Chapter 5

d$_m$ - Splitting Graph of a Graph

For $m \geq 2$, $d_m$ of a vertex in a graph is defined in this chapter and some properties are studied. Also, some graph products in $(2,k)$-regular graphs and $(r,2,k)$-regular graphs are discussed. For $m \geq 2$, $d_m$-Splitting graph of a graph is defined and some properties are also studied.

5.1 Introduction

The degree of a vertex $v$ is the number of edges incident at $v$. A graph is regular if all its vertices have the same degree. Consequently, the degree of a vertex $v$ is the number of vertices at a distance one from $v$. For a positive integer $m$ and a vertex $v$ of a graph $G$, the $d_m$-degree of $v$ in $G$, denoted by $d_m(v)$ is defined as the number of vertices at a distance $m$ from $v$. Hence $d_1(v) = d(v)$.

A concept related to the $F$-degree of a vertex was first introduced by Kocay [10], when reconstructing degree sequence on graphs. A graph $G$ is said to be $F$-regular if the $F$-degrees of all the vertices of $G$ are the same, and it is called $F$-irregular if the $F$-degrees of the vertices of $G$ are distinct[2, 10].

In a similar way, $d_m$ of a vertex in a graph is defined. For a given graph $G$, the $d_m$-degree of a vertex $v$ in $G$, denoted by $d_m(v)$ means number of vertices at a distance $m$ from $v$. A graph $G$ is said to be $d_m$-regular if $d_m$-degree of all the vertices of $G$ are the same. A graph $G$ is called $d_m$-irregular if the $d_m$-degrees of $G$ are distinct, but there is no $d_m$-irregular graph. It is observed that $(m,k)$-regular
A graph $G$ is said to be $(m,k)$-regular if $d_m(v) = k$, for all $v$ in $G$. A graph $G$ is said to be $(r, m, k)$-regular if $d(v) = r$ and $d_m(v) = k$, for all $v$ in $G$.[33]

In this chapter, the focus is on $d_2$ of a vertex in a graph and $d_2$ of a vertex in graph product, especially graph product in $(2,k)$-regular graphs and $(r,2,k)$-regular graphs. The splitting graph $S(G)$ was introduced by E. Sampath Kumar and H. B. Walikar[32].

In a similar way, Degree splitting graph $DS(G)$ was introduced by R. Ponraj and S. Somasundaram [30].

Inspired by these definitions, $d_m$- splitting graph of a graph $G$ denote by $D_mS(G)$ is defined and some properties of $D_mS(G)$ are investigated. Also, $d_2$ splitting graph of a graph $G$ denoted by $D_2S(G)$ is defined and some properties of $D_2S(G)$ are investigated.

### 5.2 $d_m$ of a vertex in graphs

In this section, $d_m$ of a vertex in graph is defined and some results about $d_m$ of a vertex in graph are proved.

**Definition 5.2.1.** Let $G$ be a graph and $d_m(v)$ be defined as the number of vertices at a distance $m$ from $v$. $N_m(v)$ denotes the set of all vertices that are at a distance $m$ from $v$ in a graph $G(m)$, a positive integer).

**Definition 5.2.2.** Let $\delta_m(v) = \min\{d_m(v) : v \in V(G)\}$ and $\Delta_m(v) = \max\{d_m(v) : v \in V(G)\}$. If $\delta_m(v) = \Delta_m(v) = d_m(v) = k$, then $G$ is a $(m,k)$-regular graph.

**Definition 5.2.3.** Let $G^*$ be a graph with vertex set the same as that of $G$. The vertices $v_1, v_2$ are adjacent in $G^*$ if and only if they are at a distance $m$ away from each other in $G[3]$.

**Theorem 5.2.4.** The $d_m$ of each vertex in $G$ is equal to the degree of the corresponding vertex in $G^*$. 
Proof. Let \( v \) be any vertex in \( G \). Let \( d_m(v) = k \), where \( k \geq 0 \). By definition of \( G^* \), \( v \) is adjacent to exactly \( k \) vertices in \( G^* \) that are at a distance \( m \) from \( v \) in \( G \). Hence \( d_m(v) \) in \( G = d(v) \) in \( G^* \).

**Theorem 5.2.5.** Any graph \( G \) with at least two vertices contains at least two vertices of the same \( d_m \).

**Proof.** Let \( G \) be a graph with at least two points. Then, \( d_m(v) \) in \( G = d(v) \) in \( G^* \), for all \( v \in V(G) \). For every graph \( G \), there exists one \( G^* \) graph and any graph with at least two vertices contains at least two vertices of the same degree. Then \( G^* \) has at least two vertices of the same degree. Without loss of generality, it is assumed that the vertices \( u \) and \( v \) have the same degree in \( G^* \). By definition of \( G^* \), \( d_m(u) = d_m(v) \) in \( G \). Hence \( G \) has at least two vertices of same \( d_m \). □

**Definition 5.2.6.** A vertex \( v \) in a graph \( G \) is called odd (even) \( d_m \) vertex if the number of vertices at a distance \( m \) from \( v \) is odd (even).

**Theorem 5.2.7.** For any graph \( G \), the number of odd \( d_m \) vertices in \( G \) is even.

**Proof.** Let \( v_1, v_2, v_3, \ldots, v_k \) be the odd \( d_m \) vertices in \( G \) and \( w_1, w_2, w_3, \ldots, w_m \) be the even \( d_m \) vertices in \( G \). By definition of \( G^* \), \( v_1, v_2, v_3, \ldots, v_k \) be the odd degree vertices in \( G^* \) and \( w_1, w_2, w_3, \ldots, w_m \) be the even degree vertices in \( G^* \).

\[
\sum_{i=1}^{k} d_m(v_i) = \sum_{i=1}^{k} d(v_i) \text{ in } G^* = 2|E(G^*)| - \sum_{i=1}^{m} d(v_i) = \text{even}.
\]

Each \( d_m(v_i) \) is odd. Hence \( k \) must be even. □

**Definition 5.2.8.** A graph \( G \) is \( d_m \)-regular if the \( d_m \) of all the vertices in \( G \) are the same, where \( d_m \) - denotes number of vertices that are at a distance \( m \) from a vertex. Hence a graph \( G \) is \( d_m \)- regular if \( d_m(v) = k \), for all \( v \in G \). The \( d_m \)-regular graph and \((m, k)\)-regular graph and semiregular graph are the same. Here after, \((m, k)\)-regular graph is used for \( d_m \)-regular graph [33].

**Theorem 5.2.9.** For any odd \( k \geq 1 \), there is no \((m, k)\)- regular graph with odd order.
Proof. Let $k \geq 1$ be odd and let $G$ be the $(m, k)$- regular graph of odd order. By definition of $G^*$, $G^*$ is an odd regular graph with odd order, which is a contradiction, since no graph of odd order is odd regular. □

**Theorem 5.2.10.** Let $G$ be a $(m, k)$- regular graph of order $n$. Then $\sum d_m(v) = nk$.

**Definition 5.2.11.** A graph $G$ is $d_m$-irregular if the $d_m$ of the vertices in the graph are distinct.

**Theorem 5.2.12.** For every $n \geq 2$, there is no non trivial $d_m$- irregular graph.

**Proof.** Let $G$ be a graph with at least two vertices. By Theorem 5.2.5, any graph $G$ with at least two vertices contains at least two vertices of the same $d_m$. Hence there is no non trivial $d_m$- irregular graph, for every $n \geq 2$. □

**Remark 5.2.13.** Let $G$ be a graph and $d_2(v)$ be defined as the number of vertices at a distance two from $v$. $N(v)$ denotes the set of all vertices adjacent to $v$. $N_2(v)$ denotes the set of all vertices, that are at a distance two from $v$ in a graph $G$.

**Theorem 5.2.14.** Let $v_1, v_2, v_3, \ldots, v_r$ be neighbours of $v$.

For any $v \in V(G)$, $N_2(v) = \bigcup_{i=1}^{r} \{N(v_i) - v\}$.

**Remark 5.2.15.** Fix $m = 2$. Consider the graph $G^*$. Two vertices $v_1, v_2$ are adjacent in $G^*$ if and only if they are at a distance two away from each other in $G[3]$. Alison Northup proved that $G$ is a $k$-semiregular if and only if $G^*$ is $k$-regular.

**Example 5.2.16.** Figure 5.1 illustrates $G^*$ in Remark 5.2.15.

![Figure 5.1](image)

**Theorem 5.2.17.** $d(v)$ in $G + d_2(v)$ in $G \leq n - 1$. 
Theorem 5.2.18. \( G^* \) is a spanning subgraph of \( G^c \).

Theorem 5.2.19. Let \( G \) be a graph of order \( n \). \( G^* \) is isomorphic to \( G^c \) if and only if \( d(v) + d_2(v) = n - 1 \).

Proof. Let \( v \) be any vertex in \( G \). Let \( d(v) = i \) in \( G \). Suppose \( d(v) + d_2(v) = n - 1 \), then \( d(v) \) in \( G^* = n - 1 - i \) is the degree of \( v \) in \( G^c \), and \( G^* \) is the spanning subgraph of \( G^c \). Hence \( G^* \) is isomorphic to \( G^c \). Conversely, suppose \( G^* \) and \( G^c \) are isomorphic, then \( d(v) \) in \( G^* = d(v) \) in \( G^c \). Hence for any vertex \( v \) in \( G \), \( d(v) \) in \( G^* = n - 1 - i \), where \( i \) is the degree of \( v \) in \( G \). Hence \( d(v) + d_2(v) = n - 1 \).

Theorem 5.2.20. If \( G_1, G_2, G_3, \ldots, G_n \) are distinct \((2, k)\)-regular graph with the same number of vertices then \( G^*_1, G^*_2, G^*_3, \ldots, G^*_n \) are the same.

Example 5.2.21. Figure 5.2 illustrates Theorem 5.2.20.

The graphs \( G_1, G_2, G_3 \) are \((2, 2)\)-regular graphs, \( G^*_1 = G^*_2 = G^*_3 = G^* \).

![Figure 5.2](image)

Theorem 5.2.22. A graph \( G \) of order \( n \) has diameter 2 if and only if \( d(v) + d_2(v) = n - 1 \).

Proof. Let \( G \) be a graph of order \( n \) with diameter two. Any vertex \( v \) in \( G \) is adjacent to some vertices and non adjacent to the remaining vertices. Since \( diam(G) = 2 \), vertices which are non adjacent to the vertex \( v \) are only at a distance two away from \( v \). Hence \( d(v) + d_2(v) = n - 1 \). Conversely, \( d(v) + d_2(v) = n - 1 \) implies that \( diam(G) = 2 \), when \( G \) is a graph of order \( n \). □
5.3 $d_2$ of a vertex in Cartesian product, Composition ( or Lexicographic product) and Join

$d_2$ of a vertex is studied in the following types of composite graphs $G \times H$, $G[H], G + H[48]$.

**Theorem 5.3.1.** Let $G$ and $H$ be connected graphs. Then, $d_2(u_1, v_1)$ in the cartesian product $G \times H = d_2(u_1) + d_2(v_1) + d(u_1)d(v_1)$.

*Proof.* $N(u_1, v_1)$ in $G \times H = \{u_1 \times N(v_1)\} \cup \{N(u_1) \times v_1\}$. Then, $N_2((u_1, v_1)$ in $G \times H = N(u_1 \times N(v_1)) \cup N(N(u_1) \times v_1) = (u_1 \times N(N(v_1))) \cup (N(N(u_1) \times v_1)) \cup N(u_1) \times N(v_1))$. Hence $d_2((u_1, v_1))$ in $G \times H = d_2(u_1) + d_2(v_1) + d(u_1)d(v_1)$.

**Theorem 5.3.2.** Let $G$ and $H$ be connected graphs. Then, $d_2(u_1, v_1)$ in the composition $G[H] = d_2(u_1)|V(H)| + |V(H)| - 1 - d(v_1)$.

*Proof.* $N(u_1, v_1)$ in $G[H] = \{N(u_1) \times V(H)\} \cup \{N(v_1)\}$ Then, $N_2(u_1, v_1)$ in $G[H] = \{(N(N(u_1)) \times V(H)) \cup \{N(N(v_1))\}$ in $G[H]$. Hence $N_2(u_1, v_1)$ in $G[H] = \{(N_2(u_1)) \times V(H)\} \cup \{N_2(v_1)\}$. Note that $G[H]$ be a graph obtained by taking $|G|$ copies of $H$ and by joining all the vertices of the $i^{th}$ and the $j^{th}$ copy of $H$ if and only if $u_i, u_j \in E(G)$. Hence any two vertices in $G[H]$ are either adjacent or at a distance two. Hence $\{N_2(v_1)\}$ in $G[H] = V(H) - \{v_1\} - N(v_1)$. Hence $d_2(u_1, v_1)$ in $G[H] = d_2(u_1)|V(H)| + |V(H)| - 1 - d(v_1)$.

**Theorem 5.3.3.** Let $G$ and $H$ be any two graphs. Then,

$$d_2(u) \text{ in } G + H = |V(G)| - 1 - d(u), \text{ for all } u \text{ in } V(G).$$

$$d_2(v) \text{ in } G + H = |V(H)| - 1 - d(v), \text{ for all } v \text{ in } V(H).$$

*Proof.* Any two vertices of $G + H$ are either adjacent or at a distance two. Let $u \in V(G)$ and $v \in V(H), N_2(u)$ in $G + H = V(G) - \{u\} - N(u)$ and $N_2(v)$ in $G + H = V(H) - \{v\} - N(v)$. Hence $d_2(u)$ in $G + H = |V(G)| - 1 - d(u), \text{ for all } u \text{ in } V(G)$ and $d_2(v)$ in $G + H = |V(H)| - 1 - d(v), \text{ for all } v \text{ in } V(H)$.

Using these theorems, $d_2$ of a vertex in graph products $G \times H, G[H], G + H$ can be easily calculated.
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5.4 Graph Products in \((2, k)\)-Regular Graphs

The graph products on \((2, k)\)-regular graphs is discussed here[45].

**Definition 5.4.1.** A graph \( G \) is \((2, k)\)-regular if \( d_2(v) = k \), for all \( v \in V(G) \), where \( d_2(v) \) is defined as the number of vertices at a distance two from \( v \).

**Note 1:** If the Cartesian product of two graphs is \((2, k)\)-regular, then it is not necessary that both are \((2, k)\)-regular.

**Example 5.4.2.** The graph \( P_3 \) is not a \((2, k)\) - regular graph, and \( K_2 \) is \((2, 0)\)-regular. Cartesian product of \( P_3 \times K_2 \) is a \((2, 2)\)-regular graph (Figure 5.3).

\[ \begin{array}{ccc}
P_3 & K_2 & P_3 \times K_2 \\
\end{array} \]

**Figure 5.3**

**Note 2:** Cartesian product of two \((2, k)\) regular graphs need not be \((2, k)\)-regular.

**Example 5.4.3.** \( H \) and \( G \) are \((2, k)\)-regular but \( H \times G \) is not \((2, k)\)-regular (Figure 5.4).

\[ \begin{array}{ccc}
H & G & H \times G \\
\end{array} \]

**Figure 5.4**

**Note 3:** Composition (Lexicographic product) of two \((2, k)\) - regular graphs need not be \((2, k)\)-regular.

**Example 5.4.4.** \( P_4 \) (Path on four vertices) is \((2, 1)\)-regular but \( P_4[P_4] \) is not \((2, k)\)-regular (Figure 5.5).
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Figure 5.5

Note 4: Join of two $(2, k)$-regular graphs need not be $(2, k)$-regular.

Example 5.4.5. $H$ and $G$ are $(2, k)$-regular but $H \cup G$ is not $(2, k)$-regular (Figure 5.6).

Figure 5.6

5.5 Graph Products in $(r, 2, k)$-Regular Graphs

The graph product on $(r, 2, k)$-regular graphs is discussed in this section[45].

Definition 5.5.1. A graph $G$ is $(r, 2, k)$-regular if $d(v) = r, d_2(v) = k$, for all $v \in V(G)$ where $d_2(v)$ is defined as the number of vertices at a distance two from $v$.

Theorem 5.5.2. Let $G$ be a connected $(r_1, 2, k_1)$-regular graph and $H$ be a connected $(r_2, 2, k_2)$-regular graph. Then, the Cartesian product $G \times H$ is $(r_1 + r_2, 2, k_1 + k_2 + r_1 r_2)$-regular.

Proof. Let $G$ be a connected $(r_1, 2, k_1)$-regular graph. Then, $d(u) = r_1$ and $d_2(u) = k_1$, for all $u \in V(G)$. Let $H$ be a connected $(r_2, 2, k_2)$-regular graph. Then $d(v) = r_2$ and $d_2(v) = k_2$, for all $v \in V(H)$. Let $(u, v) \in V(G \times H)$. Consider $d(u, v) = d(u) + d(v) = r_1 + r_2$, for all $(u, v) \in G \times H$ and $d_2(u, v) = d_2(u) + d_2(v) + d(u)d(v) = k_1 + k_2 + r_1 r_2$, for all $(u, v) \in G \times H$. Hence $G \times H$ is $(r_1 + r_2, 2, k_1 + k_2 + r_1 r_2)$-regular. $\square$
Examples

1. $K_n \times K_n$ is $(2n - 2, 2, (n - 1)^2)$-regular, since $d(u, v) = d(u) + d(v) = n - 1 + n - 1 = 2n - 2$ and $d_2(u, v) = d_2(u) + d_2(v) + d(u)d(v) = 0 + 0 + (n - 1)(n - 1) = (n - 1)^2$.

2. For $n \geq 5$, $C_n \times C_n$ is $(4, 2, 8)$-regular, since $d(u, v) = d(u) + d(v) = 4$ and $d_2(u, v) = d_2(u) + d_2(v) + d(u)d(v) = 2 + 2 \times 2 = 8$.

3. Cartesian product of two $(r, 2, r(r - 1))$-regular graphs is $(2r, 2, 3r^2 - 2r)$-regular, since $d(u, v) = d(u) + d(v) = r + r = 2r$ and $d_2(u, v) = d_2(u) + d_2(v) + d(u)d(v) = r(r - 1) + r(r - 1) + r \times r = r^2 - r + r^2 - r + r^2 = 3r^2 - 2r$.

**Theorem 5.5.3.** Let $G$ be a connected $(r_1, 2, k_1)$-regular graph of order $n_1$ and $H$ be a connected $(r_2, 2, k_2)$-regular graph of order $n_2$. Then, the composition graph $G[H]$ is $(r_1n_2 + r_2, 2, (n_2(1 + k_2)) - (1 + r_2))$-regular.

**Proof.** Let $G$ be a connected $(r_1, 2, k_1)$-regular graph of order $n_1$. Then $d(u) = r_1$ and $d_2(u) = k_1$, for all $u \in V(G)$. Let $H$ be a connected $(r_2, 2, k_2)$-regular graph of order $n_2$. Then $d(v) = r_2$ and $d_2(v) = k_2$, for all $v \in V(H)$. Let $(u, v) \in V(G[H])$. Consider $d(u, v) = d(u)|V(H)| + d(v) = r_1n_2 + r_2$, for all $(u, v) \in G[H]$ and $d_2(u, v) = d_2(u)|V(H)| + |V(H)| - 1 - d(v) = k_1n_2 + n_2 - 1 - r_2$, for all $(u, v) \in G[H]$. Hence $G[H]$ is $(r_1n_2 + r_2, 2, (n_2(1 + k_1)) - (1 + r_2))$-regular. \qed

Examples

1. $K_n[K_n]$ is $(n^2 - 1, 2, 0)$-regular, since $d(u, v) = d(u)|V(H)| + d(v) = (n - 1)n + (n - 1) = n^2 - 1$ and $d_2(u, v) = d_2(u)|V(H)| + |V(H)| - 1 - d(v) = 0 + n - 1 - (n - 1) = 0$.

2. For $n \geq 5$, $C_n[C_n]$ is $(2n + 2, 2, 3n - 3)$-regular, since $d(u, v) = d(u)|V(H)| + d(v) = 2n + 2$ and $d_2(u, v) = d_2(u)|V(H)| + |V(H)| - 1 - d(v) = 2n + n - 1 - 2 = 3n - 3$. 


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3. Since \(d(u,v) = d(u)|V(H)| + d(v) = rn2^{r-2} + r\) and \(d_2(u,v) = d_2(u)|V(H)| + |V(H)| - 1 - d(v) = (r^2 - r + 1)n2^{r-2} - (1 + r)\), composition of two \((r, 2, r(r-1))\)-regular graphs of order \(n2^{r-2}\) is \((rn2^{r-2} + r, 2, (r^2 - r + 1)n2^{r-2} - (1 + r))\)-regular graph.

Note 6: The join of two \((r, 2, k)\)-regular graphs need not be \((r, 2, k)\)-regular graph.

Example 5.5.4. \(C_4\) is \((2, 2, 1)\)-regular and \(K_2\) is \((1, 2, 0)\)-regular. Join of \(C_4\) and \(K_2\) is not a \((r, 2, k)\)-regular graph.

![Diagram](image.png)

**Figure 5.7**

**Theorem 5.5.5.** Let \(G\) be a connected \((r_1, 2, k_1)\)-regular graph of order \(n_1\) and \(H\) be a connected \((r_2, 2, k_2)\)-regular graph of order \(n_2\). Then the join \(G + H\) is a \((r, 2, k)\)-regular graph only when \(r_1 - r_2 = n_1 - n_2\).

**Proof.** Let \(G\) be a connected \((r_1, 2, k_1)\)-regular graph of order \(n_1\). Then \(d(u) = r_1\) and \(d_2(u) = k_1\), for all \(u \in V(G)\). Let \(H\) be a connected \((r_2, 2, k_2)\)-regular graph of order \(n_2\). Then \(d(v) = r_2\) and \(d_2(v) = k_2\), for all \(v \in V(H)\). The join \(G + H\) of two graphs \(G\) and \(H\) is regular only when \(d(u)\) in \(G + H = d(v)\) in \(G + H\). Then for all \(u \in V(G)\) and for all \(v \in V(H)\), \(|V(H)| + d(u) = |V(G)| + d(v)\) implies that \(n_2 + r_1 = n_1 + r_2\). Join \(G + H\) is regular only when \(n_1 - n_2 = r_1 - r_2\). The join \(G + H\) is \((2, k)\)-regular only when \(d_2(u)\) in \(G + H = d_2(v)\) in \(G + H\). Then, for all \(u \in V(G)\) and for all \(v \in V(H)\), \(|V(H)| - 1 - d(u) = |V(H)| - 1 - d(v)\) implies that \(n_1 - 1 - r_1 = n_2 - 1 - r_2\). Join \(G + H\) is \((2, k)\)-regular only when \(n_1 - n_2 = r_1 - r_2\). Join \(G + H\) is \((r, 2, k)\)-regular only when \(r_1 - r_2 = n_1 - n_2\).

**Examples:**

1. The join of \(K_n + K_n\) is \((2n - 1, 2, 0)\)-regular, since \(d(u) = 2n - 1\) and \(d_2(u) = 0\), for all \(u \in K_n\).
2. The join of \( C_n + C_n \) is \((n + 2, 2, n - 3)\)-regular, since \( d(u) = n + 2 \), for all \( u \) in \( C_n \) and \( d_2(u) = n - 3 \), for all \( u \) in \( C_n \).

3. Since \( d(u) = n^{2r-2} + r \) and \( d_2(u) = n^{2r-2} - r - 1 \), for all \( u \) in \((r, 2, r(r - 1))\)-regular graph, join of two \((r, 2, r(r - 1))\)-regular graphs is \((n^{2r-2} + r, 2, n^{2r-2} - r - 1)\)-regular.

### 5.6 \( d_m \)-Splitting Graph of a Graph

**Definition 5.6.1.** Let \( G \) be a graph with \( V(G) = V_1 \cup V_2 \cup V_3 \cdots \cup V_w \cup W \) where each \( V_i \) is a set having at least two vertices all having the same \( d_m \) and \( W = V - \bigcup_{i=1}^{w} V_i \). The \( d_m \)-splitting graph of \( G \) denoted by \( D_m S(G) \) is obtained from \( G \) by introducing new vertices \( u_1, u_2, \ldots, u_w \) and joining \( u_i \) to each vertex of \( V_i (1 \leq i \leq w) \) \( (m, \) a positive integer) 

Now, \( d_2 \)-Splitting graph of a graph is defined and few examples of \( d_2 \)-Splitting graph of a graph are given.

**Definition 5.6.2.** Let \( G \) be a graph with \( V(G) = V_1 \cup V_2 \cup V_3 \cdots \cup V_w \cup W \) where each \( V_i \) is a set having at least two vertices all having the same \( d_2 \) and \( W = V - \bigcup_{i=1}^{w} V_i \). The \( d_2 \)-splitting graph of \( G \) denoted by \( D_2 S(G) \) is obtained from \( G \) by introducing new vertices \( u_1, u_2, \ldots, u_w \) and joining \( u_i \) to each vertex of \( V_i (1 \leq i \leq w) \).

**Example 5.6.3.** Figure 5.8 and Figure 5.9 illustrate the definition 5.6.2.

(i)

![Diagram](image)

Figure 5.8

In Figure 5.8, \( V_1 = \{2, 3, 4, 5\} \), \( W = \{1\} \).
In Figure 5.9, \( V_1 = \{1, 4\}, V_2 = \{2, 3, 5\}, W = \emptyset. \)

**Theorem 5.6.4.** Trivial graph \( K_1 \) is the only graph such that \( K_1 = D_mS(K_1) \).

**Theorem 5.6.5.** For any graph \( G \neq K_1, G \) is a subgraph of \( D_mS(G) \).

**Theorem 5.6.6.** If \( G = K_c^n \), then \( D_mS(G) = K_{1,n} \).

**Theorem 5.6.7.** If \( G = W_4 \), then \( D_mS(G) = K_5 \).

**Theorem 5.6.8.** If \( G = K_n(n > 1) \), then \( D_mS(G) = K_{n+1} \).

**Theorem 5.6.9.** If \( G \) is a \((m, k)\)-regular, then \( D_mS(G) = G + K_1 \).

**Theorem 5.6.10.** If \( G \) is a connected graph with at least one edge, then \( D_mS(G) \) contains a cycle.

_Proof._ Let \( G \) be a connected graph with \( |E(G)| \geq 1 \).

**Case 1** If \( G \) contains a cycle, then \( D_mS(G) \) also contains a cycle.

**Case 2** Suppose \( G \) contains no cycle. Since \( G \) is a connected graph with \( |E(G)| \geq 1 \), \( G \) contains more than one vertex and hence \( G \) contains at least two vertices having the same \( d_m \).

Without loss of generality, let \( x \) and \( y \) be two vertices in \( G \) so \( d_m(x) = d_m(y) \). By definition of \( D_mS(G) \), it contains a vertex \( u \) so that \( u \) is adjacent to both \( x \) and \( y \).
Subcase 1  If \( x \) and \( y \) are adjacent, then \( u, x, y, u \) form a cycle in \( D_mS(G) \).

Subcase 2  If \( x \) and \( y \) are not adjacent, then they are connected by a path \( x = v_1, v_2, \ldots, v_n = y \). Since \( G \) is connected, \( u, v_1, \ldots, v_n, u \) is a cycle in \( D_mS(G) \).

\[ \square \]

Remark 5.6.11. If \( G \) is a disconnected graph with \( |E(G)| \geq 1 \), then at least one component of \( G \) has at least one edge and is connected, then by Theorem 5.6.10, \( D_mS(G) \) contains a cycle.

Theorem 5.6.12. Let \( G \) be a graph of order \( n \) which is \((m, k)\)-regular. \( D_mS(G) \) is \((m, k)\)-regular if and only if \( G = K_n \)

Proof. Suppose \( G = K_n \). Then \( G \) is \((m, 0)\)-regular. Also, \( D_mS(G) = K_{n+1} \) which is \((m, 0)\)-regular. Conversely, suppose \( D_mS(G) \) is \((m, k)\)-regular. Suppose \( G \neq K_n \) and \( G \) is \((m, k)\)-regular. Then \( k > 0 \) and \( d_m(v) = k \neq 0 \), for all \( v \in V(G) \). Let \( u \) be the vertex of \( D_mS(G) \) which is different from all vertices of \( G \), and adjacent to all the vertices of \( G \). Then \( d_m(u) = 0 \) for \( m \geq 2 \). But for \( v \in G \) with \( v \in D_mS(G) \), \( d_m(v) \neq 0 \). This shows that \( D_mS(G) \) is not \((m, k)\)-regular for any \( k \). This contradicts the hypothesis. Hence \( G = K_n \).

\[ \square \]

Theorem 5.6.13. \( D_mS(K_{n,n}) \) is a tripartite graph, \( m \geq 2 \).

Proof. Let \( V_1 = \{v_1, v_2, \ldots, v_n\} \) and \( V_2 = \{u_1, u_2, u_3, \ldots, u_n\} \) be the partition of \( V(K_{n,n}) \). Then \( d_m(v_i) = d_m(u_i) = n - 1 \), \((1 \leq i \leq n)\). By definition of \( D_mS(G) \), \( D_mS(K_{n,n}) \) contains a vertex \( u \) so that \( u \) is adjacent to all \( u_i(1 \leq i \leq n) \) and \( v_j(1 \leq j \leq n) \). Then \( u \) is adjacent to all vertices of \( K_{n,n} \). This shows that \( D_mS(K_{n,n}) \) is \( K_{1,n,n} \) and hence tripartite.

\[ \square \]

Theorem 5.6.14. \( D_mS(K_{l,n}) \) with \( l \neq n \) is a tripartite graph if \( m \geq 2 \)

Proof. Let \( V_1 = \{v_1, v_2, \ldots, v_l\} \) and \( V_2 = \{u_1, u_2, u_3, \ldots, u_n\} \) be the partition of \( V(K_{l,n}) \). Then \( d_m(v_i) = 0 \), \((1 \leq i \leq l)\). \( d_m(u_j) = 0 \), \((1 \leq j \leq n)\). By definition of \( D_mS(G) \), \( D_mS(K_{l,n}) \) contains a vertex \( u \) such that \( u \) is adjacent to all \( u_i(1 \leq i \leq n) \) and \( v_j(1 \leq j \leq l) \). Then \( u \) is adjacent with all vertices of \( K_{l,n} \). This shows that \( D_mS(K_{l,n}) \) is \( K_{1,l,n} \) and hence tripartite.

\[ \square \]

Theorem 5.6.15. If \( G \) is an Eulerian graph, then \( D_mS(G) \) is not an Eulerian graph.
Proof. Let \( G \) be an Eulerian graph. Since \( G \) is an Eulerian graph, each vertex in \( G \) is of even degree. Hence \( G \) contains at least two vertices having the same \( d_m \). Let \( x \) and \( y \) be two vertices in \( G \) such that \( d_m(x) = d_m(y) \). By definition of \( D_mS(G) \), there exits a vertex \( u \) which is adjacent to both \( x \) and \( y \). Hence the degree of \( x \) in \( D_mS(G) = (\text{degree of } x \text{ in } G) + 1 = \text{even} + 1 = \text{odd} \). Hence \( D_mS(G) \) is not an Eulerian graph.

Theorem 5.6.16. For any graph \( G \), \( \omega(D_mS(G)) \leq \omega(G) \), where \( \omega(G) \) denotes the number of components of \( G \).

Proof. Case 1 If \( G \) is a connected graph, then \( D_mS(G) \) is connected. Hence \( \omega(G) = 1 = \omega(D_mS(G)) \).

Case 2 If \( G \) is a disconnected graph, then \( G \) has more than one component. It is sufficient to prove the theorem by assuming that \( G \) has only two components \( G_1 \) and \( G_2 \). Let \( x \in V(G_1) \) and \( y \in V(G_2) \) such that \( d_m(x) = d_m(y) \) (This is possible since \( G_1 \) and \( G_2 \) are connected graphs. By definition of \( D_mS(G) \), there exists a vertex \( u \) so that \( u \) is adjacent to both \( x \) and \( y \). Hence \( \omega(D_mS(G)) = 1 < \omega(G) \). Suppose either \( x \) and \( y \) are in \( V(G_1) \) (or) \( x \) and \( y \) are in \( V(G_2) \) then \( \omega(D_mS(G)) = 2 = \omega(G) \). Hence \( \omega(D_mS(G)) \leq \omega(G) \).

5.7 \( d_2 \)-Splitting Graph of a Graph

A few properties of \( d_2 \)-splitting graph of a graph are discussed here[35].

Definition 5.7.1. Consider \( P_n \) \((n \geq 6)\) and two new vertices \( u \) and \( v \) on either side of \( P_n \). Join the vertex \( v \) to first two vertices from the left and last two vertices of \( P_n \) from the right. Join the vertex \( u \) to the remaining vertices of \( P_n \) in the middle. The resulting graph is called Shipping graph and is denoted by \( SP_n \).

Example 5.7.2. For a path on 6 vertices, the Shipping graph \( SP_6 \) is shown in Figure 5.10.
Theorem 5.7.3. If $G = P_n(n \geq 6)$, then $D_2S(G) = SP_n$.

Theorem 5.7.4. If $G = C_n$, then $D_2S(G) = W_n$.

Example 5.7.5. Figure 5.11 illustrates $D_2S$(Petersen graph)

Theorem 5.7.6. Let $G$ be a bipartite graph with bipartition $(V_1, V_2)$, where $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{v_1^1, v_2^1, \ldots, v_n^1\}$. If there is a pair of vertices $v_i$ and $v_j^1$ so that the length of the $v_i - v_j^1$ path is odd and $d_2(v_i) = d_2(v_j^1)$, then $D_2S(G)$ is not bipartite. Also, if there is no pair of vertices $v_i$ and $v_j^1$ so that $d_2(v_i) = d_2(v_j^1)$, then $D_2S(G)$ is bipartite.

Theorem 5.7.7. $D_2S(K_{m,n})$ is a bipartite graph if and only if $m \neq n$.

Proof. Let $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{v_1^1, v_2^1, \ldots, v_n^1\}$ are the partition of $V(K_{m,n})$. Then $d_2(v_i) = m - 1$, $(1 \leq i \leq m)$ and $d_2(v_j^1) = n - 1$, $(1 \leq j \leq n)$. Suppose $m \neq n$. Then $m - 1 \neq n - 1$. Hence there is no pair $v_i$ and $v_j^1$ such that $d_2(v_i) = d_2(v_j^1)$, $(1 \leq i \leq m)$ and $(1 \leq i \leq n)$. Let $V(D_2S(K_{m,n})) \setminus V(K_{m,n}) = \{u_1, u_2\}$. Let $u_1$ be adjacent to every vertex in $V_2$ and $u_2$ be adjacent
to every vertex in $V_1$. Clearly, $(V_1 \cup \{u_1\}, V_2 \cup \{u_2\})$ is a bipartition of $D_2S(K_{m,n})$. Hence $D_2S(K_{m,n})$ is a bipartite graph when $m \neq n$.

Conversely, let $D_2S(K_{m,n})$ be a bipartite graph. Suppose $m = n$, then $m-1 = n-1$. (i.e) $d_2(v) = m-1$, for all $v \in K_{m,n}$. Hence there exists a pair of adjacent vertices $v_i$ and $v_j$ such that $d_2(v_i) = d_2(v_j')$. By definition of $D_2S(K_{m,n})$, there exists a vertex $u$ which is adjacent to both $v_i$ and $v_j'$. Then $D_2S(K_{m,n})$ will contain the odd cycle $u_1v_iv_j'u_1$. This implies $D_2S(K_{m,n})$ is not a bipartite graph, which is a contradiction. Hence $m \neq n$. \hfill $\square$

**Theorem 5.7.8.** Let $G$ be a graph with $p$ vertices and $q$ edges and let $s$ be the number of vertices in $W$. Then $|E(D_2S(G))| = p + q - s$ where $W$ is as in Definition 5.6.2.

**Proof.** Let $V(G) = \{v_1, v_2, v_3, \ldots, v_p\}$ and $V(D_2S(G)) = \{u_1, u_2, u_3, \ldots, u_s\}$. Let $d'(v)$-denote the degree of a vertex $v$ in $D_2S(G)$ (clearly $d'(v) \geq d(v)$, for all $v$ in $G$).

$$|E(D_2S(G))| = \frac{1}{2} \sum d'(v) = \frac{1}{2} \left[ \sum_{i=1}^{p} (d(v_i) + 1) - s + p - s \right] = p + q - s. \hfill \square$$

**Remark 5.7.9.** If $G$ is a $(2,k)$-regular graph, then $|E(D_2S(G))| = p + q$.

**Theorem 5.7.10.** $D_2S(K_{n,n})$ is a Hamiltonian graph.

**Proof.** For $n \geq 1$, the number of vertices in $D_2S(K_{n,n}) = 2n + 1 = p \geq 3$. The minimum degree of the graph $D_2S(K_{n,n})$ is $n + 1$ and $p = 2n + 1$ and $\delta = n + 1$. Hence $\delta \geq \frac{p}{2}$. By Dirac’s theorem, $D_2S(K_{n,n})$ is a Hamiltonian graph. \hfill $\square$

**Theorem 5.7.11.** $D_2S(K_{m,n})$ is a Non-Hamiltonian graph if $m \neq n$.

**Proof.** Let $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{u_1, u_2, u_3, \ldots, u_n\}$ are the partition of $V(K_{m,n})$. Assume $m < n$. Let $V(D_2S(K_{m,n})) = \{V_1 \cup \{u_1\}\} \cup \{V_2 \cup \{u_2\}\}, u_1$ is adjacent with all the vertices of $V_2$ and $u_2$ is adjacent to all the vertices of $V_1$. Then $|V_1 \cup \{u_1\}| = m + 1, |V_2 \cup \{u_2\}| = n + 1$. $(\omega(D_2S(K_{m,n})) - \{V_1 \cup \{u_1\}\}) = n + 1 > m + 1 = |V_1 \cup \{u_1\}|$. Hence $D_2S(K_{m,n})$ is Non-Hamiltonian. \hfill $\square$

**Note 5.7.12.** $D_2S(G)$ of a disconnected graph $G$ may be connected. For instance, let $G$ be a graph with two components $G_1$ and $G_2$ such that $G_1$ and $G_2$ are $(2,k)$-regular and each vertex of $G_1$ and $G_2$ have same $d_2$. By definition of
$D_2S(G)$, there exists a vertex which is adjacent to all the vertices of $G_1$ and $G_2$ and hence $D_2S(G)$ is connected.

**Theorem 5.7.13.** Let $G$ be a connected graph. Then $\kappa(D_2S(G)) \geq \kappa(G)$.

**Proof.** Let $G$ be a connected graph with vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Let $V(D_2S(G)) - V(G) = \{u_1, u_2, u_3, \ldots, u_w\}$. Since $G$ is a connected graph with more than two vertices, $G$ contains at least two vertices having same $d_2$ and they are connected by a path. Let $v_i$ and $v_j$ be the vertices of $G$ such that $d_2(v_i) = d_2(v_j)$ and $v_i$ and $v_j$ are connected by a path. Suppose $G$ is $k$-connected. Let $S = \{v_1, v_2, v_3, \ldots, v_k\}$ be the minimum vertex cut of $G$. Since $G - S$ is disconnected, $G - S$ has at least two components. Take two components $G_1$ and $G_2$.

**Case 1** Suppose $v_i$ and $v_j$ are in the same component. Then $\kappa(D_2S(G)) = \kappa(G)$.

**Case 2** Suppose $v_i$ and $v_j$ belong to different components. Without loss of generality, let $v_i \in G_1$ and $v_j \in G_2$. Then there is no $v_i$-$v_j$ path in $G - S$. But $v_i$ and $v_j$ are connected by a path $v_i u_i v_j$ in $(D_2S(G) - S)$. That is, $D_2S(G) - S$ is connected. Hence $\kappa(D_2S(G)) \geq \kappa(G)$.

**Conclusion and Scope:** For further investigation, the following open problems are suggested.

1. $d_m$ ($m > 2$) of vertex in the graph product may be investigated.

2. The nature of the graph product other than Cartesian, Composition and Join on $(r, 2, k)$-regular graphs may be investigated.

3. $d_m$-Splitting graph $D_mS(G)$, for $m > 2$ may be investigated.