Chapter 2

\((r, m, k)\) - Regular Graphs

In this chapter, \((r, m, k)\)-regular graph is defined and the few properties possessed by \((r, m, k)\)-regular graph are studied. Also, \((r, 2, k)\)-regular graph is defined and a method to construct \((r, 2, k)\)-regular graph containing a given graph is suggested. Also, it is shown that some minimal \((r, 2, k)\)-regular graphs contain a given graph as an induced subgraph, and some minimal \((r, 2, k)\)-regular graphs contain a given graph and its complement as induced subgraphs.

2.1 Introduction

The concept of distance degree regular graph was introduced and studied by G. S. Bloom, J. K. Kennedy and L. V. Quintas [5, 8, 18]. A graph \(G\) is said to be distance \(d\)-regular if every vertex of \(G\) has the same number of vertices at a distance \(d\) from it.

If each vertex of \(G\) has exactly \(k\) number of vertices at a distance \(m\) from it, then this graph is called as \((m, k)\)-regular graph. An \(r\)-regular graph which is \((m, k)\)-regular is called \((r, m, k)\)-regular graph. In otherwords, a graph is said to be \(G\) is said to be \((r, m, k)\)-regular if each vertex in the graph \(G\) is at a distance one from exactly \(r\)-vertices in \(G\) and at a distance \(m\) from exactly \(k\)-vertices in \(G\).

The concept of \((m, k)\)-regular graph is a natural extension of the idea of regu-
regular graph. The \((1, r)\)-regular graphs are nothing but the usual \(r\)-regular graphs. A graph \(G\) is said to be \((2, k)\)-regular graph if each vertex of \(G\) is at a distance two away from exactly \(k\) vertices.

A graph \(G\) is said to be \((r, 2, k)\)-regular if each vertex in the graph \(G\) is at a distance one from exactly \(r\)-vertices in \(G\) and at a distance 2 from exactly \(k\) vertices in \(G\).

It should be noted that \((m, k)\)-regular graph may be regular or non-regular.

**Example 2.1.1.** Among the following four \((2, k)\)-regular graphs given in Figure 2.1, the first two are regular whereas the last two are non-regular.

![Figure 2.1](image)

An induced subgraph of \(G\) is a subgraph \(H\) of \(G\) such that \(E(H)\) consists of all edges of \(G\) whose end points belong to \(V(H)\).

In 1936, König proved that if \(G\) is any graph, whose largest degree is \(r\), then there is an \(r\)-regular graph \(H\) containing \(G\) as an induced subgraph\[20]\). In 1963, Paul Erdos and Paul Kelly determined the smallest number of new vertices which must be added to a given graph \(G\) to obtain such a graph\[12]\).

In 1987, Y. Alavi, Gary Chartrand, E. R. K. Chung, Paul Erdos, R. L. Graham, O. R. Oellerman proved that every graph of order \(n \geq 2\) is an induced subgraph of a highly irregular graph of order \(4n - 4\)\[1\]. A connected graph \(G\) is highly irregular if every vertex of \(G\) is adjacent only to vertices with distinct degrees.
The concept of semiregular graph was introduced and studied by Alison Northup [3]. A connected graph \( G \) is \textit{semiregular} if each vertex in the graph is at a distance two away from exactly the same number of vertices.

The above results have been the inspiration to suggest a method to construct \((m + n - 1, 2, mn - 1)\)-regular graph of order \(2mn\) containing a given graph \( G \) of order \( n \geq 2 \) as an induced subgraph, for any \( m \geq 1 \). Also, it is proved that a \(((m + n - 2), 2, (m - 1)(n - 1))\)-regular graph of smallest order \( mn \) containing a given graph \( G \) of order \( n \geq 2 \) as an induced subgraph for any \( m > 1 \). Further, it is shown that a \((m + 2(n - 1), 2, (m - 1)(2n - 1))\)-regular graph of smallest order \( 2mn \) containing a given graph \( G \) of order \( n \geq 2 \), and its complement \( G^c \) as induced subgraphs, for any \( m \geq 1 \).

\section{(m, k)-Regular Graphs}

\textbf{Definition 2.2.1.} A graph \( G \) is \((m, k)\)-\textit{regular} if each vertex in graph \( G \) is at a distance \( m \) from exactly \( k \) vertices. Then, a graph \( G \) is said to be \((m, k)\)-\textit{regular} if \( d(v) = r \) and \( d_m(v) = k \), for all \( v \) in \( G \).

\textbf{Remark 2.2.2.} There exists two types of \((m, k)\)-regular graphs. They are non-regular graphs which are \((m, k)\)-regular and regular graphs which are \((m, k)\)-regular.

\textbf{Definition 2.2.3.} A graph \( G \) is \((2, k)\)-\textit{regular} if each vertex in graph \( G \) is at a distance two from exactly \( k \) vertices. A graph \( G \) is said to be \((2, k)\)-\textit{regular} if \( d(v) = r \) and \( d_2(v) = k \), for all \( v \) in \( G \).[34]

\textbf{Example 2.2.4.} (i) Regular graphs which are \((2, k)\)-regular.

(a) The first two graphs shown in Figure 2.1 are regular graphs which are \((2, k)\)-regular.

(b) Any complete \( m \)-partite graph \( K_{n_1,n_2,n_3,...,n_m} \) is \((2, k)\)-regular iff \( n_1 = n_2 = n_3 = n_4 = \cdots = n_m = k + 1 \).

(ii) Non-regular graphs which are \((2, k)\) - regular.
(a) Graphs shown in Figure 2.2 are non-regular graphs which are \((2, k)\)-regular.

(b) Book graph \(B_n = S_n \times P_2, (n \geq 2)\) (where \(S_n\) is the star graph of order \(n\) and \(P_2\) is the path of order 2) is \((2, (n - 1))\)-regular graph.

(c) Let \(H_{n,n}\) denote the bipartite graph having two partite sets \(V_1 = \{v_1, v_2, v_3, v_4, \ldots, v_n\}\) and \(V_2 = \{u_1, u_2, u_3, \ldots, u_n\}\) and edge set \(E(H_{n,n}) = \bigcup_{i=1}^{n} E_i\), where \(E_i = \{v_i u_j : n - i + 1 \leq j \leq n\}\). This graph \(H_{n,n}\) is \((2, (n - 1))\)-regular graph[1].

(d) Splitting graph \(S(C_n), (n \geq 5)\) is \((2, 5)\)-regular[49].

(e) Sunflower graph \(SF_n, (n \geq 5)\) is \((2, 4)\) - regular graph[49].

(iii) The \((2, k)\)-regular graph which is not \((3, k)\)-regular and \((2, k)\)-regular graph which is \((3, k)\)-regular.

(a) \(S(C_5)\) is a \((2, 5)\)-regular graph which is not \((3, k)\)-regular.

(b) \(S(C_6)\) is \((3, 4)\)-regular which is \((2, 5)\)-regular.

It is observed that \((2, k)\) - regular and \(k\) - semiregular graph are the same. The concept of the semiregular graph was introduced and studied by Alison Northup [3]. A graph \(G\) is said to be \(k\)-semiregular graph if each vertex of \(G\) is at distance two from exactly \(k\) vertices of \(G\). The following facts on semiregular graph are known from literature[3].
Fact 2.2.5. The \( n \)-Barbell graph is \( n \)-semiregular, for all \( n \geq 0 \).

Fact 2.2.6. All connected vertex transitive graphs are semiregular.

Fact 2.2.7. A connected graph is 0-semiregular if and only if it is a complete graph \( K_n \) for \( n \geq 1 \).

Fact 2.2.8. Every finite 1-semiregular graph has an even number of vertices.

Fact 2.2.9. A connected graph is 1-semiregular if and only if it is \( P_4 \) or complement of \( P_4 \) for \( n \geq 2 \).

Fact 2.2.10. A finite tree is semiregular if and only if it is \( P_1 \) or member of the Barbell class.

Fact 2.2.11. Let \( G \) be a \( n \)-semiregular graph. Let \( G^* \) be defined as the graph with the same vertex set as \( G \) such that \( v_1 \) and \( v_2 \) are connected in \( G^* \) if and only if they are at a distance two away from each other in \( G \). Then \( G^* \) is \( n \)-regular.

Fact 2.2.12. If \( G \) is an \( n \)-regular graph, let \( G_1 \) be defined by inserting two vertices on to each edge of \( G \). Then \( G_1 \) is an \( n \)-semiregular graph.

Also, Alison Northup presented an algorithm for determining whether a graph is \( k \)-semiregular or not.

Now the existence of some \((2, k)\)-regular graphs is discussed below.

Example 2.2.13. For any \( n \geq 2 \), Complete graph \( K_n \), with a pendant vertex attached to each vertex of \( K_n \) is of order \( 2n \). This graph is a \((2, n - 1)\)-regular graph with clique number \( n \) and independent number \( n \) having the smallest order \( 2n \).

Graphs in Figure 2.4 are \((2, n - 1)\)-regular graphs with clique number \( n \) and independent number \( n \) for \( n = 3, 4, 5 \).

![Figure 2.4](image)
Theorem 2.2.14. For any $l > 1$, there exists non-regular graph which is $(2, 2l - 1)$-regular.

Proof. Let $G_l$ be a graph obtained from two disjoint copies of $K_{l,l}$ by adding a matching between two partite sets of size $l$. This graph $G_l$ is non-regular graph which is $(2, 2l - 1)$-regular of order $4l$. □

Theorem 2.2.15. For any $m \geq 1$, there exists a non-regular graph which is $(2, m)$-regular.

Proof. Let $G_m$ be a graph obtained from two disjoint copies of $K_{1,m}$ by adding a matching between the two partite sets of size $m$. This graph $G_m$ is non-regular graph of order $2m + 2$ which is $(2, m)$-regular. □

Theorem 2.2.16. For any $n \geq 1$, there exists a non-regular graph of order $2n$ which is $(2, n - 1)$-regular.

Proof. Let $F_n$ be a connected graph of order $n$ with the property that for every pair $u, v$ of distinct vertices $d(u) \neq d(v)$ with exactly one exception. In particular, let $V(F_n) = \{u_1, u_2, u_3, \ldots, u_n\}$, where

$$d(u_i) = \begin{cases} n - i & \text{if } 1 \leq i \leq \lceil n/2 \rceil \\ n - i + 1 & \text{if } \lceil n/2 \rceil + 1 \leq i \leq n \end{cases}$$

Let $F'_n$ be another copy of $F_n$, where $V(F'_n) = \{v_1, v_2, \ldots, v_n\}$ and $v_i$ corresponds to $u_i$, $(1 \leq i \leq n)$. For $n > 1$, define $G_n$ to consist of $F_n$ and $F'_n$ together with the edges $u_iv_i$, where $1 \leq i \leq \lceil n/2 \rceil$. This graph $G_n$ is non-regular graph of order $2n$ which is $(2, n - 1)$-regular. □

Example 2.2.17. Figure 2.5 illustrates the Theorem 2.2.16 for $n = 4$.

![Figure 2.5](image-url)
2.3 \((r, m, k)\)-Regular Graphs

**Definition 2.3.1.** A graph \(G\) is \((r, m, k)\)-regular if each vertex in the graph \(G\) is at a distance one from exactly \(r\) vertices and at a distance \(m\) from exactly \(k\) vertices.

**Example 2.3.2.** (*) \((r, 2, k)\)-regular and \((r, 3, k)\)-regular graphs.

\[\begin{array}{ccc}
(5,2,0)\text{-regular} & (4,2,1)\text{-regular} & (4,2,3)\text{-regular} \\
(3,3,0)\text{-regular} & (3,3,4)\text{-regular} \\
\end{array}\]

(*) Complete graph \(K_m\), \((m > 1)\) vertices is \((m - 1, 1, m - 1)\)-regular.

(*) Any \(r\)-regular graph with diameter less than \(m\) is \((r, m, 0)\)-regular graph.

(*) Complete graph \(K_m\) vertices is \((m - 1, 2, 0)\)-regular

(*) Complete bipartite graph \(K_{n,n}\) is \((n, 2, (n - 1))\)-regular graph.

(*) Cycle \(C_{2m}\) is \((2, m, 1)\)-regular.

(*) Cycle \(C_{2m+1}\) is \((2, m, 2)\)-regular.

(*) Petersen graph is \((3, 2, 6)\)-regular.

**Theorem 2.3.3.** Any \((r, m, k)\)-regular graph has at least \(k + r + 1\) vertices.

**Proof.** Let \(G\) be a \((r, m, k)\)-regular graph. Then, each vertex \(v\) is adjacent with \(r\)-vertices and non-adjacent with at least \(k\)-vertices. Hence \(G\) has at least \(k + r + 1\) vertices. \(\square\)
Theorem 2.3.4. Any \((m, k)\)-regular graph with maximum degree \(r\) has at least \(k + r + 1\) vertices.

Theorem 2.3.5. If \(r\) and \(k\) are odd, then \((r, m, k)\)-regular graph has at least \(k + r + 2\) vertices.

Proof. If \(r\) and \(k\) are odd, then there is no \((r, m, k)\)-regular graph of order \(r + k + 1\), since odd regular graphs have only even number of vertices. \(\square\)

2.4 \((r, 2, k)\)-Regular Graphs

Examples are given for the existence of some \((r, 2, k)\)-regular graphs and a few properties possessed by \((r, 2, k)\)-regular graph are studied in this section\([34, 38]\).

Definition 2.4.1. A graph \(G\) is said to be \((r, 2, k)\)-regular if each vertex in the graph \(G\) is at a distance one from exactly \(r\)-vertices in \(G\) and at a distance two from exactly \(k\) vertices in \(G\) \((d(v) = r \text{ and } d_2(v) = k, \text{ for all } v \in V(G))\).

Example 2.4.2. \((r, 2, k)\)-regular graphs.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{regular_graphs}
\caption{\((3, 2, 0)\)-regular, \((3, 2, 2)\)-regular, \((3, 2, 4)\)-regular}
\end{figure}

Theorem 2.4.3. For any \(r \geq 1\), there exists an \((r, 2, r - 1)\)-regular graph of order \(2r\).

Proof. For any \(r \geq 1\), the complete bipartite graph \(K_{r,r}\) is an \((r, 2, r - 1)\)-regular graph of order \(2r\). \(\square\)

Theorem 2.4.4. For any \(r \geq 1\), there exists an \((r, 2, 2(r - 1))\) - regular graph of order \(4r - 2\).

Proof. For any \(r \geq 1\), let \(F_r\) be the graph obtained from two disjoint copies of \(K_{r-1,r}\) by adding a matching between the two partite sets of size \(r\). This graph \(F_r\) is \((r, 2, 2(r - 1))\)- regular graph of order \(4r - 2\). \(\square\)
**Theorem 2.4.5.** For any $r \geq 2$ and $k \geq 1$, $G$ is a $(r, 2, k)$-regular graph of order $r + k + 1$ if and only if $\text{diam}(G) = 2$.

**Proof.** Suppose $G$ is a $(r, 2, k)$-regular graph with $r + k + 1$ vertices such that $r \geq 2$ and $k \geq 1$. Let $v$ be any vertex of $G$, and $v$ is adjacent to $r$-vertices $v_1, v_2, v_3, \ldots, v_r$. Then $d(v, v_i) = 1$, $(1 \leq i \leq r)$. Also, $v$ is at a distance two away from exactly $k$ vertices $u_1, u_2, u_3, \ldots, u_k$. Then $d(v, u_i) = 2$, $(1 \leq i \leq k)$. Hence $\text{diam}(G) = \max\{d(u, v)/u, v \in V(G)\} = 2$.

Conversely, let $\text{diam}(G) = 2$ and $G$ be a $r$-regular graph with $n$ vertices. Let $v$ be any vertex in $V(G)$ and $d(v) = r$, for all $v \in V(G)$. Then $v$ is adjacent with $r$ vertices and non-adjacent with $(n - r - 1)$ vertices. Since $\text{diam}(G) = 2$, then remaining $n - r - 1$ vertices are at a distance two away from $v$. Hence $G$ is a $(r, 2, n - r - 1)$-regular graph of order $n = r + (n - r - 1) + 1 = r + k + 1$. \(\square\)

### 2.5 Some $(r, 2, k)$-Regular Graphs Containing a Given Graph

König proved that if $G$ is any graph, whose largest degree is $r$, then it is possible to add new vertices and to draw new edges joining either two new vertices or a new vertex to an existing point, so that the resulting graph $H$ is a regular graph containing $G$ as an induced subgraph [20]. Paul Erdös and Paul Kelly [12] determined the smallest number of new vertices which must be added to a given graph $G$ to obtain such a graph. A method that may be considered as an analogue to König’s theorem for $(r, 2, k)$-regular graph is suggested[37].

**Example 2.5.1.** For any $n \geq 1$, the smallest order of $(n, 2, (n - 1))$-regular graph containing a complete bipartite graph $K_{n,n}$ of order $2n$ is $K_{n,n}$ itself.

**Theorem 2.5.2.** Every graph $G$ of order $n \geq 2$ is an induced subgraph of $(n + 1, 2, 2n)$-regular graph of order $5n$.

**Proof.** Let $G$ be a given graph of order $n \geq 2$ with the vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Let $G_t$ denote a copy of $G$ with the vertex set $V(G_t) =$
\{v^t_1, v^t_2, v^t_3, \ldots, v^t_n: 1 \leq t \leq 5\}. H_1 be the graph with vertex set \(V(H_1) = \bigcup_{t=1}^{5} V(G_t) = \{v^t_i: 1 \leq i \leq n, 1 \leq t \leq 5\}\) and edge set \(E(H_1) = \bigcup_{t=1}^{5} E(G_t) \bigcup_{t=1}^{4} \{v^t_jv^{t+1}_i, v^5_jv^1_i: v^1_jv^1_i \notin E(G_1), 1 \leq j \leq n, j+1 \leq i \leq n\} \bigcup_{k=1}^{n} \{v^i_kv^{i+1}_k, v^5_kv^1_k: 1 \leq i \leq 4\}\). The resulting graph \(H_1\) contains \(G\) as an induced subgraph. Moreover, in \(H_1\), \(d(v^t_i) = n + 1\) and \(d_2(v^t_i) = 2n\), for \(1 \leq i \leq n\), \((1 \leq t \leq 5)\). Hence \(H_1\) is \((n+1, 2, 2n)\) - regular graph of order \(5n\) containing a given graph \(G\) as an induced subgraph.

**Corollary 2.5.3.** Every graph \(G\) of order \(n \geq 2\) is an induced subgraph of \((n+1, 2, 2n-2)\)-regular graph of order \(3n\).

**Proof.** If 3 copies of \(G\) are taken instead of taking 5 copies of \(G\) in Theorem 2.5.2, then \(H_1\) is \((n+1, 2, 2n-2)\)-regular graph containing a graph \(G\) of order \(n \geq 2\) as an induced subgraph. \(\square\)

**Example 2.5.4.** Figure 2.8 illustrates the Corollary 2.5.3, for \(n = 3\).

In Figure 2.8, the graph \(G\) is induced by the vertices \(x, y, z\).

**Corollary 2.5.5.** Every graph \(G\) of order \(n \geq 2\) is an induced subgraph of \((n+1, 2, 2n-1)\)-regular graph of order \(4n\).

**Proof.** If 4 copies of \(G\) are taken instead of taking 5 copies of \(G\) in Theorem 2.5.2, then \(H_1\) is \((n+1, 2, 2n-1)\)-regular graph containing a graph \(G\) of order \(n \geq 2\) as an induced subgraph. \(\square\)

**Remark 2.5.6.** If we take 6, 7, 8, \ldots\ copies of \(G\) are taken instead of 5 copies of \(G\) in Theorem 2.5.2, then \(H_1\) is only \((n+1, 2, 2n)\)-regular graph of order \(6n, 7n, 8n, \ldots\). The construction given in 2.5.2 gives \((n+1, 2, 2n-2)\)-regular graph, \((n+1, 2, 2n-1)\)-regular graph, \((n+1, 2, 2n)\)-regular graph only. So, another construction for \((r, 2, k)\)-regular graph containing a given graph as an induced subgraph is given.
**Theorem 2.5.7.** For any \( m \geq 1 \), every graph \( G \) of order \( n \geq 2 \) is an induced subgraph of a \(((n + m - 1), 2, (mn - 1))\)-regular graph of order \( 2mn \).

**Proof.** Let \( G \) be a given graph of order \( n \geq 2 \) with the vertex set \( V(G) = \{v_1, v_2, v_3, \ldots, v_n\} \). Let \( G_t \) denote a copy of \( G \) with the vertex set \( V(G_t) = \{v^t_1, v^t_2, v^t_3, \ldots, v^t_m: 1 \leq t \leq m\} \). Let \( G_{r+m} \) denote a copy of \( G \) with the vertex set \( V(G_{r+m}) = \{u^r_1, u^r_2, u^r_3, \ldots, u^r_r: 1 \leq r \leq m\} \). Let \( H_2 \) be a graph with vertex set \( V(H_2) = \{v^t_i, u^r_i : 1 \leq i \leq n, 1 \leq t \leq m\} \) and the edge set \( E(H_2) = \bigcup_{t=1}^{2m} E(G_t) \bigcup \{v^t_i u^r_i, u^r_j v^t_i : v^t_j v^t_i \notin E(G_t), 1 \leq j \leq n, j + 1 \leq i \leq n\} \bigcup_{k=1}^{n} \{v^t_k u^{t+k}_k : 1 \leq i \leq m \text{ and } 0 \leq j \leq m - 1\} \) (super scripts are taken modulo \( m \)). The resulting graph \( H_2 \) contains \( G \) as an induced subgraph. Moreover, in \( H_2 \), \( d(v^t_i) = d(u^r_i) = m + n - 1, (1 \leq t \leq m) \) and \( d_2(v^t_i) = d_2(u^r_i) = mn - 1, (1 \leq i \leq n) \). Hence \( H_2 \) is \(((m + n - 1), 2, mn - 1)\)-regular graph. Thus, for any graph of order \( n \geq 2 \), there exists \(((m + n - 1), 2, mn - 1)\)-regular graph \( H_2 \) of order \( 2mn \) containing a given graph as an induced subgraph. \( \square \)

**Example 2.5.8.** Figure 2.9 illustrates Theorem 2.5.7, for \( m = 2 \) and \( n = 3 \).

![Figure 2.9](image)

In Figure 2.9, the graph \( G \) is induced by the vertices \( x, y, z \).

**Corollary 2.5.9.** Every graph \( G \) of order \( n \geq 2 \), is an induced subgraph of \((n, 2, n - 1)\)-regular graph of order \( 2n \).

**Proof.** This result is the particular case of Theorem 2.5.7, for \( m = 1 \). Let \( G \) be a graph of order \( n \geq 2 \) with vertex set \( V(G) = \{v_1, v_2, v_3, \ldots, v_n\} \). Let \( G_1 \) denote a copy of \( G \) with the vertex set \( V(G_1) = \{v^1_1, v^1_2, v^1_3, \ldots, v^1_n\} \). Let \( G_2 \) denote a copy of \( G \) with the vertex set \( V(G_2) = \{u^1_1, u^1_2, u^1_3, \ldots, u^1_n\} \). Let \( H_2 \) be a graph with the vertex set \( V(H_2) = V(G_1) \cup V(G_2) \) and edge set \( E(H_2) = E(G_1) \cup E(G_2) \cup \{v^1_j v^1_i: v^1_j v^1_i \notin E(G_1), 1 \leq j \leq n, j + 1 \leq i \leq n\} \bigcup_{k=1}^{n} \{v^1_k u^1_k\} \). The resulting graph \( H_2 \) contains \( G \) as an induced subgraph.
In $H_2$, $d(v_i^1) = d(u_i^1) = n$ and $d_2(v_i^1) = d_2(u_i^1) = n - 1$, $(1 \leq i \leq n)$. Hence $H_2$ is $(n, 2, n - 1)$-regular graph of order $2n$ containing a given graph $G$ of order $n \geq 2$, as an induced subgraph. \hfill \Box

**Example 2.5.10.** Graphs in Figure 2.10 illustrates Corollary 2.5.9 for $n = 3$. 

![Figure 2.10](image)

In Figure 2.10, the graph $G$ is induced by the vertices $x, y, z$.

**Corollary 2.5.11.** Every graph $G$ of order $n \geq 2$, is an induced subgraph of $(n + 1, 2, 2n - 1)$-regular graph of order $4n$.

**Corollary 2.5.12.** Every graph $G$ of order $n \geq 2$, is an induced subgraph of $(n + 2, 2, 3n - 1)$-regular graph of order $6n$.

**Remark 2.5.13.** If $m = 1, 2, 3, 4, 5, \ldots, n$, then there are $(n, 2, n - 1), (n + 1, 2, 2n - 1), (n + 2, 2, 3n - 1), (n + 3, 2, 4n - 1), (n + 4, 2, 5n - 1), \ldots (2n, 2, n^2 - 1)$-regular graphs of order $2n, 4n, 6n, 8n, 10n, \ldots, 2n^2, \ldots$ containing a given graph $G$ of order $n \geq 2$ as an induced subgraph.

### 2.6 Minimal $(r, 2, k)$-Regular Graphs Containing a Given Graph as an Induced Subgraph

Smallest $(r, 2, k)$-regular graphs containing a given graph as an induced subgraphs are constructed in this section[40].

**Theorem 2.6.1.** For any $m > 1$, every graph $G$ of order $n \geq 2$ is an induced subgraph of a $(n + m - 2, 2, (m - 1)(n - 1))$-regular graph $H_3$ of order $mn$.

**Proof.** Let $G$ be a given graph of order $n \geq 2$ with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let $G_t$ denote a copy of $G$ with $V(G_t) = \{v_1^t, v_2^t, v_3^t \cdots, v_n^t : 1 \leq t \leq m\}$. Let $H_3$ be
the graph with vertex set \( V(H_3) = \{v^t_i : 1 \leq j \leq n, 1 \leq t \leq m \} = \bigcup_{t=1}^m V(G_t) \) and the edge set \( E(H_3) = \bigcup_{t=1}^m E(G_t) = \bigcup_{t=1}^{m-1} \{v^t_iv^{t+1}_j, v^mv^n_1 : v^t_i v^n_j \notin E(G_t), 1 \leq j \leq n, j+1 \leq i \leq n \} \cup \{v^k_iv^{k+j}_j : 1 \leq i \leq m-1, 1 \leq j \leq m-i \} \).

The resulting graph \( H_3 \) contains \( G \) as an induced subgraph. Moreover in \( H_3 \), \( d(v^t_i) = m + n - 2 \), \((1 \leq i \leq n), (1 \leq t \leq m) \). Then, \( H_3 \) is \((m + n - 2)\)-regular graph with \( mn \) vertices.

To find the \( d_2 \)-degree of each vertex in \( H_3 \), the following cases are examined:

**Case 1** When \( t = 1 \). If \( v \in V(G_1) \), then \( v = v^t_j \), for some \( j \). Let \( v^t_j \in V(H_3) - N[v^t_i] \). Then \( v^t_j \) and \( v^n_1 \) are non-adjacent vertices in \( H_3 \). By the construction, \( v^t_j \) is adjacent to \( v^t_i \) and \( v^n_1 \) is adjacent to \( v^t_i \). Then \( d(v^t_j, v^n_1) = 2 \). This implies that \( v^t_j \in N_2(v^n_1) \). Hence \( V(H_3) - N[v^t_i] \subseteq N_2(v^n_1) \). If \( v^t_j \in N_2(v^n_1) \), then \( v^t_j \) is non-adjacent with \( v^n_1 \). This implies that \( v^t_j \in V(H_3) - N[v^t_i] \). Hence \( N_2(v^t_j) = V(H_3) - N[v^t_i] \), \((1 \leq i \leq n) \) and \( d_2(v^t_i) = (m - 1)(n - 1), (1 \leq i \leq n) \).

**Case 2** When \( 2 \leq t \leq m - 1 \).

If \( v \in V(G_t) \), then \( v = v^t_j \), for some \( j \). Let \( v^t_j \in V(H_3) - N[v^t_i] \), Then \( v^t_j \) and \( v^n_1 \) are non-adjacent vertices in \( H_3 \). By the construction, \( v^t_j \) is adjacent to \( v^t_i \) and \( v^n_1 \) is adjacent to \( v^t_i \). Then \( d(v^t_j, v^n_1) = 2 \). This implies that \( v^t_j \in N_2(v^n_1) \). Hence \( V(H_3) - N[v^t_i] \subseteq N_2(v^n_1) \). If \( v^t_j \in N_2(v^n_1) \), then \( v^t_j \) is non-adjacent with \( v^n_1 \). This implies that \( v^t_j \in V(H_3) - N[v^t_i] \). Hence \( N_2(v^t_j) = V(H_3) - N[v^t_i] \), \((1 \leq i \leq n) \) and \( d_2(v^t_j) = (m - 1)(n - 1), (1 \leq i \leq n) \).

**Case 3** When \( t = m \). If \( v \in V(G_m) \), then \( v = v^m_j \), for some \( j \). Let \( v^m_j \in V(H_3) - N[v^t_i] \), then \( v^m_j \) and \( v^n_1 \) are non-adjacent vertices in \( H_3 \). By the construction, \( v^m_j \) is adjacent to \( v^m_i \) and \( v^n_1 \) is adjacent to \( v^m_i \). Then \( d(v^m_j, v^n_1) = 2 \). This implies that \( v^m_j \in N_2(v^n_1) \). Hence \( V(H_3) - N[v^m_i] \subseteq N_2(v^n_1) \). If \( v^m_j \in N_2(v^n_1) \), then \( v^m_j \) is non-adjacent with \( v^n_1 \). This implies that \( v^m_j \in V(H_3) - N[v^m_i] \). Hence \( N_2(v^m_j) = V(H_3) - N[v^m_i] \) for \( 1 \leq i \leq n \) and \( d_2(v^m_j) = (m - 1)(n - 1), for 1 \leq i \leq n \).

Similarly, for \( 2 \leq t \leq m \), \( N_2(v^t_j) = V(H_3) - N[v^t_i] \) and \( d_2(v^t_j) = (m - 1)(n - 1), for 1 \leq i \leq n \). Hence \( H_3 \) is a \((m + n - 2, 2, (m - 1)(n - 1))\)-regular graph of order \( mn \) containing a given graph \( G \) as an induced subgraph.

For any graph \( G \) of order \( n \geq 2 \), there exists a \((m + n - 2, 2, (m - 1)(n - 1))\)-regular graph \( H_3 \) of order \( mn \) containing a given graph as an induced subgraph. \( \square \)
**Corollary 2.6.2.** For any \( m > 1 \), the smallest order of a \((n+m-2, 2, (m-1)(n-1))\)-regular graph \( H_3 \) containing a given graph \( G \) of order \( n \geq 2 \) as an induced subgraph is \( mn \).

*Proof.* The graph \( H_3 \) constructed in Theorem 2.6.1 is \((n+m-2, 2, (m-1)(n-1))\)-regular graph of smallest order \( mn \). Suppose \( H_3 \) is \((n+m-2, 2, (m-1)(n-1))\)-regular graph of order \( mn - 1 \). That is, for each \( v_i \in H_3 \), \( d_2(v_i) = (m-1)(n-1) \) and \( d(v_i) = n+m-2, (1 \leq i \leq 2n) \). Hence \( H_3 \) has at least \( (m-1)(n-1) + n + m - 2 + 1 = mn \) vertices, which is a contradiction. \( \square \)

**Corollary 2.6.3.** Every graph \( G \) of order \( n \geq 2 \) is an induced subgraph of \((n, 2, (n-1))\)-regular graph of smallest order \( 2n \).

**Example 2.6.4.** Figure 2.11 illustrates Corollary 2.6.3 for \( n = 3 \) and \( m = 2 \).

\[
\begin{array}{ccc}
  x & y & z \\
  \circlearrowright & & \\
  \end{array}
\quad
\begin{array}{ccc}
  x & y & z \\
  \circlearrowright & & \\
  \end{array}
\quad
\begin{array}{ccc}
  x & y & z \\
  \circlearrowright & & \\
  \end{array}
\quad
\begin{array}{ccc}
  x & y & z \\
  \circlearrowright & & \\
  \end{array}
\]

*Figure 2.11*

In Figure 2.11, the graph \( G \) is induced by the vertices \( x, y, z \).

**Corollary 2.6.5.** Every graph \( G \) of order \( n \geq 2 \) is an induced subgraph of \((n+1, 2, 2(n-1))\)-regular graph of smallest order \( 3n \).

**Example 2.6.6.** Figure 2.12 illustrates the Corollary 2.6.5 for \( n = 3 \) and \( m = 3 \).

\[
\begin{array}{ccc}
  x & y & z \\
  \circlearrowleft & & \\
  \end{array}
\quad
\begin{array}{ccc}
  x & y & z \\
  \circlearrowleft & & \\
  \end{array}
\quad
\begin{array}{ccc}
  x & y & z \\
  \circlearrowleft & & \\
  \end{array}
\quad
\begin{array}{ccc}
  x & y & z \\
  \circlearrowleft & & \\
  \end{array}
\]

*Figure 2.12*

In Figure 2.12, the graph \( G \) is induced by the vertices \( x, y, z \).

**Corollary 2.6.7.** Every graph \( G \) of order \( n \geq 2 \) is an induced subgraph of \((n+2, 2, 3(n-1))\)-regular graph of smallest order \( 4n \).
CHAPTER 2.\((r, m, k)\) - REGULAR GRAPH

Remark 2.6.8. If \(m = 2, 3, 4, 5, \ldots n, \ldots\), then there are \((n, 2, (n-1)), (n+1, 2, 2(n-1)), (n+2, 2, 3(n-1)), (n+3, 2, 4(n-1)), \ldots, (2n-2, 2, (n-1)^2)\) \ldots regular graphs of smallest order \(2n, 3n, 4n, 5n, \ldots, n^2, \ldots\) respectively containing any graph \(G\) of order \(n \geq 2\) as an induced subgraph.

2.7 Minimal \((r, 2, k)\)-Regular Graphs Containing a Given Graph and its Complement as Induced Subgraphs

Smallest \((r, 2, k)\)-regular graphs containing a given graph and its complement as induced subgraphs are constructed in this section.

Theorem 2.7.1. For a graph \(G\) of order \(n \geq 2\), there exists a \((m + 2n - 2, 2, (m - 1)(2n - 1))\)-regular graph \(H_4\) of order \(2mn\) such that \(G\) and \(G^c\) are the induced subgraphs of \(H_4\).

Proof. Let \(G\) be a graph of order \(n \geq 2\), \(G\) and \(G^c\) have the same vertex set \(\{v^1_i : 1 \leq i \leq n\}\). Take a graph \(G'\) which is isomorphic to \(G^c\). The vertex set of \(G'\) is denoted as \(\{u^1_i : 1 \leq i \leq n\}\) and \(u^1_i\) corresponds to \(v^1_i(1 \leq i \leq n)\). Let \(G_1 = G \cup G'\). Then \(V(G_1) = \{v^1_i, u^1_i : 1 \leq i \leq n\}\). Let \(G_t(2 \leq t \leq m)\) be \((m-1)\) copies of \(G_1\) with the vertex set \(V(G_t) = \{v^t_i, u^t_i : 1 \leq i \leq n, 2 \leq t \leq m\}\) and \(v^t_i, u^t_i(2 \leq t \leq m)\) correspond to \(v^1_i, u^1_i(1 \leq i \leq n)\) respectively. The desired graph \(H_4\) has the vertex set \(V(H_4) = \bigcup_{t=1}^{m} V(G_t)\), and edge set \(E(H_4) = \bigcup_{t=1}^{m} E(G_t) \bigcup \{v^t_i v^{t+1}_j, v^t_i v^t_j : v^t_i v^{t+1}_j \notin E(G_1), 1 \leq j \leq n, j + 1 \leq i \leq n\} \bigcup \{v^t_k v^{t+j}_k : 1 \leq i \leq m - 1, 1 \leq j \leq m - i\} \bigcup \{u^t_i u^{t+1}_j, u^t_j u^t_i \notin E(G_1), 1 \leq j \leq n, j + 1 \leq i \leq n\} \bigcup \{u^t_k u^{t+j}_k : 1 \leq i \leq m - 1, 1 \leq j \leq m - i\} \bigcup \{v^t_i v^{t+1}_j, v^t_j v^t_i : 1 \leq i, j \leq n\}\).

The resulting graph \(H_4\) contains \(G_1\) as an induced subgraph. Moreover in \(H_4\), \(d(v^1_i) = m + 2(n - 1), (1 \leq i \leq n), (1 \leq t \leq m)\). Then, \(H_4\) is \(m + 2(n - 1)\) regular graph with \(2mn\) vertices. Hence \(H_4\) contains \(G\) and \(G^c\) as induced subgraphs. In \(H_4\), \(d(v_i) = d(v^1_i) = d(u_i) = d(u^1_i) = m + 2n - 2, (1 \leq i \leq n)\).
To find the $d_2$ degree of each vertex in $H_4$, the following cases are examined.

**Case 1** When $t = 1$. If $v \in V(G_1)$, then $v \in V(G)$ (or ) $v \in V(G')$.

**Subcase 1** If $v \in V(G)$, then $v = v^1_j$, for some $j$. Let $v^i_j \in V(H_4) - N[v^1_i]$. Then $v^1_j$ and $v^i_j$ are non-adjacent vertices in $H_4$. By the construction, $v^1_j$ is adjacent to $v^2_j$ and $v^i_j$ is adjacent to $v^1_i$. Then $d(v^1_j, v^i_j) = 2$. Hence $v^1_j \in N_2(v^1_i)$. This implies that $V(H_4) - N[v^1_i] \subseteq N_2(v^1_i)$. If $v^1_i \in N_2(v^1_j)$, then $v^1_i$ is non-adjacent with $v^1_j$. This implies that $v^1_j \in V(H_4) - N[v^1_i]$. Hence $N_2(v^1_i) = V(H_4) - N[v^1_i]$, $(1 \leq i \leq n)$ and $d_2(v^1_i) = (m - 1)(2n - 1)$, $(1 \leq i \leq n)$.

**Subcase 2** If $v \in V(G')$, then $v = u^1_i$, for some $j$. Let $u^1_i \in V(H_4) - N[u^1_i]$. Then, $u^1_j$ and $u^1_i$ are non-adjacent vertices in $H_4$. By the construction, $u^1_j$ is adjacent to $u^2_i$ and $u^2_i$ is adjacent to $u^1_i$. Then $d(u^1_j, u^1_i) = 2$. Hence $u^1_j \in N_2(u^1_i)$. This implies that $V(H_4) - N[u^1_i] \subseteq N_2(u^1_i)$. If $u^1_i \in N_2(u^1_j)$, then $u^1_i$ is non-adjacent with $u^1_j$. Hence $u^1_j \in V(H_4) - N[u^1_i]$. This implies that $N_2(u^1_i) = V(H_4) - N[u^1_i]$, $(1 \leq i \leq n)$ and $d_2(u^1_i) = (m - 1)(2n - 1)$, $(1 \leq i \leq n)$.

**Case 2** When $2 \leq t \leq m - 1$. If $v \in V(G_t)$, then $v = v^t_j$ (or) $v = u^t_j$, for some $j$.

**Subcase 1** If $v = v^t_j$ and if $v^t_j \in V(H_4) - N[v^1_i]$, then $v^t_j$ and $v^i_j$ are non-adjacent vertices in $H_4$. By the construction, $v^t_j$ is adjacent to $v^i_j$ and $v^i_j$ is adjacent to $v^t_i$. Then $d(v^t_j, v^i_j) = 2$. Hence $v^t_j \in N_2(v^1_i)$. This implies that $V(H_4) - N[v^1_i] \subseteq N_2(v^1_i)$. If $v^t_j \in N_2(v^1_i)$, then $v^t_j$ is non-adjacent with $v^1_i$. Hence $v^t_j \in V(H_4) - N[v^1_i]$. This implies that $v^t_j \in V(H_4) - N[v^1_i]$. Hence $N_2(v^1_i) = V(H_4) - N[v^1_i]$, $(1 \leq i \leq n)$ and $d_2(v^1_i) = (m - 1)(2n - 1)$, $(1 \leq i \leq n)$.

**Subcase 2** If $v = u^t_j$ and if $u^t_j \in V(H_4) - N[u^1_i]$, then $u^t_j$ and $u^1_i$ are non-adjacent vertices in $H_4$. By the construction, $u^t_j$ is adjacent to $u^t_i$, and $u^t_i$ is adjacent to $u^1_i$. Then $d(u^t_j, u^1_i) = 2$. Hence $u^t_j \in N_2(u^1_i)$. This implies that $V(H_4) - N[u^1_i] \subseteq N_2(u^1_i)$. If $u^t_j \in N_2(u^1_i)$, then $u^t_j$ is non-adjacent with $u^1_i$. Hence $u^t_j \in V(H_4) - N[u^1_i]$. This implies that $N_2(u^1_i) = V(H_4) - N[u^1_i]$, $(1 \leq i \leq n)$ and $d_2(u^1_i) = (m - 1)(2n - 1)$, $(1 \leq i \leq n)$.

**Case 3** When $t = m$. If $v \in V(G_m)$, then $v = v^m_j$ (or) $v = u^m_j$, for some $j$.

**Subcase 1** If $v = v^m_j$ and if $v^m_j \in V(H_4) - N[v^1_i]$, then $v^m_j$ and $v^1_i$ are non-adjacent vertices in $H_4$. By the construction, $v^m_j$ is adjacent to $v^m_i$ and $v^m_i$ is adjacent to $v^1_i$. Then $d(v^m_j, v^1_i) = 2$. Hence $v^m_j \in N_2(v^1_i)$. This implies that $V(H_4) - N[v^1_i] \subseteq N_2(v^1_i)$. If $v^m_j \in N_2(v^1_i)$, then $v^m_j$ is non-adjacent with $v^1_i$. Hence $v^m_j \in V(H_4) - N[v^1_i]$. This implies that $N_2(v^1_i) = V(H_4) - N[v^1_i]$, $(1 \leq i \leq n)$ and $d_2(v^1_i) = (m - 1)(2n - 1)$, $(1 \leq i \leq n)$.
(1 ≤ i ≤ n).

**Subcase 2**  If \( v = u^m_j \) and if \( u^m_j \in V(H_4) - N[u^1_i] \), then \( u^m_j \) and \( u^1_i \) are non-adjacent vertices in \( H_4 \). By our construction, \( u^m_j \) is adjacent to \( u^m_i \) and \( u^m_i \) is adjacent to \( u^1_i \). Then \( d(u^m_j, u^1_i) = 2 \). Hence \( u^m_j \in N_2(v_i) \). This implies that \( V(H_4) - N[u^i_i] \subseteq N_2(u^i_i) \).

If \( u^m_j \in N_2(u^1_i) \), then \( u^m_j \) is non-adjacent with \( u^1_i \). Hence \( u^m_j \in V(H_4) - N[u^1_i] \). This implies that \( N_2(u^1_i) = V(H_4) - N[u^1_i] \), \( 1 ≤ i ≤ n \) and \( d(u^1_i) = (m-1)(2n-1) \), \( 1 ≤ i ≤ n \). Similarly, for \( 1 ≤ t ≤ m \) \( d_2(v^1_t) = d_2(u^1_i) = (m-1)(2n-1) \), \( 1 ≤ i ≤ n \).

\( H_4 \) is \((m+2(n-1), 2, (m-1)(2n-1))\)-regular graph of order \( 2mn \) containing a given graph \( G \) of order \( n ≥ 2 \) and its complement as induced subgraphs. ∎

**Corollary 2.7.2.** For any \( m ≥ 1 \), the smallest order of \((m+2n-2, 2, (m-1)(2n-1))\)-regular graph containing a given graph of order \( n ≥ 2 \) and its complement is \( 2mn \).

*Proof.* For the graph \( H_4 \) constructed in Theorem 2.7.1 is \((m+2n-2, 2, (m-1)(2n-1))\)-regular graph of order \( 2mn \). Suppose \( H_4 \) is \((m+2n-2, 2, (m-1)(2n-1))\)-regular graph of order \( 2mn - 1 \). Then, for each \( v_i \in H_4 \), \( d(v_i) = (m-1)(2n-1) \) and \( d(v_i) = m+2n-2 \). Hence \( H_4 \) has at least \((m-1)(2n-1) + m+2n-2) + 1 = 2mn \) vertices, which is a contradiction. ∎

**Corollary 2.7.3.** Every graph \( G \) of order \( n ≥ 2 \), and its complement \( G^c \) are the induced subgraphs of \((2n, 2, 2(n-1))\)-regular graph of smallest order \( 4n \).

**Example 2.7.4.** Figure 2.13 illustrates Corollary 2.7.3 for \( G = K_3 \).

![Figure 2.13](image)

In Figure 2.13, the graph \( G \) is induced by the vertices \( x, y, z \).

**Corollary 2.7.5.** Every graph \( G \) of order \( n ≥ 2 \), and its complement \( G^c \) are the induced subgraphs of \((2n+1, 2, 2(2n-1))\)-regular graph of smallest order \( 6n \).

**Example 2.7.6.** Figure 2.14 illustrates Corollary 2.7.5 for \( G = K_2 \) and \( G = K_3 \).
In Figure 2.14, the graph $G$ and $G^c$ are induced by the vertices $x, y$ for $G = K_2$.

In the second graph, the graph $G$ and $G^c$ are induced by the vertices $x, y, z$ for $G = K_3$.

**Corollary 2.7.7.** Every graph $G$ of order $n \geq 2$, and its complement $G^c$ are the induced subgraphs of $(2^n + 2, 2, 3(2^n - 1))$-regular graph of smallest order $8n$.

**Corollary 2.7.8.** Every graph $G$ of order $n \geq 2$, and its complement $G^c$ are the induced subgraph of $(2^n + 3, 2, 4(2^n - 1))$-regular graph of smallest order $10n$.

**Remark 2.7.9.** If $m = 2, 3, 4, 5, \ldots$, then there are $(2n, 2, (2n-1)), (2n+1, 2, 2(2n-1)), (2n + 2, 2, 3(2n - 1)), (2n + 3, 2, 4(2n - 1)), \ldots$ regular graphs of the smallest order $4n, 6n, 8n, 10n, 12n \ldots$ respectively containing any graph $G$ of order $n \geq 2$ and its complement as induced subgraphs.

## 2.8 Topological Indices of the graphs $H_3$ and $H_4$

The topological indices Wiener Index $W$, Hyper Wiener Index $WW$, Degree Distance $DD$, Variance of degrees, The first Zagreb index, The second Zagreb Index and the third Zagreb Index of the graphs $H_3$ and $H_4$, which were constructed in Theorems 2.6.1 and 2.7.1 are calculated in this section.

Topological index $\text{Top}(G)$ of a graph $G$ is a number with this property that for every graph $H$ isomorphic to $G$, $\text{Top}(G) = \text{Top}(H)$. For historical background, computational techniques and mathematical properties of Zagreb indices and Wiener, Hyper Wiener one can refer to [53, 22, 27, 16, 8, 15, 21].
The graph $H_3$ is $(m + n - 2, 2, (m - 1)(n - 1))$-regular graph having $mn$ vertices and $(1/2)mn(m + n - 2)$ edges with diameter 2. Also, for each $v \in H_3$, $d_2(v) = (m - 1)(n - 1)$ and $d(v) = m + n - 2$.

The graph $H_4$ is $(m + 2n - 2, 2, (m - 1)(2n - 1))$-regular graph having $2mn$ vertices and $mn(m + 2n - 2)$ edges with diameter 2. Also, for each $v \in H_4$, $d_2(v) = (m - 1)(2n - 1)$ and $d(v) = m + 2n - 2$.

Computation of $W$, $WW$ and $DD$ for $H_3$ and $H_4$ is done by using the following theorem[22].

**Theorem 2.8.1.** Let $G$ be a graph with $n$ vertices, $m$ edges and with diameter 2, then

1. $W(G) = n(n - 1) - m$.  
2. $WW(G) = 3/2(n(n - 1)) - 2m$.  
3. $DD(G) = 4(n - 1)m - M_1(G)$.

**Wiener index:** The Wiener index $W$ is the first and important topological index in chemistry which was introduced by H. Wiener in 1947 to study the boiling points of parafins. This index is useful to describe molecular structures and also crystal lattice that depends on its $W$ value.

**Definition 2.8.2.** The *Wiener index* $W(G)$ of a finite, connected graph $G$ is defined to be $W(G) = \frac{1}{2} \sum d(u,v)$, where $d(u,v)$ denotes the distance between $u$ and $v$ in $G$.

**Wiener Index of a graph $H_3$**

$W(H_3) = \frac{(mn)(mn - 1) - (1/2)(mn)(m + n - 2)}{2mn(2mn - 1) - ((mn)(m + 2(n - 1))}$

$= \frac{(mn/2)(2mn - 2 - m - n + 2)}{mn(4mn - 2 - m - 2n + 2)}$

$= \frac{(mn)(4mn - (m + 2n))}{(mn)(4mn - (m + 2n))}$

**Wiener Index of a graph $H_4$**

$W(H_4) = \frac{(mn)(mn - 1) - (1/2)(mn)(m + n - 2))}{2mn(2mn - 1) - ((mn)(m + 2(n - 1))}$

$= \frac{(mn/2)(2mn - 2 - m - n + 2)}{mn(4mn - 2 - m - 2n + 2)}$

$= \frac{(mn)(4mn - (m + 2n))}{(mn)(4mn - (m + 2n))}$
**Hyper Wiener index:** The Hyper Wiener index $WW$ was introduced by Randic. The Hyper Wiener Index $WW$ is used as a structure descriptor for predicting physicochemical properties of organic compounds.

**Definition 2.8.3.** The Hyper Wiener index $WW(G)$ of a finite, connected graph $G$ is defined to be $WW(G) = \frac{1}{2}(W_1(G) + W_2(G))$, where $W_1(G) = W(G)$ and $W_\lambda(G) = \sum d_G(k)(k^\lambda)$ is called the Wiener-type invariant of $G$ associated to a real number.

Hyper Wiener Index of a graph $H_3 = WW(H_3)$

\begin{align*}
HH_1 & = \frac{3}{2}mn(mn - 1) - 2(mn/2)(m + n - 2) \\
HH_2 & = (mn/2)(3mn - 3 - 2m - 2n + 4) \\
HH_3 & = (mn/2)(3mn - (2m + 2n) + 1)
\end{align*}

Hyper Wiener Index of a graph $H_4 = WW(H_4)$

\begin{align*}
HH_1 & = (3/2)(2mn(2mn - 1) - 2mn(m + 2(n - 1))) \\
HH_2 & = (mn)(6mn - 3 - 2m - 4n + 4) \\
HH_3 & = (mn)(6mn - (2m + 4n) + 1)
\end{align*}

**Zagreb indices:** The Zagreb indices were introduced by Gutman and Trinajstic [16, 19, 27]

**Definition 2.8.4.** The oldest and most investigated topological graph indices are defined as: First Zagreb index $M_1(G) = \sum_{v \in V(G)}(d_G(v))^2$, second Zagreb index $M_2(G) = \sum_{uv \in E(G)}(d_G(u)d_G(v))$ and third Zagreb index $M_3(G) = \sum |d(u) - d(v)|$, $uv \in E(G)$.

Zagreb Indices of a graph $H_3$:

1. $M_1(H_3) = \sum d(u)d(u)$
   \[= \sum d(u)^2\]
   \[= mn((m + n - 2)^2)\]

2. $M_2(H_3) = \sum d(u)d(v)$, $uv \in E(H_3)$
   \[= (mn/2)(m + n - 2)(m + n - 2)(m + n - 2)\]
   \[= (mn/2)(m + n - 2))^3\]
3. $M_3(H_3) = \sum |d(u) - d(v)|, uv \in E(H_3)$
   $= \sum |(m + n - 2) - (m + n - 2)| = 0.$

Zagreb Indices of a graph $H_4$:

1. $M_1(H_4) = \sum d(u)d(u)$
   $= \sum d(u)^2$
   $= 2mn((m + 2n - 2)^2)$

2. $M_2(H_4) = \sum d(u)d(v), uv \in E(H_4)$
   $= (mn)(m + 2(n - 1))\cdot (m + 2(n - 1)(m + 2(n - 1))$
   $= (mn)((m + 2(n - 1))^3)$

3. $M_3(H_4) = \sum |d(u) - d(v)|, uv \in E(H_4)$
   $= \sum |(m + 2(n - 1)) - (m + 2(n - 1))| = 0.$

**Definition 2.8.5. Degree Distance:** The degree distance (Schultz index) of $G$ was introduced by Dobrynin and Kochetova and Gutman as a weighted version of the Wiener index defined as $DD(G) = \sum (d(u) + d(v))d(u,v)$. It is to be noted that $DD(G)$ and $W(G)$ are closely mutually related for certain classes of molecular graphs [8, 15].

Degree Distance of a graph $H_3$ is

$$DD(H_3) = 4(mn - 1)(mn/2)(m + n - 2) - M_1(H_3)$$
$$= mn(m + n - 2)[2(mn - 1) - (m + n - 2)]$$
$$= mn(m + n - 2)[2mn - (m + n)]$$

Degree Distance of a graph $H_4$ is

$$DD(H_4) = 4(2mn - 1)(mn)(m + 2(n - 1)) - M_1(H_4)$$
$$= 2mn(m + 2n - 2)[2(2mn - 1) - (m + 2n - 2)]$$
$$= 2mn(m + 2n - 2)[4mn - (m + 2n)]$$

**Definition 2.8.6.** The status, or distance sum, of a vertex $v$ in a graph is defined by $s(v) = \sum d(u,v)$, where $d(u,v)$ is the distance between the vertices $u$ and $v$ and $u \neq v$. The status sequence of a graph consists of a list of the stati of all the
vertices[21]. Since diameter of $H_3$ is two, the status of a vertex $v$ in $H_3$ is

\[
s(v) = (m + n - 2) + 2(m - 1)(n - 1)
= m + n - 2 + 2mn - 2m - 2n + 2
= 2mn - (m + n)
\]

Since diameter of $H_4$ is two, the status of a vertex $v$ in $H_4$ is

\[
s(v) = (m + 2(n - 1) + 2(m - 1)(2n - 1)
= m + 2n - 2 + 4mn - 2m - 4n + 2
= 4mn - 2(m + n)
\]

**Definition 2.8.7.** A graph is said to be **self median, or SM**, if the stati of its vertices are all equal. Every vertex in $H_3$ has the same status $2mn - (m + n)$, then the graph $H_3$ is a Self Median graph.

Every vertex in $H_4$ has the same status $4mn - 2(m + n)$, then the graph $H_4$ is a Self Median graph.

**Conclusion and Scope:** For further investigation, the following open problem is suggested.

1. Construct $(r, m, k)$-regular graphs containing a given graph $G$ of order $n \geq 2$, as an induced subgraph, for $m \geq 3$.

2. Construct $(r, m, k)$-regular graphs containing a given graph $G$ of order $n \geq 2$, as an induced subgraph, for all values of $k$. 