{(3)}) \text{ in } G_1 \cup \{x_{i+2}^{(6)}, x_i^{(6)}, x_{i+1}^{(5)}, x_{i+1}^{(7)}, x_{i+3}^{(6)}, x_i^{(5)}, x_i^{(7)}\} \text{ in } G_1.
\]

Note that \( x_i^{(6)} \) is the common element in \( N(x_{i+1}^{(6)}) \text{ in } G'_1 \) and \( N(N(x_i^{(3)})) \text{ in } G'_1 \).

\[
d_2(x_i^{(3)}) \text{ in } G_1 = d_2(x_i^{(3)}) \text{ in } G_1 + (|N(x_{i+1}^{(6)})| \text{ in } G'_1 + |N(N(x_i^{(3)}))| \text{ in } G'_1) - 1.
\]

\[
= 9 + (4 + 4) - 1 = 16 = (5 - 1)(5 - 1), (0 \leq i \leq 3).
\]

To find \( d_2 \) of each vertex in \( C^{(6)} \), where \( C^{(6)} \) is the cycle induced by the vertices \( \{x_i^{(6)} : 0 \leq i \leq 3\} \) in \( G_2 \).

\[
N_2(x_{i+1}^{(6)}) \text{ in } G_2 = N_2(x_{i+1}^{(6)}) \text{ in } G_1 \cup N(x_i^{(3)}) \text{ in } G_1 \cup N(N(x_{i+1}^{(6)})) \text{ in } G_1.
\]

\[
= N_2(x_{i+1}^{(6)}) \text{ in } G_1 \cup \{x_i^{(3)}, x_{i+3}^{(3)}, x_i^{(4)}, x_i^{(2)}\} \text{ in } G_1
\]

\[
\quad \cup \{x_{i+3}^{(3)}, x_i^{(3)}, x_i^{(4)}, x_i^{(2)}\} \text{ in } G_1
\]

\[
= N_2(x_{i+1}^{(6)}) \text{ in } G_1 \cup \{x_{i+2}^{(3)}, x_{i+3}^{(3)}, x_i^{(4)}, x_i^{(2)}\} \text{ in } G_1.
\]

Note that \( x_{i+3}^{(3)} \) is the common element in \( N(x_i^{(1)}) \text{ in } G_1 \) and \( N(N(x_{i+1}^{(6)})) \text{ in } G_1 \).

\[
d_2(x_{i+1}^{(6)}) \text{ in } G_2 = d_2(x_{i+1}^{(6)}) \text{ in } G'_1 + (|N(x_i^{(3)})| \text{ in } G_1 + |N(N(x_{i+1}^{(6)}))| \text{ in } G_1) - 1.
\]

\[
d_2(x_{i+1}^{(6)}) \text{ in } G_2 = 9 + 4 + 4 - 1 = 9 + 8 - 1 = (5 - 1)(5 - 1), (0 \leq i \leq 3).
\]
To find the edges $x_i^{(4)}x_i^{(5)}$, $(1 \leq i \leq 3)$.

$$N(x_i^{(4)}) = \{x_{i+1}^{(4)}, x_{i+3}^{(4)}, x_i^{(3)}, x_{i+3}^{(1)}\} \quad \text{in } G_1 \text{ and } |N(x_i^{(4)})| = 4, \text{ in } G_1.$$  
$$N(N(x_i^{(4)})) = \{x_{i+1}^{(5)}, x_{i+3}^{(5)}, x_i^{(6)}, x_{i+1}^{(8)}\} \quad \text{in } G'_1 \text{ and } |N(N(x_i^{(4)}))| = 4, \text{ in } G'_1.$$  
$$N(x_i^{(5)}) = \{x_{i+2}^{(5)}, x_{i+3}^{(5)}, x_i^{(6)}, x_{i+1}^{(8)}\} \quad \text{in } G'_1 \text{ and } |N(x_i^{(5)})| = 4, \text{ in } G'_1.$$  
$$N(N(x_i^{(5)})) = \{x_{i+2}^{(4)}, x_{i+3}^{(4)}, x_i^{(3)}, x_{i+3}^{(1)}\} \quad \text{in } G_1 \text{ and } |N(N(x_i^{(5)}))| = 4, \text{ in } G_1.$$  

To find $d_2$ of each vertex in $C^{(4)}$, where $C^{(4)}$ is the cycle induced by the vertices $\{x_i^{(4)}: 0 \leq i \leq 3\}$ in $G_2$.

$$N_2(x_i^{(4)}) \quad \text{in } G_2 = N_2(x_i^{(4)}) \quad \text{in } G_1 \cup N(x_i^{(5)}) \quad \text{in } G'_1 \cup N(N(x_i^{(4)})) \quad \text{in } G'_1$$

$$= N_2(x_i^{(4)}) \quad \text{in } G_1 \cup \{x_{i+2}^{(5)}, x_{i+3}^{(5)}, x_i^{(6)}, x_{i+1}^{(8)}\} \quad \text{in } G_1$$

$$\cup \{x_{i+1}^{(5)}, x_{i+3}^{(5)}, x_i^{(6)}, x_{i+1}^{(8)}\} \quad \text{in } G'_1$$

$$= N_2(x_i^{(5)}) \quad \text{in } G_1 \cup \{x_{i+2}^{(5)}, x_{i+3}^{(5)}, x_i^{(6)}, x_{i+1}^{(8)}\} \quad \text{in } G'_1$$

Note that $x_{i+3}^{(5)}$ is the common element in $N(x_i^{(5)})$ in $G'_1$ and $N(N(x_i^{(4)}))$ in $G'_1$.

$$d_2(x_i^{(4)}) \quad \text{in } G_2 = d_2(x_i^{(4)}) \quad \text{in } G_1 + (|N(x_i^{(5)})| \quad \text{in } G'_1 + |N(N(x_i^{(4)}))| \quad \text{in } G'_1) - 1.$$

$$= 9 + 4 + 4 - 1 = 16 = (5 - 1)(5 - 1), (0 \leq i \leq 3).$$

To find $d_2$ of each vertex in $C^{(5)}$, where $C^{(5)}$ is the cycle induced by the vertices $\{x_i^{(5)}: 0 \leq i \leq 3\}$ in $G_2$.

$$N_2(x_i^{(5)}) \quad \text{in } G_2 = N_2(x_i^{(5)}) \quad \text{in } G'_1 \cup N(x_i^{(4)}) \quad \text{in } G_1 \cup N(N(x_i^{(5)})) \quad \text{in } G_1$$

$$= N_2(x_i^{(5)}) \quad \text{in } G'_1 \cup \{x_{i+1}^{(4)}, x_{i+3}^{(4)}, x_i^{(3)}, x_{i+3}^{(1)}\} \quad \text{in } G_1$$

$$\cup \{x_{i+2}^{(4)}, x_{i+3}^{(4)}, x_i^{(3)}, x_{i+3}^{(1)}\} \quad \text{in } G_1$$

$$= N_2(x_i^{(5)}) \quad \text{in } G'_1 \cup \{x_{i+1}^{(4)}, x_{i+3}^{(4)}, x_i^{(3)}, x_{i+3}^{(1)}\} \quad \text{in } G'_1$$

Note that $x_{i+3}^{(4)}$ is the common element in $N(x_i^{(4)})$ in $G_1$ and $N(N(x_i^{(5)}))$ in $G_1$.

$$d_2(x_i^{(5)}) \quad \text{in } G_2 = d_2(x_i^{(5)}) \quad \text{in } G'_1 + (|N(x_i^{(4)})| \quad \text{in } G_1 + |N(N(x_i^{(5)}))| \quad \text{in } G_1) - 1$$

$$d_2(x_i^{(5)}) \quad \text{in } G_2 = 9 + 4 + 4 - 1 = 9 + 8 - 1 = (5 - 1)(5 - 1), (0 \leq i \leq 3).$$

In $G_2$, $d_2(x_i^{(t)}) = (5 - 1)(5 - 1)$, $(0 \leq i \leq 3)$, $(1 \leq t \leq 8)$. Hence $G_2$ is a $(5, 2, (5 - 1)(5 - 1))$ regular graph on $4 \times 2^{5-2} = 32$ vertices with the vertex set
$V(G_2) = \{x_i^{(t)} : 1 \leq t \leq 2^{k-2}, 0 \leq i \leq 3\}$ and $E(G_2) = E(G_1) \cup E(G'_1) \cup \{x_i^{(1)} x_{i+1}^{(8)}, x_i^{(2)} x_i^{(7)}, x_i^{(3)} x_i^{(6)}, x_i^{(4)} x_i^{(5)}\}$. Then the result is true for $r = 5$. Let us assume that this result is true for $r = m + 3$. Then there exists a $(m + 3, 2, (m + 2)^2)$-regular graph on $4 \times 2^{m+1}$ vertices with the vertex set $V(G_m) = \{x_i^{(t)} : 1 \leq t \leq 2^{m+1}, 0 \leq i \leq 3\}$ and $E(G_m) = E(G_{m-1}) \cup E(G'_{m-1}) \cup \{x_i^{(t)} x_{i+1}^{2m+1-t} : 0 \leq i \leq 3\}$. Hence for $t$, $(1 \leq t \leq 2^{m+1})$, $d_2(x_i^{(t)}) = (m + 2)(m + 2)$ and $d(x_i^{(t)}) = m + 3$, $(0 \leq i \leq 3)$. Take another copy of $G_m$ as $G'_m$ with the vertex set $V(G'_m) = \{x_i^{(t)} : 2m+1 + 1 \leq t \leq 2m+2, 0 \leq i \leq 3\}$ and each $x_i^{(t)}$, $(2m+1 + 1 \leq t \leq 2m+2)$, corresponds to $x_i^{(t)}$, $(1 \leq t \leq 2^{m+1})$, $(0 \leq i \leq 3)$. The desired graph $G_{m+1}$ has the vertex set $V(G_{m+1}) = V(G_m) \cup V(G'_m)$ and edge set $E(G_{m+1}) = E(G_m) \cup E(G'_m) \cup \{x_i^{(t)} x_{i+1}^{2m+2-t} : 0 \leq i \leq 3\}$. Now, the resulting graph $G_{m+1}$ is $(m + 4)$ regular graph having $4 \times 2^{m+2}$ vertices.

Consider the edges $\bigcup_{t=1}^{2m+1} \{x_i^{(t)} x_{i+1}^{2m+2-t} : 0 \leq i \leq 3\}$.

For $(1 \leq t \leq 2^{m+1})$, $d_2$ of each vertex in $C^{(t)}$, where $C^{(t)}$ is the cycle induced by the vertices $\{x_i^{(t)} : 0 \leq i \leq 3\}$ in $G_{m+1}$.

$$d_2(x_i^{(t)}) \text{ in } G_{m+1} = d_2(x_i^{(t)}) \text{ in } G_m + |N(x_i^{(t)})| \text{ in } G'_m$$

$$+ |N(N(x_i^{(t)}))| \text{ in } G_m.$$ 

$$= (m + 2)(m + 2) + ((m + 3) + (m + 3)) - 1,$$

$$\text{for } (0 \leq i \leq 3).$$

$$= (m + 2)(m + 2) + 2m + 5.$$

$$= m^2 + 6m + 9 = (m + 3)(m + 3).$$

For $(2^{m+1} + 1 \leq t \leq 2^{m+2})$, $d_2$ of each vertex in $C^{(2m+2-t)}$, where $C^{(2m+2-t)}$ is the cycle induced by the vertices $\{x_i^{(2m+2-t)} : 0 \leq i \leq 3\}$ in $G_{m+1}$.

$$d_2(x_i^{(2m+2-t)}) \text{ in } G_{m+1} = d_2(x_i^{(2m+2-t)}) \text{ in } G_m + |N(x_i^{(t)})| \text{ in } G_m$$

$$+ |N(N(x_i^{(t)}))| \text{ in } G_m.$$ 

$$= (m + 3)(m + 2) + (m + 3) + (m + 3) - 1,$$

$$\text{for } (0 \leq i \leq 3).$$

$$= (m + 3)(m + 3).$$

In $G_{m+1}$, $d_2(x_i^{(t)}) = (m + 3)(m + 3), (0 \leq i \leq 3), (1 \leq t \leq 2^{m+2})$. Then there
exists a $((m + 4), 2, (m + 3)(m + 3))$-regular graph on $4 \times 2^{m+2}$ vertices with the
vertex set $V(G_{m+1}) = \{x_i(t) : 1 \leq t \leq 2^{m+2}, 0 \leq i \leq 3\}$ and $E(G_{m+1}) = E(G_m) \cup$
$E(G'_m) \bigcup_{t=1}^{2^{m+1}} \{x_i(t)x_{i+t(mod 2)}^{2m+2-t+1} : 0 \leq i \leq 3\}$. Hence for $1 \leq t \leq 2^{m+2}$, $d_2(x_i(t)) = (m + 3)(m + 3)$ and $d(x_i(t)) = m + 4, (0 \leq i \leq 3)$. If the result is true for $r = m + 3$, then it is true for $r = m + 4$. Hence the result is true for all $r \geq 2$. \hfill \Box

3.11 $(r, 2, (r - n)(r - 1))$- Regular Graphs

A method is given in this section to construct an $(r, 2, (r - n)(r - 1))$-regular graph
with $(n + 1) \times 2^{-n}$ vertices, for any $r \geq n \geq 2[47]$.

Definition 3.11.1. A graph $G$ is called $(r, 2, (r - n)(r - 1))$-regular graph, $(r \geq n)$
if each vertex in the graph $G$ is at a distance one from $r$ vertices and each vertex in
the graph $G$ is at a distance two from exactly $(r - n)(r - 1)$ vertices.

The $(r, 2, k)$-regular graphs have been constructed for $k = (r - 2)(r - 1)$ [38] and
$k = (r - 3)(r - 1)[39]$.

Theorem 3.11.2. For any $r \geq 2$, there is an $(r, 2, (r - 2)(r - 1))$-regular graph on
$3 \times 2^{r-2}$ vertices [38].

Theorem 3.11.3. For any $r \geq 3$, there is an $(r, 2, (r - 3)(r - 1))$-regular graph
on $4 \times 2^{r-3}$ vertices [39]. These constructions are used to construct $(r, 2, (r -
)n)(r - 1)$-regular graphs, for any $r \geq n$, and the existence of $(r, 2, m(r - 1))$-regular
graph for any positive integer $m \leq r$ is proved.

Theorem 3.11.4. For any $r \geq n \geq 2$, there exists an $(r, 2, (r - n)(r - 1))$-regular
graph on $(n + 1) \times 2^{-n}$ vertices.

Proof. If $r = n$, complete graph on $(n + 1)$ vertices is the required graph. Let us
prove this result by induction on $r$.

Let $G$ be a graph with vertex set $V(G) = \{x_i(1), x_i(2) : 0 \leq i \leq n\}$ and edge set
$E(G) = \{x_i(1)x_i(2) : 0 \leq i \leq n\} \bigcup_{i=0}^{n-1} \{x_i(1)x_{i+j}^{1} : 1 \leq j \leq n - i\} \bigcup_{i=0}^{n-1} \{x_i(2)x_{i+j}^{2} : 1 \leq j \leq
CHAPTER 3. \((r, 2, m(r-1))\) - REGULAR GRAPH

\(n - i\). 

For \((0 \leq i \leq n)\), (subscripts are taken modulo \(n\)).

\[
N_2(x_i^{(1)}) = \{x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)} , \ldots , x_{i+n}^{(2)}\} \text{ and } d_2(x_i^{(1)}) = n.
\]

\[
N_2(x_i^{(2)}) = \{x_i^{(1)}, x_{i+1}^{(1)}, x_{i+2}^{(1)}, \ldots , x_{i+n}^{(1)}\} \text{ and } d_2(x_i^{(1)}) = n.
\]

Hence \(G\) is \((n+1), 2, ((n+1) - (n))(n+1 - 1))\)-regular graph on \((n+1) \times 2^{n+1-n} = 2(n+1)\) vertices.

**Step 1**

Take another copy of \(G\) as \(G'\). Let \(V(G') = \{x_i^{(3)}, x_i^{(4)} : 0 \leq i \leq n\}\) and \(E(G') = \{x_i^{(3)} x_i^{(4)} : 0 \leq i \leq n\} \cup \{x_i^{(4)} x_{i+j}^{(4)} : 1 \leq j \leq n-i\} \cup \{x_i^{(3)} x_{i+j}^{(3)} : 1 \leq i \leq n-i\}\).

The desired graph \(G_1\) has the vertex set \(V(G_1) = V(G) \cup V(G')\) edge set \(E(G_1) = E(G) \cup E(G') \cup \{x_i^{(1)} x_{i+1}^{(1)}, x_i^{(2)} x_i^{(3)} : 0 \leq i \leq n\}\) (subscripts are taken modulo \(n\)). Now, the resulting graph \(G_1\) is \((n + 2)\) regular graph having \((n+1) \times 2^{n+2-(n)} = 4(n+1)\) vertices.

**Next, consider the edges** \(x_i^{(1)} x_i^{(4)}\), \((0 \leq i \leq n)\).

\[
N(x_i^{(1)}) = \{x_{i+1}^{(1)}, x_{i+2}^{(1)}, x_{i+3}^{(1)} , \ldots , x_{i+n}^{(1)}\} \text{ in } G \text{ and } |N(x_i^{(1)})| = n + 1 \text{ in } G.
\]

\[
N(N(x_i^{(1)})) = \{x_i^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)} , \ldots , x_{i+n}^{(4)}\} \text{ in } G' \text{ and } |N(N(x_i^{(1)}))| = n + 1 \text{ in } G'.
\]

\[
N(x_i^{(4)}) = \{x_i^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)} , x_{i+n}^{(4)}\} \text{ in } G' \text{ and } |N(x_i^{(4)})| = n + 1 \text{ in } G'.
\]

\[
N(N(x_i^{(4)})) = \{x_i^{(1)}, x_{i+1}^{(1)}, x_{i+2}^{(1)}, x_{i+3}^{(1)} , \ldots , x_{i+n}^{(1)}\} \text{ in } G \text{ and } |N(N(x_i^{(4)}))| = n + 1 \text{ in } G.
\]

\(d_2\) of each vertex in \(C^{(1)}\), where \(C^{(1)}\) is the cycle induced by the vertices \(\{x_i^{(1)} : 0 \leq i \leq n\}\)

\[
N_2(x_i^{(1)}) \text{ in } G_1 = N_2(x_i^{(1)}) \text{ in } G \cup N(x_i^{(4)}) \text{ in } G' \cup N(N(x_i^{(1)})) \text{ in } G' \]

\[
= N_2(x_i^{(1)}) \text{ in } G \cup \{x_i^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)} , \ldots , x_{i+n}^{(4)}\} \text{ in } G' \]

\[
\quad \cup \{x_i^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)} , \ldots , x_{i+n}^{(4)}\} \text{ in } G'.
\]

\[
= N_2(x_i^{(1)}) \text{ in } G \cup \{x_i^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)} , \ldots , x_{i+n}^{(4)}\} \text{ in } G'.
\]

It is noted that \(x_{i+2}^{(4)}, x_{i+3}^{(4)}, x_i^{(4)} , \ldots , x_{i+n}^{(4)}\) are the common elements in \(N(x_i^{(4)})\) in \(G'\)
and \( N(N(x_i^{(1)})) \) in \( G' \).

\[
d_2(x_i^{(1)}) \text{ in } G_1 = d_2(x_i^{(1)}) \text{ in } G + (d(x_i^{(1)}) \text{ in } G' + |N(N(x_i^{(1)}))| \text{ in } G') - n.
\]

\[
= n + (n + 1 + n + 1) - (n) = 2(n + 1)
\]

\[
= [(n + 2) - (n)](n + 2 - 1), (0 \leq i \leq n).
\]

\[d_2 \text{ of each vertex } \text{ in } C^{(4)}, \text{ where } C^{(4)} \text{ is the cycle induced by the vertices } \{x_i^{(4)} : 0 \leq i \leq n\}\]

\[
N_2(x_i^{(4)}) \text{ in } G_1 = N_2(x_i^{(4)}) \text{ in } G' \cup N(x_i^{(1)}) \text{ in } G \cup N(N(x_i^{(1)})) \text{ in } G
\]

\[
= N_2(x_i^{(4)}) \text{ in } G' \cup \{x_i^{(1)}, x_{i+1}, x_{i+2}, x_{i+3}, \ldots, x_{i+n}, x_i^{(2)}\} \text{ in } G
\]

\[
\cup \{x_i^{(1)}, x_{i+2}, x_{i+3}, \ldots, x_{i+n}, x_i^{(2)}\} \text{ in } G.
\]

\[
= N_2(x_i^{(4)}) \text{ in } G' \cup \{x_i^{(1)}, x_{i+1}, x_{i+2}, x_{i+3}, \ldots, x_{i+n}, x_i^{(2)}, x_{i+1}\} \text{ in } G.
\]

Note that \( x_i^{(1)}, x_{i+1}, x_{i+2}, x_{i+3}, \ldots, x_{i+n} \) are the common elements in \( N(x_i^{(1)}) \) in \( G \) and \( N(N(x_i^{(1)})) \) in \( G \).

\[
d_2(x_i^{(4)}) \text{ in } G_1 = d_2(x_i^{(4)}) \text{ in } G' + (d(x_i^{(1)}) \text{ in } G + |N(N(x_i^{(1)}))| \text{ in } G') - n.
\]

\[
= n + (n + 1 + n + 1) - (n) = 2(n + 1)
\]

\[
= [(n + 2) - (n)](n + 2 - 1), (0 \leq i \leq n).
\]

Next, consider the edges \( x_i^{(2)}x_i^{(3)}, (0 \leq i \leq n) \).

\[
N(x_i^{(2)}) = \{x_i^{(2)}, x_{i+2}, x_{i+3}, \ldots, x_{i+n}, x_i^{(1)}\} \text{ in } G \text{ and } |N(x_i^{(2)})| = n + 1 \text{ in } G.
\]

\[
N(N(x_i^{(2)})) = \{x_i^{(3)}, x_{i+3}, x_{i+3}, \ldots, x_{i+n}, x_i^{(4)}\} \text{ in } G' \text{ and } |N(N(x_i^{(2)}))| = n + 1 \text{ in } G'.
\]

\[
N(x_i^{(3)}) = \{x_i^{(3)}, x_{i+2}, x_{i+3}, \ldots, x_{i+n}, x_i^{(4)}\} \text{ in } G' \text{ and } |N(x_i^{(3)})| = n + 1 \text{ in } G'.
\]

\[
N(N(x_i^{(3)})) = \{x_i^{(2)}, x_{i+2}, x_{i+3}, \ldots, x_{i+n}, x_i^{(1)}\} \text{ in } G \text{ and } |N(N(x_i^{(3)}))| = n + 1 \text{ in } G.
\]

\[d_2 \text{ of each vertex } \text{ in } C^{(2)}, \text{ where } C^{(2)} \text{ is the cycle induced by the vertices } \{x_i^{(2)} : 0 \leq i \leq n\}\]

\[
N_2(x_i^{(2)}) \text{ in } G_1 = N_2(x_i^{(2)}) \text{ in } G \cup N(x_i^{(3)}) \text{ in } G' \cup N(N(x_i^{(2)})) \text{ in } G'
\]

\[
= N_2(x_i^{(2)}) \text{ in } G \cup \{x_i^{(3)}, x_{i+3}, x_{i+3}, \ldots, x_{i+n}, x_i^{(4)}\} \text{ in } G'
\]

\[
\cup \{x_i^{(3)}, x_{i+2}, x_{i+3}, \ldots, x_{i+n}, x_i^{(4)}, x_{i+1}\} \text{ in } G'.
\]

\[
= N_2(x_i^{(2)}) \text{ in } G \cup \{x_i^{(3)}, x_{i+3}, x_{i+3}, \ldots, x_{i+n}, x_i^{(4)}, x_{i+1}\} \text{ in } G'.
\]
Note that $x_{i+1}^{(3)}, x_{i+2}^{(3)}, x_{i+3}^{(3)}, \ldots, x_{i+n}^{(3)}$ are the common elements in $N(x_i^{(3)})$ in $G'$ and $N(N(x_i^{(3)}))$ in $G'$.

$$d_2(x_i^{(2)}) \text{ in } G_1 = d_2(x_i^{(2)}) \text{ in } G + (d(x_i^{(3)}) \text{ in } G' + |N(N(x_i^{(3)}))| \text{ in } G') - n.$$  
$$= n + (n + 1 + n + 1) - (n) = 2(n + 1)$$  
$$= [(n + 2) - (n)](n + 2 - 1), (0 \leq i \leq n).$$

$d_2$ of each vertex in $C^{(3)}$, where $C^{(3)}$ is the cycle induced by the vertices $\{x_i^{(3)} : 0 \leq i \leq n\}$

$$N_2(x_i^{(3)}) \text{ in } G_1 = N_2(x_i^{(3)}) \text{ in } G' \cup N(x_i^{(2)}) \text{ in } G \cup N(N(x_i^{(3)})) \text{ in } G$$  
$$= N_2(x_i^{(3)}) \text{ in } G' \cup \{x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}, \ldots, x_{i+n}^{(2)}, x_i^{(1)}\} \text{ in } G$$  
$$\cup \{x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}, \ldots, x_{i+n}^{(2)}, x_i^{(1)}\} \text{ in } G$$  
$$= N_2(x_i^{(3)}) \text{ in } G' \cup \{x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}, \ldots, x_{i+n}^{(2)}, x_i^{(4)}, x_i^{(4)}\} \text{ in } G'.$$

Note that $x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}, \ldots, x_{i+n}^{(2)}$ are the common elements in $N(x_i^{(2)})$ in $G$ and $N(N(x_i^{(3)}))$ in $G$.

$$d_2(x_i^{(3)}) \text{ in } G_1 = d_2(x_i^{(3)}) \text{ in } G' + (d(x_i^{(2)}) \text{ in } G + |N(N(x_i^{(3)}))| \text{ in } G) - n.$$  
$$= n + (n + 1 + n + 1) - (n) = 2(n + 1)$$  
$$= [(n + 2) - (n)](n + 2 - 1), (0 \leq i \leq n).$$

In $G_1$, $d_2(x_i^{(t)}) = [(n + 2) - (n)](n + 2 - 1), (0 \leq i \leq n), (1 \leq t \leq 4).$

$G_1$ is $(n + 2), 2, ((n + 2) - (n))(n + 2 - 1)$-regular having $(n + 1) \times 2^{n+2-(n)} = 4(n + 1)$ vertices with the vertex set $V(G_1) = \{x_i^{(t)} / (1 \leq t \leq 2^{n+2}), (0 \leq i \leq n)\}$ and $E(G_1) = E(G) \cup E(G') \cup \{x_i^{(1)}, x_{i+1}^{(2)}, x_i^{(4)}, x_i^{(3)} / (0 \leq i \leq n)\}$. Hence the result is true for $r = n + 2$.

**Step 2** Take another copy of $G_1$ as $G'_1$ with the vertex set $V(G'_1) = \{x_i^{(t)} : (2^{5-3} + 1 \leq t \leq 2^{5-2}), (0 \leq i \leq n)\}$ and each $x_i^{(t)}(2^{5-3} + 1 \leq t \leq 2^{5-3})$, corresponds to $x_i^{(t)}, (1 \leq t \leq 2^{5-3}), (0 \leq i \leq n).$

The desired graph $G_2$ has the vertex set $V(G_2) = V(G_1) \cup V(G'_1)$ and edge set $E(G_2) = E(G_1) \cup E(G'_1) \cup \{x_i^{(1)}, x_{i+1}^{(2)}, x_i^{(3)}, x_{i+1}^{(4)} / 0 \leq i \leq n\}$ (subscripts are taken modulo $n$).

Now, the resulting graph $G_2$ is $(n+3)$ regular graph having $(n+1) \times 2^{n+3-n} = 8(n+1)$
vertices.

Consider the edges $x_i^{(1)} x_{i+1}^{(8)}$, $(0 \leq i \leq n)$.

\[ N(x_i^{(1)}) = \{x_{i+1}^{(1)}, x_{i+2}^{(1)}, \ldots, x_{i+n}^{(1)}, x_i^{(2)} x_{i+1}^{(4)}\} \text{ in } G_1 \text{ and } |N(x_i^{(1)})| = n + 2 \text{ in } G_1. \]

\[ N(N(x_i^{(1)})) = \{x_i^{(8)}, x_{i+2}^{(8)}, \ldots, x_{i+n}^{(8)}, x_i^{(7)} x_{i+1}^{(5)}\} \text{ in } G'_1 \text{ and } |N(N(x_i^{(1)}))| = n + 2 \text{ in } G'_1. \]

\[ N(x_{i+1}^{(8)}) = \{x_i^{(8)}, x_{i+2}^{(8)}, \ldots, x_{i+n}^{(8)}, x_i^{(7)} x_{i+1}^{(5)}\} \text{ in } G'_1 \text{ and } |N(x_{i+1}^{(8)})| = n + 2 \text{ in } G'_1. \]

\[ N(N(x_{i+1}^{(8)})) = \{x_{i+1}^{(1)}, x_{i+2}^{(1)}, \ldots, x_{i+n}^{(1)}, x_i^{(2)} x_{i+1}^{(4)}\} \text{ in } G_1 \text{ and } |N(N(x_{i+1}^{(8)}))| = n + 2 \text{ in } G_1. \]

$d_2$ of each vertex in $C^{(1)}$, where $C^{(1)}$ is the cycle induced by the vertices $\{x_i^{(1)} : 0 \leq i \leq n\}$

\[ N_2(x_i^{(1)}) \text{ in } G_2 = N_2(x_i^{(1)}) \text{ in } G_1 \cup N(x_{i+1}^{(8)}) \text{ in } G'_1 \cup N(N(x_i^{(1)})) \text{ in } G'_1 \]

\[ = N_2(x_i^{(1)}) \text{ in } G_1 \cup \{x_i^{(8)}, x_{i+2}^{(8)}, \ldots, x_{i+n}^{(8)}, x_i^{(7)} x_{i+1}^{(5)}\} \text{ in } G'_1 \]

\[ \cup \{x_i^{(8)}, x_{i+2}^{(8)}, \ldots, x_{i+n}^{(8)}, x_i^{(7)} x_{i+1}^{(5)}\} \text{ in } G'_1. \]

Note that $x_i^{(8)}, x_{i+2}^{(8)}, \ldots, x_{i+n}^{(8)}$ are the common elements in $N(x_{i+1}^{(8)})$ in $G'_1$ and $N(N(x_i^{(1)}))$ in $G'_1$.

\[ d_2(x_i^{(1)}) \text{ in } G_1 = d_2(x_i^{(1)}) \text{ in } G_1 + (d(x_{i+1}^{(8)}) \text{ in } G'_1 + |N(N(x_i^{(1)}))| \text{ in } G'_1) - n. \]

\[ = 2(n + 1) + (n + 2 + n + 2) - (n) = 3(n + 2) \]

\[ = [(n + 3) - (n)](n + 3 - 1), (0 \leq i \leq n). \]

$d_2$ of each vertex in $C^{(8)}$, where $C^{(8)}$ is the cycle induced by the vertices $\{x_i^{(8)} : 0 \leq i \leq n\}$

\[ N_2(x_{i+1}^{(8)}) \text{ in } G_2 = N_2(x_{i+1}^{(8)}) \text{ in } G'_1 \cup N(x_i^{(1)}) \text{ in } G_1 \cup N(N(x_{i+1}^{(8)})) \text{ in } G_1 \]

\[ = N_2(x_{i+1}^{(8)}) \text{ in } G'_1 \cup \{x_{i+1}^{(1)}, x_{i+2}^{(1)}, \ldots, x_{i+n}^{(1)}, x_i^{(2)} x_{i+1}^{(4)}\} \text{ in } G_1 \]

\[ \cup \{x_{i+1}^{(1)}, x_{i+2}^{(1)}, \ldots, x_{i+n}^{(1)}, x_i^{(2)} x_{i+1}^{(4)}\} \text{ in } G_1. \]

It is noted that $x_{i+1}^{(1)}, x_{i+2}^{(1)}, \ldots, x_{i+n}^{(1)}$ are the common elements in $N(x_i^{(1)})$ in $G_1$

and $N(N(x_{i+1}^{(8)}))$ in $G_1$.

\[ d_2(x_{i+1}^{(8)}) \text{ in } G_2 = d_2(x_{i+1}^{(8)}) \text{ in } G'_1 + (d(x_i^{(1)}) \text{ in } G_1 + |N(N(x_{i+1}^{(8)}))| \text{ in } G_1) - n. \]

\[ = 2(n + 1) + (n + 2 + n + 2) - (n) = 3(n + 2) \]

\[ = [(n + 3) - (n)](n + 3 - 1), (0 \leq i \leq n). \]
Next, consider the edges $x_i^{(2)}x_i^{(7)}$, $(0 \leq i \leq n)$.

\[ N(x_i^{(2)}) = \{x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}, \ldots, x_{i+n}^{(2)}, x_i^{(3)}\} \text{ in } G_1 \text{ and } |N(x_i^{(2)})| = n + 2 \text{ in } G_1. \]

\[ N(N(x_i^{(2)})) = \{x_{i+1}^{(7)}, x_{i+2}^{(7)}, x_{i+3}^{(7)}, \ldots, x_{i+n}^{(7)}, x_i^{(6)}\} \text{ in } G'_1 \text{ and } |N(N(x_i^{(2)}))| = n + 2 \text{ in } G'_1. \]

\[ N(x_i^{(7)}) = \{x_{i+1}^{(7)}, x_{i+2}^{(7)}, x_{i+3}^{(7)}, \ldots, x_{i+n}^{(7)}, x_i^{(6)}\} \text{ in } G_1 \text{ and } |N(x_i^{(7)})| = n + 2 \text{ in } G_1. \]

\[ N(N(x_i^{(7)})) = \{x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}, \ldots, x_{i+n}^{(2)}, x_i^{(3)}\} \text{ in } G_1 \text{ and } |N(N(x_i^{(7)}))| = n + 2 \text{ in } G_1. \]

$d_2$ of each vertex in $C(2)$, where $C(2)$ is the cycle induced by the vertices \( \{x_i^{(2)} : 0 \leq i \leq n\} \)

\[ N_2(x_i^{(2)}) \text{ in } G_2 = N_2(x_i^{(2)}) \text{ in } G_1 \cup N(x_i^{(7)}) \text{ in } G'_1 \cup N(N(x_i^{(2)})) \text{ in } G'_1 \]

\[ = N_2(x_i^{(2)}) \text{ in } G_1 \cup \{x_{i+1}^{(7)}, x_{i+2}^{(7)}, x_{i+3}^{(7)}, \ldots, x_{i+n}^{(7)}, x_i^{(6)}\} \text{ in } G'_1 \]

\[ \cup \{x_{i+1}^{(7)}, x_{i+2}^{(7)}, x_{i+3}^{(7)}, \ldots, x_{i+n}^{(7)}, x_i^{(6)}\} \text{ in } G'_1. \]

Note that $x_{i+1}^{(7)}, x_{i+2}^{(7)}, x_{i+3}^{(7)}, \ldots, x_{i+n}^{(7)}$ are the common elements in $N(x_i^{(7)})$ in $G_1$ and $N(N(x_i^{(2)}))$ in $G'_1$.

\[ d_2(x_i^{(2)}) \text{ in } G_2 = d_2(x_i^{(2)}) \text{ in } G_1 + (d(x_i^{(7)}) \text{ in } G'_1 + |N(N(x_i^{(2)}))| \text{ in } G'_1) - n. \]

\[ = 2(n + 1) + (n + 2 + n + 2) - n = 3(n + 2) \]

\[ = [(n + 3) - (n)](n + 3 - 1), (0 \leq i \leq n). \]

$d_2$ of each vertex in $C(7)$, where $C(7)$ is the cycle induced by the vertices \( \{x_i^{(7)} : 0 \leq i \leq n\} \)

\[ N_2(x_i^{(7)}) \text{ in } G_2 = N_2(x_i^{(7)}) \text{ in } G'_1 \cup N(x_i^{(2)}) \text{ in } G_1 \cup N(N(x_i^{(7)})) \text{ in } G_1 \]

\[ = N_2(x_i^{(7)}) \text{ in } G'_1 \cup \{x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}, \ldots, x_{i+n}^{(2)}, x_i^{(3)}\} \text{ in } G_1 \]

\[ \cup \{x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}, \ldots, x_{i+n}^{(2)}, x_i^{(3)}\} \text{ in } G_1. \]

Note that $x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}, \ldots, x_{i+n}^{(2)}$ are the common elements in $N(x_i^{(2)})$ in $G_1$ and $N(N(x_i^{(7)}))$ in $G_1$.

\[ d_2(x_i^{(7)}) \text{ in } G_2 = d_2(x_i^{(7)}) \text{ in } G'_1 + (d(x_i^{(2)}) \text{ in } G_1 + |N(N(x_i^{(7)}))| \text{ in } G_1) - n. \]

\[ = 2(n + 1) + (n + 2 + n + 2) - n = 3(n + 2) \]

\[ = [(n + 3) - (n)](n + 3 - 1), (0 \leq i \leq n). \]
Next, consider the edges $x_i^{(3)} x_{i+1}^{(6)}$, $(0 \leq i \leq n)$.

$N(x_i^{(3)}) = \{x_{i+1}^{(3)}, x_{i+2}^{(3)}, x_{i+3}^{(3)}, \ldots, x_{i+n}^{(3)}, x_i^{(4)}, x_i^{(2)}\}$ in $G_1$ and $|N(x_i^{(3)})| = n + 2$ in $G_1$.

$N(N(x_i^{(3)})) = \{x_i^{(6)}, x_i^{(6)}, x_i^{(6)}, \ldots, x_{i+n}^{(6)}, x_i^{(5)}, x_i^{(7)}\}$ in $G_1'$ and $|N(N(x_i^{(3)}))| = n + 2$ in $G_1'$.

$N(x_i^{(6)}) = \{x_i^{(6)}, x_{i+2}^{(6)}, x_{i+3}^{(6)}, \ldots, x_{i+n}^{(6)}, x_i^{(5)}, x_i^{(7)}\}$ in $G_1'$ and $|N(x_i^{(6)})| = n + 2$ in $G_1'$.

$N(N(x_i^{(6)})) = \{x_{i+1}^{(3)}, x_{i+3}^{(3)}, x_{i+5}^{(3)}, x_{i+1}^{(4)}, x_{i+1}^{(2)}\}$ in $G_1$ and $|N(N(x_i^{(6)}))| = n + 2$ in $G_1$.

$d_2$ of each vertex in $C^{(3)}$, where $C^{(3)}$ is the cycle induced by the vertices

$\{x_i^{(3)} : 0 \leq i \leq n\}$

$N_2(x_i^{(3)})$ in $G_2 = N_2(x_i^{(3)})$ in $G_1 \cup N(x_{i+1}^{(6)})$ in $G_1' \cup N(N(x_i^{(3)}))$ in $G_1'$

$= N_2(x_i^{(3)})$ in $G_1 \cup \{x_i^{(6)}, x_{i+2}^{(6)}, x_{i+3}^{(6)}, \ldots, x_{i+n}^{(6)}, x_i^{(5)}, x_i^{(7)}\}$ in $G_1'$

$= N_2(x_i^{(3)})$ in $G_1 \cup \{x_i^{(6)}, x_{i+2}^{(6)}, x_{i+3}^{(6)}, \ldots, x_{i+n}^{(6)}, x_i^{(5)}, x_i^{(7)}\}$ in $G_1'$.

Note that $x_i^{(7)}, x_{i+2}^{(7)}, x_{i+3}^{(7)}, \ldots, x_{i+n}^{(7)}$ are the common elements in $N(x_i^{(7)})$ in $G_1$ and $N(N(x_i^{(6)}))$ in $G_1'$.

$d_2(x_i^{(2)})$ in $G_2 = d_2(x_i^{(2)})$ in $G_1 + (d(x_i^{(7)})$ in $G_1' + |N(N(x_i^{(2)}))| in G_1') - n.$

$= 2(n + 1) + (n + 2 + n + 2) - (n) = 3(n + 2)$

$= [(n + 3) - (n)](n + 3 - 1), (0 \leq i \leq n).$

$d_2$ of each vertex in $C^{(6)}$, where $C^{(6)}$ is the cycle induced by the vertices

$\{x_i^{(6)} : 0 \leq i \leq n\}$

$N_2(x_{i+1}^{(6)})$ in $G_2 = N_2(x_{i+1}^{(6)})$ in $G_1' \cup N(x_{i+1}^{(3)})$ in $G_1 \cup N(N(x_{i+1}^{(6)}))$ in $G_1$

$= N_2(x_{i+1}^{(6)})$ in $G_1' \cup \{x_{i+1}^{(3)}, x_{i+2}^{(3)}, x_{i+3}^{(3)}, \ldots, x_{i+n}^{(3)}, x_i^{(4)}, x_i^{(2)}\}$ in $G_1$

$= N_2(x_{i+1}^{(6)})$ in $G_1' \cup \{x_{i+1}^{(3)}, x_{i+2}^{(3)}, x_{i+3}^{(3)}, \ldots, x_{i+n}^{(3)}, x_i^{(4)}, x_i^{(2)}\}$ in $G_1$.

Note that $x_{i+1}^{(3)}, x_{i+2}^{(3)}, x_{i+3}^{(3)}, \ldots, x_{i+n}^{(3)}$ are the common elements in $N(x_{i+1}^{(1)})$ in $G_1$ and

$N(N(x_{i+1}^{(6)}))$ in $G_1$.

$d_2(x_{i+1}^{(6)})$ in $G_2 = d_2(x_{i+1}^{(6)})$ in $G_1' + (d(x_i^{(3)}) in G_1 + |N(N(x_{i+1}^{(6)}))| in G_1) - n.$

$d_2(x_{i+1}^{(6)}) in G_2 = 2(n + 1) + (n + 2 + n + 2) - (n) = 3(n + 2)$

$= [(n + 3) - (n)](n + 3 - 1), (0 \leq i \leq n).$
Next, consider the edges $x_i^{(4)}, x_i^{(5)}, (0 \leq i \leq n)$.

$$N(x_i^{(4)}) = \{x_{i+1}^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)}, \ldots, x_{i+n}^{(4)}, x_i^{(3)}, x_i^{(1)}\} \text{ in } G_1 \text{ and } |N(x_i^{(4)})| = n + 2, \text{ in } G_1.$$  

$$N(N(x_i^{(4)})) = \{x_{i+1}^{(5)}, x_{i+2}^{(5)}, x_{i+3}^{(5)}, \ldots, x_{i+n}^{(5)}, x_i^{(6)}, x_i^{(8)}\} \text{ in } G'_1 \text{ and } |N(N(x_i^{(4)}))| = n + 2, \text{ in } G'_1.$$  

$$N(x_i^{(5)}) = \{x_{i+1}^{(5)}, x_{i+2}^{(5)}, x_{i+3}^{(5)}, \ldots, x_{i+n}^{(5)}, x_i^{(6)}, x_i^{(8)}\} \text{ in } G'_1 \text{ and } |N(x_i^{(5)})| = n + 2, \text{ in } G'_1.$$  

$$N(N(x_i^{(5)})) = \{x_{i+1}^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)}, \ldots, x_{i+n}^{(4)}, x_i^{(3)}, x_i^{(1)}\} \text{ in } G_1 \text{ and } |N(N(x_i^{(5)}))| = n + 2, \text{ in } G_1.$$  

$d_2$ of each vertex in $C^{(4)}$, where $C^{(4)}$ is the cycle induced by the vertices $\{x_i^{(4)} : 0 \leq i \leq n\}$

$$N_2(x_i^{(4)}) \text{ in } G_2 = N_2(x_i^{(4)}) \text{ in } G_1 \cup N(x_i^{(5)}) \text{ in } G'_1 \cup N(N(x_i^{(4)})) \text{ in } G'_1$$  

$$= N_2(x_i^{(4)}) \text{ in } G_1 \cup \{x_{i+1}^{(5)}, x_{i+2}^{(5)}, x_{i+3}^{(5)}, \ldots, x_i^{(6)}, x_i^{(8)}\} \text{ in } G'_1$$  

$$+ \{x_{i+1}^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)}, \ldots, x_i^{(3)}, x_i^{(1)}\} \text{ in } G_1.$$  

Note that $x_{i+1}^{(5)}, x_{i+2}^{(5)}, x_{i+3}^{(5)}, \ldots, x_i^{(5)}$ are the common elements in $N(x_i^{(5)})$ in $G'_1$ and $N(N(x_i^{(4)}))$ in $G'_1$.

$$d_2(x_i^{(4)}) \text{ in } G_2 = d_2(x_i^{(4)}) \text{ in } G_1 + (d(x_i^{(5)}) \text{ in } G'_1 + |N(N(x_i^{(4)}))| \text{ in } G'_1) - n.$$  

$$= 2(n + 1) + (n + 2 + n + 2) - (n) = 3(n + 2)$$  

$$= [(n + 3) - (n)](n + 3 - 1), (0 \leq i \leq n).$$

$d_2$ of each vertex in $C^{(5)}$, where $C^{(5)}$ is the cycle induced by the vertices $\{x_i^{(5)} : 0 \leq i \leq n\}$

$$N_2(x_i^{(5)}) \text{ in } G_2 = N_2(x_i^{(5)}) \text{ in } G'_1 \cup N(x_i^{(4)}) \text{ in } G_1 \cup N(N(x_i^{(5)})) \text{ in } G_1$$  

$$= N_2(x_i^{(5)}) \text{ in } G'_1 \cup \{x_{i+1}^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)}, x_i^{(3)}, x_i^{(1)}\} \text{ in } G_1$$  

$$+ \{x_{i+1}^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)}, x_i^{(3)}, x_i^{(1)}\} \text{ in } G_1.$$  

It is observed that $x_{i+1}^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)}, \ldots, x_i^{(4)}$ are the common elements in $N(x_i^{(4)})$ in $G_1$ and $N(N(x_i^{(5)}))$ in $G_1$.

$$d_2(x_i^{(5)}) \text{ in } G_2 = d_2(x_i^{(5)}) \text{ in } G'_1 + (d(x_i^{(4)}) \text{ in } G_1 + |N(N(x_i^{(5)}))| \text{ in } G_1) - n.$$  

$$d_2(x_i^{(5)}) \text{ in } G_2 = 2(n + 1) + (n + 2 + n + 2) - (n) = 3(n + 2)$$  

$$= [(n + 3) - (n)](n + 3 - 1), (0 \leq i \leq n).$$
In $G_2$, for $1 \leq t \leq 8$, $d_2(x_i^{(t)}) = [(n + 3) - (n)](n + 3 - 1)$, $(0 \leq i \leq n)$. $G_2$ is $(n + 3, 2, ((n + 3) - (n))(n + 3 - 1))$-regular on $(n + 1) \times 2^{n+3-n} = 8(n+1)$ vertices with the vertex set $V(G_2) = \{x_i^{(t)} : 1 \leq t \leq 2^{n+3-n}, 0 \leq i \leq n\}$ and edge set $E(G_2) = E(G_1) \cup E(G_1') \cup \{x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, x_i^{(4)}, x_i^{(5)} : 0 \leq i \leq n\}$. Hence the result is true for $r = n + 3$.

Let us assume that this result is true for $r = m + n + 1$.

Then there exists a $(m + n + 1, 2, (m + 1)(m + n))$-regular graph on $(n + 1) \times 2^{m+1}$ vertices with the vertex set $V(G_m) = \{x_i^{(t)} : 1 \leq t \leq 2^{m+1}, 0 \leq i \leq n\}$ and edge set $E(G_m) = E(G_{m-1}) \cup E(G_{m-1}') \cup \{x_i^{(t)} : 1 \leq t \leq 2^{m+1}, 0 \leq i \leq n\}$.

Then for $1 \leq t \leq 2^{m+1}$, $d_2(x_i^{(t)}) = (m+1)(m+n)$, and $d(x_i^{(t)}) = m+n+1,(0 \leq i \leq n)$.

Take another copy of $G_m$ as $G'_m$ with the vertex set $V(G'_m) = \{x_i^{(t)} : 2^{m+1} + 1 \leq t \leq 2^{m+2}, (0 \leq i \leq n)\}$ and each $x_i^{(t)}(2^{m+1} + 1 \leq t \leq 2^{m+2})$, corresponds to $x_i^{(t)}, (1 \leq t \leq 2^{m+1})$, for $(0 \leq i \leq n)$.

The desired graph $G_{m+1}$ has the vertex set $V(G_{m+1}) = V(G_m) \cup V(G_m')$ and edge set $E(G_{m+1}) = E(G_m) \cup E(G_m') \cup \{x_i^{(t)} : 1 \leq t \leq 2^{m+2}, 0 \leq i \leq n\}$.

Now, the resulting graph $G_m+1$ is $(m + n + 2)$ regular graph having $(n + 1) \times 2^{m+2}$ vertices.

Consider the edges $\bigcup_{t=1}^{2^{m+1}} \{x_i^{(t)} : 2^{m+2-t+1} : 0 \leq i \leq n\}$.

For $(1 \leq t \leq 2^{m+1})$, $d_2$ of each vertex in $C^{(t)}$, where $C^{(t)}$ is the cycle induced by the vertices $\{x_i^{(t)} : 0 \leq i \leq n\}$.

\begin{align*}
N_2(x_i^{(t)}) & \text{ in } G_{m+1} = N_2(x_i^{(t)}) \text{ in } G \cup N(x_i^{2^{m+2-t+1}+t+2}) \text{ in } G'_m \cup N(N(x_i^{(t)})) \text{ in } G_m' \\
d_2(x_i^{(t)}) & \text{ in } G_{m+1} = d_2(x_i^{(t)}) \text{ in } G_m + d(x_i^{2^{m+2-t+1}+t+2}) \text{ in } G'_m + |N(N(x_i^{(t)}))| \text{ in } G'_m \\
& = (m+1)(m+n) + ((m+n+1) + (m+n+1)) - n, (0 \leq i \leq n). \\
& = (m+2)(m+n+1), (0 \leq i \leq n).
\end{align*}

$d_2$ of each vertex in $C^{(2^{m+2-t+1})}$, where $C^{(2^{m+2-t+1})}$ is the cycle induced by
the vertices \( \{x_i^{(2^{m+2} - t + 1)} : 0 \leq i \leq n\} \).

\[
N_2(x_i^{2^{m+2} - t + 1} \mod 2) \in G_{m+1} = N_2(x_i^{2^{m+2} - t + 1} \mod 2) \in G'_m
+ N(N(x_i^{(t)} \mod 2) \in G_m + |N(N(x_i^{2^{m+2} - t + 1} \mod 2))| \in G_m
\]

\[
d_2(x_i^{2^{m+2} - t + 1} \mod 2) \in G_{m+1} = (m+1)(m+n) + ((m+n+1) + (m+n+1)) - n
= (m+2)(m+n+1), (0 \leq i \leq n).
\]

In \( G_{m+1} \), \( d_2(x_i^{(t)}) = (m+2)(m+n+1), (0 \leq i \leq n),(1 \leq t \leq 2^{m+2}) \)
Then there exists a \( (m + n + 2, 2, (m + 2)(m + n + 1)) \)-regular graph on \( (n+1) \times 2^{m+2} \)
vertices with the vertex set \( V(G_m) = \{x_i^{(t)} : 1 \leq t \leq 2^{m+2}, 0 \leq i \leq n\} \) and
\( E(G_{m+1}) = E(G_m) \cup E(G'_m) \bigcup \{x_i^{(t)} x_i^{2^{m+2} - t + 1} \ : \ 0 \leq i \leq n\} \).

Then for \( 1 \leq t \leq 2^{m+2}, d_2(x_i^{(t)}) = (m+2)(m+n+1) \) and \( d(x_i^{(t)}) = m+n+2, \)
\( 0 \leq i \leq n \).

If the result is true for \( r = m+n+1 \), then it is true for \( r = m+n+2 \).
Hence the result is true for all \( r \geq n \).

Then for any \( r \geq n \geq 2 \), there is an \( (r, 2, (r-n)(r-1)) \)-regular graph on \( (n+1) \times 2^{r-n} \)
vertices.

\( \square \)

**Corollary 3.11.5.** For any \( r \geq 2 \), there is an \( (r, 2, (r-2)(r-1)) \)-regular graph
on \( 3 \times 2^{r-2} \) vertices [38].

**Corollary 3.11.6.** For any \( r \geq 3 \), there is an \( (r, 2, (r-3)(r-1)) \)-regular graph
on \( 4 \times 2^{r-3} \) vertices [39].

**Corollary 3.11.7.** For any \( r \geq 4 \), there is an \( (r, 2, (r-4)(r-1)) \)-regular graph
on \( 5 \times 2^{r-4} \) vertices.

**Summary 3.11.8.** In Theorem 3.11.2, if \( n = 2, 3, 4, \ldots, r \), then there are \( (r, 2, (r, 2)(r-1)) \)-regular graph, \( (r, 2, (r-3)(r-1)) \)-regular graph, \( (r, 2, (r-4)(r-1)) \)-regular graph, \( (r, 2, (r-5)(r-1)) \)-regular graph, \( \ldots \), \( (r, 2, 4(r-1)) \)-regular graph, \( (r, 2, 3(r-1)) \)-regular graph, \( (r, 2, 2(r-1)) \)-regular graph, \( (r, 2, (r-1)) \)-regular graph and \( (r, 2, 0) \)-regular graph.
3.12 \((r, 2, m(r-1))\)-Regular Graphs

The existence of an \((r, 2, m(r-1))\)-regular graph, for any positive integer \(m \leq r\) is proved in this section[42].

**Definition 3.12.1.** A graph \(G\) is \((r, 2, m(r-1))\)-regular if each vertex in the graph \(G\) is at a distance one from \(r\) vertices and each vertex in the graph \(G\) is at a distance two from exactly \(m(r-1)\) vertices.

The motivation is to construct the \((r, 2, k)\)-regular graphs for all values of \(k\) from 0 to \(r(r-1)\), for any \(r\). With this motivation, some \((r, 2, k)\)-regular graphs in [33, 34, 38, 39] have been constructed. Now, let us look at the graphs in the following constructions.

1. For any \(n \geq 5, (n \neq 6, 8)\) and any \(r > 1\), there exists an \((r, 2, r(r-1))\)-regular graph on \(n \times 2^{r-2}\) vertices with girth five. The graphs given in Figure 3.12, illustrate the construction in [33], for \(r = 2, 3, 4\).

![Figure 3.12](image)

2. For any \(r \geq 2\), there is an \((r, 2, (r-1)(r-1))\)-regular graph on \(4 \times 2^{r-2}\) vertices. Then the graphs given in Figure 3.13, illustrate the construction in [37], for \(r = 2, 3, 4\).
3. For any $r \geq 2$, there is an $(r, 2, (r - 2)(r - 1))$-regular graph on $3 \times 2^{r-2}$ vertices. Then the graphs given in Figure 3.14, illustrate the construction in [38], for $r = 2, 3, 4$.

4. For any $r \geq 3$, there is an $(r, 2, (r - 3)(r - 1))$-regular graph on $4 \times 2^{r-3}$ vertices. Then the graphs given in Figure 3.15, illustrate the construction in [39], for $r = 3, 4, 5$.

5. For any $r \geq n \geq 2$, there exists an $(r, 2, (r - n)(r - 1))$-regular graph on $(n + 1) \times 2^{r-n}$ vertices [47].
Theorem 3.12.2. For any positive integer \( m \leq r \), there is an \((r, 2, m(r-1))\)-regular graph.

Proof. If \( m \leq r \), then \( m = 0, 1, 2, 3, 4, 5, \ldots, (r-4), (r-3), (r-2), (r-1), r \).

If \( m = r \), then there is an \((r, 2, r(r-1))\)-regular graph, since for any \( n \geq 5, (n \neq 6, 8) \) and any \( r > 1 \), there exists an \((r, 2, r(r-1))\)-regular graph[33].

If \( m = r-1 \), then there is an \((r, 2, (r-1)(r-1))\)-regular graph, since for any \( r \geq 2 \), there is a \((r, 2, (r-1)(r-1))\)-regular graph on \( 4 \times 2^{r-2} \) vertices [37].

If \( m = r-2 \), then there is an \((r, 2, (r-2)(r-1))\)-regular graph, since for any \( r \geq 2 \), there is an \((r, 2, (r-2)(r-1))\)-regular graph on \( 3 \times 2^{r-2} \) vertices [38].

If \( m = r-3 \), then there is an \((r, 2, (r-3)(r-1))\)-regular graph, since for any \( r \geq 3 \), there is an \((r, 2, (r-3)(r-1))\)-regular graph on \( 4 \times 2^{r-3} \) vertices [39].

If \( m = r-n \), then for any \( r \geq n \geq 2 \), there exists an \((r, 2, (r-n)(r-1))\)-regular graph on \((n+1) \times 2^{-n} \) vertices[43].

If \( m = 1 \), then there is an \((r, 2, (r-1))\)-regular graph of order \( 2r \). Then the complete bipartite graph \( K_{r,r} \) is the required graph.

If \( m = 0 \), then there is an \((r, 2, 0)\)-regular graph of order \( (r+1) \).

Then the complete graph on \( r+1 \) vertices is the required graph.

Hence for any positive integer \( m \leq r \), there is an \((r, 2, m(r-1))\)-regular graph. \( \square \)

From Theorem 3.12.2, the graphs up to \( r = 4 \) are listed out in the following table.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( m )</th>
<th>((r, 2, m(r-1))-\text{Regular Graph})</th>
</tr>
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<tr>
<td>1</td>
<td>1</td>
<td>(1,2,0)-regular graph</td>
</tr>
<tr>
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<td>1</td>
<td>(2,2,1)-regular graph</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>(2,2,2)-regular graph</td>
</tr>
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<td>(4,2,9)-regular graph</td>
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<tr>
<td>4</td>
<td>4</td>
<td>(4,2,12)-regular graph</td>
</tr>
</tbody>
</table>
CHAPTER 3. \((r, 2, m(r - 1))\) - REGULAR GRAPH

**Remark 3.12.3.** From the table, it is noted that if \(r = 1, 2\) then there exists \((r, 2, k)\) - regular graphs for all values of \(k\). But if \(r = 3\), then existing graphs are \((3, 2, 2)\)-regular, \((3, 2, 4)\)-regular and \((3, 2, 6)\)-regular. There is no \((3, 2, 3)\)- regular graph and \((3, 2, 5)\)-regular graph due to the constructions in this chapter. There exists \((r, 2, k)\)-regular graphs for \(k\) is a multiple of \(r - 1\).

**Conclusion and Scope:** For further investigation, the following Open problem posed.

1. Can \((r, 2, k)\)- regular graphs for all values of \(k\) lies between 0 and \(r(r - 1)\) be constructed?.

2. For \(m > 2\), construct \((r, m, k)\)-regular graphs for all values of \(k\) lies between 0 and \(r(r - 1)^{m-1}\)

3. Both \(r\) and \(k\) are odd and \(k < r\). Is there any \((r, 2, k)\)-regular graph?