CHAPTER - 3

ANTI (T, S)-FUZZY IDEALS OF A HEMIRING AND TRANSLATIONS

3.1 Introduction: In this chapter, we introduce the concept of anti (T, S)-fuzzy ideals of a hemiring, translations of anti (T, S)-fuzzy subhemiring and establish some results on these.

3.1.1 Definition: A (T, S)-norm is a binary operations T: $[0, 1]\times[0, 1] \to [0, 1]$ and S: $[0, 1]\times[0, 1] \to [0, 1]$ satisfying the following requirements;

(i) $T(0, x) = 0$, $T(1, x) = x$ (boundary condition)

(ii) $T(x, y) = T(y, x)$ (commutativity)

(iii) $T(x, T(y, z)) = T(T(x,y), z)$(associativity)

(iv) if $x \leq y$ and $w \leq z$, then $T(x, w) \leq T(y, z)$(monotonicity).

(v) $S(0, x) = x$, $S(1, x) = 1$ (boundary condition)

(vi) $S(x, y) = S(y, x)$(commutativity)

(vii) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity)

(viii) if $x \leq y$ and $w \leq z$, then $S(x, w) \leq S(y, z)$(monotonicity).

3.1.2 Definition: Let (R , + , .) be a hemiring. A fuzzy subset A of R is said to be an anti (T, S)-fuzzy ideal (anti fuzzy ideal with respect to (T, S)-norm) of R if it satisfies the following conditions:

(i) $\mu_A(x + y) \leq S(\mu_A(x), \mu_A(y))$,

(ii) $\mu_A(xy) \leq T(\mu_A(x), \mu_A(y))$, for all x and y $\in$ R.

3.1.3 Definition: Let (R , + , .) be a hemiring. An anti (T, S)-fuzzy ideal A of R is said to be an anti (T, S)-fuzzy normal ideal (ATSFNI) of R if $\mu_A(xy) = \mu_A(yx)$, for all x and y $\in$ R.

3.1.4 Definition: Let A be an anti (T, S)-fuzzy ideal of a hemiring (R, +, .) and a in R. Then the pseudo anti (T, S)-fuzzy coset $(aA)^p$ is defined by $((a\mu_A)^p)(x) = p(a)\mu_A(x)$, for every $x \in R$ and for some $p \in P$.

3.1.5 Definition: Let A be a fuzzy subset of X and $\alpha \in [\ 0,\ 1 - \mathrm{Sup}\{\ A(x) : x \in X,\ 0 < A(x) < 1\ \}]$. Then $T = T_\alpha^A$ is called a **translation** of A if $T(x) = A(x) + \alpha$, for all $x \in X$.

3.2 - PROPERTIES OF ANTI (T, S)-FUZZY IDEALS OF A HEMIRING:

3.2.1 Theorem: Union of any two anti (T, S)-fuzzy ideal of a hemiring R is an anti (T, S)-fuzzy ideal of R.

Proof: Let A and B be any two anti (T, S)-fuzzy ideals of a hemiring R and x and y $\in$ R. Let $A = \{ (x, \mu_A(x)) / x \in R \}$ and $B = \{ (x, \mu_B(x)) / x \in R \}$ and $C = A \cup B = \{ (x, \mu_C(x)) / x \in R \}$, where $\max\{ \mu_A(x), \mu_B(x)\} = \mu_C(x)$.

Now, $\mu_C(x + y) = \max \{ \mu_A(x + y), \mu_B(x + y) \}$

$\leq \max \{ S (\mu_A(x), \mu_A(y)), S (\mu_B(x), \mu_B(y)) \}$

$\leq S (S (\mu_A(x), \mu_B(x)), S (\mu_A(y), \mu_B(y)))$

$= S (\mu_C(x), \mu_C(y))$.

Therefore, $\mu_C(x + y) \leq S (\mu_C(x), \mu_C(y))$, for all x and y $\in$ R.

And, $\mu_C(xy) = \max \{ \mu_A(xy), \mu_B(xy) \}$

$\leq \max \{ T (\mu_A(x), \mu_A(y)), T (\mu_B(x), \mu_B(y)))$

$\leq T\,(\,T\,(\mu_A(x), \mu_B(x)\,), T\,(\mu_A(y), \mu_B(y)\,)\,)$

$= T\,(\,\mu_C(x), \mu_C(y)\,).$

Therefore, $\mu_C(xy) \leq T\,(\,\mu_C(x), \mu_C(y)\,)$, for all x and y $\in$ R.

Therefore C is an anti (T, S)-fuzzy ideal of a hemiring R.

Hence the union of any two anti (T, S)-fuzzy ideals of a hemiring R is an anti (T, S)-fuzzy ideal of R. □

3.2.2 Theorem: The union of a family of anti (T, S)-fuzzy ideals of hemiring R is an anti (T, S)-fuzzy ideal of R.

Proof: The argument is trivial.

3.2.3 Theorem: If A and B are any two anti (T, S)-fuzzy ideals of the hemirings R_1 and R_2 respectively, then anti-product $A\times B$ is an anti (T, S)-fuzzy ideal of $R_1\times R_2$.

Proof: Let A and B be two anti (T, S)-fuzzy ideals of the hemirings R_1 and R_2 respectively. Let x_1 and $x_2 \in R_1$, y_1 and $y_2 \in R_2$.

Then (x_1, y_1) and $(x_2, y_2) \in R_1\times R_2$.

Now, $\mu_{A\times B}\,[\,(x_1, y_1) + (x_2, y_2)\,] = \mu_{A\times B}\,(\,x_1 + x_2, y_1 + y_2\,)$

$= \max\,\{\,\mu_A(x_1 + x_2\,),\ \mu_B(\,y_1 + y_2\,)\,\}$

$\leq \max\,\{\,S\,(\mu_A(x_1), \mu_A(x_2)\,), S\,(\mu_B(y_1), \mu_B(y_2)\,)\,\}$

$\leq S\,(\,S\,(\mu_A(x_1), \mu_B(y_1)\,), S\,(\mu_A(x_2), \mu_B(y_2)\,)\,)$

$= S\,(\mu_{AxB}\,(x_1, y_1), \mu_{AxB}\,(x_2, y_2)\,).$

Therefore, $\mu_{A\times B}\,[(x_1, y_1) + (x_2, y_2)] \leq S\,(\mu_{A\times B}\,(x_1, y_1), \mu_{A\times B}\,(x_2, y_2)\,)$.

Also, $\mu_{A\times B}\,[\,(x_1, y_1)(x_2, y_2)\,] = \mu_{A\times B}(x_1x_2, y_1y_2\,)$

$= \max \{ \mu_A(x_1x_2), \mu_B(y_1y_2) \}$

$\leq \max \{ T (\mu_A(x_1), \mu_A(x_2)), T (\mu_B(y_1), \mu_B(y_2)) \}$

$\leq T (T (\mu_A(x_1), \mu_B(y_1)), T (\mu_A(x_2), \mu_B(y_2)) \}$

$= T (\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)).$

Therefore, $\mu_{A\times B} [(x_1, y_1)(x_2, y_2)] \leq T (\mu_{A\times B} (x_1, y_1), \mu_{A\times B} (x_2, y_2)).$

Hence A×B is an anti (T, S)-fuzzy ideal of hemiring of $R_1\times R_2$. □

3.2.4 Theorem: If A is an anti (T, S)-fuzzy subhemiring of a hemiring (R, +, ·), then $\mu_A(x) \geq \mu_A(0)$, for $x \in R$, the zero $0 \in R$.

Proof: For $x \in R$, and 0 is the zero element of R. Now, $\mu_A(x) = \mu_A(x+0) \leq S (\mu_A(x), \mu_A(0))$, for all $x \in R$. So, $\mu_A(x) \geq \mu_A(0)$ is only possible.

3.2.5 Theorem: Let A and B be anti (T, S)-fuzzy ideal of the hemirings R_1 and R_2 respectively. Suppose that 0_1 and 0_2 are the zero element of R_1 and R_2 respectively. If A×B is an anti (T, S)-fuzzy ideal of $R_1\times R_2$, then at least one of the following two statements must hold.

(i) $B(0_1) \leq A(x)$, for all $x \in R_1$,

(ii) $A(0_1) \leq B(y)$, for all $y \in R_2$.

Proof: Let A×B be an anti (T, S)-fuzzy ideal of $R_1\times R_2$.

By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a in R_1 and b in R_2 such that $A(a) < B(0_2)$ and$B(b) < A(0_1)$.

We have, $A\times B(a, b) = \max\{ A(a), B(b) \}$

$< \max\{ B(0_2), A(0_1)\}$

$= \max\{ A(0_1), B(0_2) \}$

$= A \times B(0_1, 0_2)$.

Thus $A \times B$ is not an anti (T, S)-fuzzy ideal of $R_1 \times R_2$.

Hence either $B(0_2) \leq A(x)$, for all x in R_1 or $A(0_1) \leq B(y)$, for all $y \in R_2$.

3.2.6 Theorem: Let A and B be two fuzzy subsets of the hemirings R_1 and R_2 respectively and $A \times B$ is an anti (T, S)-fuzzy ideal of $R_1 \times R_2$. Then the following are true:

(i) if $A(x) \geq B(0_2)$, then A is an anti (T, S)-fuzzy ideal of R_1.

(ii) if $B(x) \geq A(0_1)$, then B is an anti (T, S)-fuzzy ideal of R_2.

(iii) either A is an anti (T, S)-fuzzy ideal of R_1 or B is an anti (T, S)-fuzzy ideal of R_2.

Proof: Let $A \times B$ be an anti (T, S)-fuzzy ideal of $R_1 \times R_2$ and x and $y \in R_1$ and $0_2 \in R_2$. Then $(x, 0_2)$ and $(y, 0_2) \in R_1 \times R_2$.

Now, using the property that $A(x) \geq B(0_2)$, for all $x \in R_1$.

We get, $A(x+y) = \max\{A(x+y), B(0_2+0_2)\}$

$= A \times B((x+y), (0_2+0_2))$

$= A \times B[(x, 0_2)+(y, 0_2)]$

$\leq S(A \times B(x, 0_2), A \times B(y, 0_2))$

$= S(\max\{A(x), B(0_2)\}, \max\{A(y), B(0_2)\})$

$= S(A(x), A(y))$.

Therefore, $A(x+y) \leq S(A(x), A(y))$, for all x and $y \in R_1$.

Also, $A(xy) = \max\{A(xy), B(0_2 0_2)\}$

$= A \times B((xy), (0_2 0_2))$

$= A \times B[(x, 0_2)(y, 0_2)]$

$\leq T\ (A\times B(x, 0_2\ ), A\times B(y, 0_2\ )\ )$

$= T\ (\ \max\{A(x), B(0_2)\ \}, \max\{A(y), B(0_2\ )\ \}\ )$

$= T\ (A(x), A(y)\ ).$

Therefore, $A(xy) \leq T\ (A(x), A(y)\ )$, for all x and y $\in R_1$.

Hence A is an anti (T, S)-fuzzy ideal of R_1. Thus (i) is proved.

Now, $B(x) \geq A(0_2)$, for all $x \in R_2$,

let x and y in R_2 and $0_1 \in R_1$. Then $(0_1, x)$ and $(0_1, y) \in R_1\times R_2$.

We get, $B(x+y) = \max\{B(x+y), A(0_1+0_1\ )\}$

$= \max\{A(0_1+0_1\ ), B(x+y)\ \}$

$= A\times B(\ (0+0_1\ \ ), (x+y)\ )$

$= A\times B[\ (0_1, x\ ) + (0_1, y\ )\ ]$

$\leq S\ (A\times B(0_1, x\ ), A\times B(0_1, y\ )\ )$

$= S\ (\ \max\{A(0_1), B(x)\ \}, \max\{A(0_1\ ), B(y)\ \}\ )$

$= S\ (B(x), B(y)\ ).$

Therefore, $B(\ x+y\ ) \leq S\ (B(x), B(y)\ )$, for all x and y $\in R_2$.

Also, $B(xy\ ) = \max\{B(xy), A(0_1\ 0_1\ )\}$

$= \max\{A(0_1\ 0_1\ ), B(xy)\}$

$= A\times B(\ (0_1\ 0_1\ ),\ (xy)\ )$

$= A\times B[\ (0_1, x\ )(0_1, y\ )\ ]$

$\leq T\ (A\times B(0_1, x\ ), A\times B(\ 0_1, y\ )\ )$

$= T\ (\ \max\{\ A(0_1\ ), B(x)\ \}, \max\{\ A(0_1\ ), B(y)\}\ )$

$= T\ (\ B(x), B(y)\ ).$

Therefore, $B(xy) \leq T\ (\ B(x), B(y)\ )$, for all x and y $\in R_2$.

Hence B is an anti (T, S)-fuzzy ideal of a hemiring R_2. Thus (ii) is proved.

(iii) is clear. □

3.2.7 Theorem: Let A be a fuzzy subset of a hemiring R and V be the anti-strongest fuzzy relation of R. Then A is an anti (T, S) fuzzy ideal of R if and only if V is an anti (T, S)-fuzzy ideal of R×R.

Proof: Suppose that A is an anti (T, S)-fuzzy ideal of a hemiring R.

Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in R\times R$.

We have,
$$\begin{aligned}
\mu_V(x+y) &= \mu_V[(x_1, x_2) + (y_1, y_2)] \\
&= \mu_V(x_1+y_1, x_2+y_2) \\
&= \max\{\mu_A(x_1+y_1), \mu_A(x_2+y_2)\} \\
&\le \max\{S(\mu_A(x_1), \mu_A(y_1)), S(\mu_A(x_2), \mu_A(y_2))\} \\
&\le S(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\}) \\
&= S(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) \\
&= S(\mu_V(x), \mu_V(y)).
\end{aligned}$$

Therefore, $\mu_V(x+y) \le S(\mu_V(x), \mu_V(y))$, for all x and $y \in R\times R$.

And,
$$\begin{aligned}
\mu_V(xy) &= \mu_V[(x_1, x_2)(y_1, y_2)] \\
&= \mu_V(x_1y_1, x_2y_2) \\
&= \max\{\mu_A(x_1y_1), \mu_A(x_2y_2)\} \\
&\le \max\{T(\mu_A(x_1), \mu_A(y_1)), T(\mu_A(x_2), \mu_A(y_2))\} \\
&\le T(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\}) \\
&= T(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) \\
&= T(\mu_V(x), \mu_V(y)).
\end{aligned}$$

Therefore, $\mu_V(xy) \leq T(\mu_V(x), \mu_V(y))$, for all x and y $\in$ R×R.

This proves that V is an anti (T, S)-fuzzy ideal of R×R.

Conversely assume that V is an anti (T, S)-fuzzy ideal of R×R, then for any x = (x_1, x_2) and y = $(y_1, y_2) \in$ R×R, we have

$$\begin{aligned}
\max\{\mu_A(x_1+y_1), \mu_A(x_2+y_2)\} &= \mu_V(x_1+y_1, x_2+y_2) \\
&= \mu_V[(x_1, x_2) + (y_1, y_2)] \\
&= \mu_V(x+y) \\
&\leq S(\mu_V(x), \mu_V(y)) \\
&= S(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) \\
&= S(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\}).
\end{aligned}$$

If $x_2 = 0$, $y_2 = 0$, we get, $\mu_A(x_1+y_1) \leq S(\mu_A(x_1), \mu_A(y_1))$, for all x_1 and $y_1 \in$ R.

$$\begin{aligned}
\text{And, } \max\{\mu_A(x_1y_1), \mu_A(x_2y_2)\} &= \mu_V(x_1y_1, x_2y_2) \\
&= \mu_V[(x_1, x_2)(y_1, y_2)] \\
&= \mu_V(xy) \\
&\leq T(\mu_V(x), \mu_V(y)) \\
&= T(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) \\
&= T(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\}).
\end{aligned}$$

If x_2=0, y_2=0, we get $\mu_A(x_1y_1) \leq T(\mu_A(x_1), \mu_A(y_1))$, for all x_1 and $y_1 \in$ R.

Therefore A is an anti (T, S)-fuzzy ideal of R. □

3.2.8 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring (R, +, .). If $\mu_A(x+y) = 1$, then either $\mu_A(x) = 1$ or $\mu_A(y) = 1$, for all x and y $\in$ R.

Proof: The argument is trivial.

In the next theorem we introduce a new composition operation to get a S- fuzzy ideal of R

3.2.9 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring H and f is a isomorphism from hemiring R onto H. Then A◦f is an anti (T, S)-fuzzy ideal of R.

Proof: Let x and y in R and A be an anti (T, S)-fuzzy ideal of a hemiring H.

Then we have, $(\mu_A \circ f)(x+y) = \mu_A(f(x+y))$

$= \mu_A(f(x)+f(y))$, as f is an isomorphism

$\leq S(\mu_A(f(x)), \mu_A(f(y)))$,

$\leq S((\mu_A \circ f)(x), (\mu_A \circ f)(y))$,

which implies that $(\mu_A \circ f)(x+y) \leq S((\mu_A \circ f)(x), (\mu_A \circ f)(y))$.

And, $(\mu_A \circ f)(xy) = \mu_A(f(xy))$

$= \mu_A(f(x)f(y))$, as f is an isomorphism

$\leq T(\mu_A(f(x)), \mu_A(f(y)))$,

$\leq T((\mu_A \circ f)(x), (\mu_A \circ f)(y))$,

which implies that $(\mu_A \circ f)(xy) \leq T((\mu_A \circ f)(x), (\mu_A \circ f)(y))$.

Therefore (A◦f) is an anti (T, S)-fuzzy ideal of a hemiring R. □

3.2.10 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring H and f is an anti-isomorphism from a hemiring R onto H, then A◦f is an anti (T, S)-fuzzy ideal of R.

Proof: Let x and y $\in$ R and A be an anti (T, S)-fuzzy ideal of a hemiring H.

Then we have,

$(\mu_A \circ f)(x+y) = \mu_A(f(x+y))$

$= \mu_A(f(y)+f(x))$, as f is an anti-isomorphism

$\leq S (\mu_A(f(x)), \mu_A(f(y)))$,

$\leq S ((\mu_A \circ f)(x), (\mu_A \circ f)(y))$,

which implies that $(\mu_A \circ f)(x+y) \leq S ((\mu_A \circ f)(x), (\mu_A \circ f)(y))$.

Now, $(\mu_A \circ f)(xy) = \mu_A(f(xy))$

$= \mu_A(f(y)f(x))$, as f is an anti-isomorphism

$\leq T (\mu_A(f(x)), \mu_A(f(y)))$,

$\leq T ((\mu_A \circ f)(x), (\mu_A \circ f)(y))$,

which implies that $(\mu_A \circ f)(xy) \leq T ((\mu_A \circ f) (x), (\mu_A \circ f) (y))$.

Therefore $A \circ f$ is an anti (T, S)-fuzzy ideal of a hemiring R. □

3.2.11 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring (R, +, .), then the pseudo anti (T, S)-fuzzy coset $(aA)^p$ is an anti (T, S)-fuzzy ideal of a hemiring R, for every a in R.

Proof: Let A be an anti (T, S)-fuzzy ideal of a hemiring R.

For every x and y $\in$ R, we have,

$((a\mu_A)^p)(x+y) = p(a)\mu_A(x+y)$

$\leq p(a) S (\mu_A(x), \mu_A(y))$

$= S (p(a)\mu_A(x), p(a)\mu_A(y))$

$= S (((a\mu_A)^p)(x), ((a\mu_A)^p)(y))$.

Therefore, $((a\mu_A)^p)(x+ y) \leq S (((a\mu_A)^p)(x), ((a\mu_A)^p)(y))$.

Now, $((a\mu_A)^p)(xy) = p(a)\mu_A(xy)$

$\leq p(a) T (\mu_A(x), \mu_A(y))$

$$= T\,(\,p(a)\mu_A(x),\,p(a)\mu_A(y)\,)$$

$$= T\,((\,(a\mu_A)^p\,)(x),\,(\,(a\mu_A)^p\,)(y)\,).$$

Therefore, $(\,(a\mu_A)^p\,)(\,xy\,) \leq T\,(\,(\,(a\mu_A)^p\,)(x),\,(\,(a\mu_A)^p\,)(y)\,)$.

Hence $(aA)^p$ is an anti (T, S)-fuzzy ideal of a hemiring R. □

3.2.12 Theorem: Let (R, + , .) and (R^l, +, .) be any two hemirings, then the homomorphic image of an anti (T, S)-fuzzy ideal of R is an anti (T, S)-fuzzy ideal of R^l.

Proof: Let (R, +, .) and (R^l, +, .) be any two hemirings. Let $f : R \rightarrow R^l$ be a homomorphism. Then, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and $y \in R$. Let $V = f(A)$, where A is an anti (T, S)-fuzzy ideal of R. We have to prove that V is an anti (T, S)-fuzzy ideal of R^l. Now, for $f(x), f(y) \in R^l$,

$$\mu_v(\,f(x) + f(y)) = \mu_v(\,f(x+y)\,), \text{ as f is a homomorphism}$$

$$\leq \mu_A(x+y)$$

$$\leq S\,(\mu_A(x),\,\mu_A(y)\,)\,,$$

which implies that $\mu_v(f(x) + f(y)) \leq S\,(\mu_v(\,f(x)\,),\,\mu_v(\,f(y)\,)\,)$.

Again, $\mu_v(\,f(x)f(y)\,) = \mu_v(\,f(xy)\,)$, as f is a homomorphism

$$\leq \mu_A(xy)$$

$$\leq T\,(\mu_A(x),\,\mu_A(y)\,)\,,$$

which implies that $\mu_v(\,f(x)f(y)\,) \leq T\,(\mu_v(\,f(x)\,),\,\mu_v(\,f(y)\,)\,)$.

Hence V is an anti (T, S)-fuzzy ideal of R^l. □

3.2.13 Theorem: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. The homomorphic preimage of an anti (T, S)-fuzzy ideal of R^{l} is an anti (T, S)-fuzzy ideal of R.

Proof: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. Let f : R → R^{l} be a homomorphism. Then, f(x+y) = f(x) + f(y) and f(xy) = f(x)f(y), for all x and y ∈ R. Let V = f(A), where V is an anti (T, S)-fuzzy ideal of R^{l}. We have to prove that A is an anti (T, S)-fuzzy ideal of R. Let x and y ∈ R.

Then, $\mu_A(x+ y) = \mu_v(f(x + y))$, since $\mu_v(f(x)) = \mu_A(x)$

$= \mu_v(f(x) + f(y))$, as f is a homomorphism

$\leq S (\mu_v(f(x)), \mu_v(f(y)))$

$= S (\mu_A(x), \mu_A(y))$, since $\mu_v(f(x)) = \mu_A(x)$

which implies that $\mu_A(x+ y) \leq S (\mu_A(x), \mu_A(y))$.

Again, $\mu_A(xy) = \mu_v(f(xy))$, since $\mu_v(f(x)) = \mu_A(x)$

$= \mu_v(f(x)f(y))$, as f is a homomorphism

$\leq T (\mu_v(f(x)), \mu_v(f(y)))$

$= T (\mu_A(x), \mu_A(y))$, since $\mu_v(f(x)) = \mu_A(x)$

which implies that $\mu_A(xy) \leq T (\mu_A(x), \mu_A(y))$.

Hence A is an anti (T, S)-fuzzy ideal of R. □

3.2.14 Theorem: Let (R, +, .) and (R^{l}, +, .) be any two hemirings, then the anti-homomorphic image of an anti (T, S)-fuzzy ideal of R is an anti (T, S)-fuzzy ideal of R^{l}.

Proof: Let (R, +, .) and (R^l, +, .) be any two hemirings. Let $f : R \to R^l$ be an anti-homomorphism. Then, f(x+y) = f(y) + f(x) and f(xy) = f(y) f(x), for all x and y in R. Let V = f(A), where A is an anti (T, S)-fuzzy ideal of R. We have to prove that V is an anti (T, S)-fuzzy ideal of R^l.

Now, for f(x), f(y) in R^l,

$$\begin{aligned}\mu_v(f(x) + f(y)) &= \mu_v(f(y + x)), \text{ as f is an anti-homomorphism}\\ &\leq \mu_A(y + x)\\ &\leq S (\mu_A(y), \mu_A(x))\\ &= S (\mu_A(x), \mu_A(y)),\end{aligned}$$

which implies that $\mu_v(f(x) + f(y)) \leq S (\mu_v(f(x)), \mu_v(f(y)))$.

Again, $\mu_v(f(x)f(y))= \mu_v(f(yx))$, as f is an anti-homomorphism

$$\begin{aligned}&\leq \mu_A(yx)\\ &\leq T (\mu_A(y), \mu_A(x))\\ &= T (\mu_A(x), \mu_A(y)),\end{aligned}$$

which implies that $\mu_v(f(x)f(y)) \leq T (\mu_v(f(x)), \mu_v(f(y)))$.

Hence V is an anti (T, S)-fuzzy ideal of R^l. □

3.2.15 Theorem: Let (R, + , .) and (R^l, +, .) be any two hemirings, then the anti-homomorphic preimage of an anti (T, S)-fuzzy ideal of R^l is an anti (T, S)-fuzzy ideal of R.

Proof: Let (R, +, .) and (R^l, +, .) be any two hemirings. Let $f : R \to R^l$ be an anti-homomorphism. Then, f(x+y) = f(y) +f(x) and f(xy) = f(y) f(x), for all x and y $\in$ R. Let V = f(A), where V is an anti (T, S)-fuzzy ideal of R^l. We have to prove that A is an anti (T, S)-fuzzy ideal of R. Let x and y $\in$ R.

Then, $\mu_A(x+ y) = \mu_v(f(x + y))$, since $\mu_v(f(x)) = \mu_A(x)$

$= \mu_v(f(y) + f(x))$, as f is an anti-homomorphism

$\leq S (\mu_v(f(y)), \mu_v(f(x)))$

$= S (\mu_v(f(x)), \mu_v(f(y)))$

$= S (\mu_A(x), \mu_A(y))$, since $\mu_v(f(x)) = \mu_A(x)$

which implies that $\mu_A(x+y) \leq S (\mu_A(x), \mu_A(y))$.

Again, $\mu_A(xy) = \mu_v(f(xy))$, since $\mu_v(f(x)) = \mu_A(x)$

$= \mu_v(f(y)f(x))$, as f is an anti-homomorphism

$\leq T (\mu_v(f(y)), \mu_v(f(x)))$

$= T (\mu_v(f(x)), \mu_v(f(y)))$

$= T (\mu_A(x), \mu_A(y))$, since $\mu_v(f(x)) = \mu_A(x)$

which implies that $\mu_A(xy) \leq T (\mu_A(x), \mu_A(y))$.

Hence A is an anti (T, S)-fuzzy ideal of R. □

3.2.16 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring R, A^+ be a fuzzy set in R defined by $A^+(x) = A(x)+1 – A(0)$, for all x in R , then A^+ is an anti (T, S)-fuzzy ideal of a hemiring R.

Proof : Let x and y in R. We have, $A^+(x+y) = A(x+y) +1- A(0)$

$\leq S (A(x), A(y)) + 1- A(0) \leq S((A(x) +1- A(0)), (A(y) +1- A(0)))$

$= S(A^+(x), A^+(y))$. Therefore, $A^+(x+y) \leq S(A^+(x), A^+(y))$, for all $x, y \in R$.

Similarly, $A^+(xy) = A(xy) +1- A(0) \leq T (A(x), A(y))+1- A(0)$

$\leq T ((A(x) +1- A(0)), (A(y) +1- A(0))) = T (A^+(x), A^+(y))$.

Therefore, $A^+(xy) \leq T (A^+(x), A^+(y))$, for all $x, y \in R$.

Hence A^+ is an anti (T, S)-fuzzy ideal of a hemiring R. □

3.2.17 Theorem : Let A be an anti (T, S)-fuzzy ideal of a hemiring R, A^+ be a fuzzy set in R defined by $A^+(x) = A(x) + 1 - A(0)$, for all $x \in R$, then there exists $0 \in R$ such that $A(0) = 1$ if and only if $A^+(x) = A(x)$.

Proof : The argument is trivial.

3.2.18 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring R, A^+ be a fuzzy set in R defined by $A^+(x) = A(x) + 1 - A(0)$, for all $x \in R$. Then there exists $x \in R$ such that $A^+(x) = 1$ if and only if $x = 0$.

Proof: The argument is trivial.

3.2.19 Theorem : Let A be an anti (T, S)-fuzzy ideal of a hemiring R, A^+ be a fuzzy set in R defined by $A^+(x) = A(x) + 1 - A(0)$, for all $x \in R$. Then $(A^+)^+ = A^+$.

Proof: Let x and y in R. We have, $(A^+)^+(x) = A^+(x) + 1 - A^+(0) = \{ A(x) + 1 - A(0) \} + 1 - \{A(0) + 1 - A(0) \} = A(x) + 1 - A(0) = A^+(x)$. Hence $(A^+)^+ = A^+$.

3.2.20 Theorem : Let A be an anti (T, S)-fuzzy ideal of a hemiring R. Then A^0 is an anti (T, S)-fuzzy ideal of the hemiring R .

Proof: For any $x \in R$, we have $A^0(x+y) = A(x+y)A(0) \leq [A(0)]\ S\ (A(x), A(y)) \leq S\ ([A(x)A(0)], [A(y)A(0)]) = S\ (A^0(x), A^0(y))$. That is $A^0(x+y) \leq S(A^0(x), A^0(y))$, for all $x, y \in R$. Similarly, $A^0(xy) = A(xy)A(0) \leq [A(0)]\ T(A(x), A(y)) \leq T([A(x)A(0)], [A(y)A(0)]) = T(A^0(x), A^0(y))$. That is $A^0(xy) \leq T(A^0(x), A^0(y))$, for all $x, y \in R$. Hence A^0 is an anti (T, S)-fuzzy ideal of the hemiring R. □

3.2.21 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring R. Then for α in [0,1] such that $\mu_A(0) \leq \alpha$, A_α is a lower level ideal of R.

Proof: For all x and y in $A_{\alpha,}$ we have, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$. Now, $\mu_A(x-y) \leq S(\mu_A(x), \mu_A(y)) \leq S(\alpha, \alpha) = \alpha$,which implies that $\mu_A(x-y) \leq \alpha$.

And, $\mu_A(xy) \leq T(\mu_A(x), \mu_A(y)) \leq T(\alpha, \alpha) = \alpha$, which implies that $\mu_A(xy) \leq \alpha$.

Therefore, $\mu_A(x-y) \leq \alpha$ and $\mu_A(xy) \leq \alpha$. Therefore, x - y and xy in A_α.

Hence A_α is a lower level ideal of a hemiring R. □

3.2.22 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring R. If any two lower level ideals of A belongs to R, then their intersection is also lower level ideal of A $\in$ R.

Proof : Let $\alpha_1, \alpha_2 \in [0,1]$.

Case (i): If $\alpha_1 < \mu_A(x) < \alpha_{2,}$ then $A_{\alpha 1} \subseteq A_{\alpha 2}$.

Therefore, $A_{\alpha 1} \cap A_{\alpha 2} = A_{\alpha 1}$, but $A_{\alpha 1}$ is a lower level ideal of A.

Case (ii): If $\alpha_1 > \mu_A(x) > \alpha_2$, then $A_{\alpha 2} \subseteq A_{\alpha 1}$.

Therefore, $A_{\alpha 1} \cap A_{\alpha 2} = A_{\alpha 2,}$ but $A_{\alpha 2}$ is a lower level ideal of A.

Case (iii): If $\alpha_1 = \alpha_2$, then $A_{\alpha 1} = A_{\alpha 2.}$

Hence, intersection of any two lower level ideals is a lower level ideal of A.

3.2.23 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring R. If $\alpha_i \in [0,1]$ and $A_{\alpha i}$, $i \in I$ is a collection of lower level ideals of A, then their intersection is also a lower level ideal of A.

Proof: The argument is trivial.

3.2.24 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring R. If any two lower level ideals of A belongs to R, then their union is also a lower level ideal of A in R.

Proof: Let $\alpha_1, \alpha_2 \in [0,1]$.

Case (i): If $\alpha_1 < \mu_A(x) < \alpha_2$, then $A_{\alpha 1} \subseteq A_{\alpha 2}$.

Therefore, $A_{\alpha 1} \cup A_{\alpha 2} = A_{\alpha 2}$, but $A_{\alpha 2}$ is a lower level ideal of A.

Case (ii): If $\alpha_1 > \mu_A(x) > \alpha_2$, then $A_{\alpha 2} \subseteq A_{\alpha 1}$.

Therefore , $A_{\alpha 1} \cup A_{\alpha 2} = A_{\alpha 1}$, but $A_{\alpha 1}$ is a lower level ideal of A.

Case (iii): If $\alpha_1 = \alpha_2$, then $A_{\alpha 1} = A_{\alpha 2}$.

Hence, union of any two lower level ideal is also a lower level ideal of A. □

3.2.25 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring R. If $\alpha_i \in [0,1]$ and $A_{\alpha i}$, $i \in I$ is a collection of lower level ideals of A, then their union is also a lower level ideal of A.

Proof: The argument is trivial.

3.2.26 Theorem: The homomorphic image of a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R is a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R^l.

Proof: Let (R, +, .) and (R^l , +, .) be any two hemirings and $f : R \rightarrow R^l$ be a homomorphism. That is, f(x+y)=f(x)+f(y) and f(xy)=f(x)f(y), for all x and y $\in$ R. Let V = f(A), where A is an anti (T, S)-fuzzy ideal of a hemiring R. Clearly V is an anti (T, S)-fuzzy ideal of a hemiring R^l. Let x and y $\in$ R, implies f(x) and f(y) $\in R^l$. Let A_α is a lower level ideal of A. That is, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$; $\mu_A(x+y) \leq \alpha$, $\mu_A(xy) \leq \alpha$. We have to prove that f (A_α) is a lower level ideal of V. Now, $\mu_V(f(x)) \leq \mu_A(x) \leq \alpha$, which implies that $\mu_V(f(x)) \leq \alpha$; and $\mu_V(f(y)) \leq \mu_A(y) \leq \alpha$, which implies that $\mu_V(f(y)) \leq \alpha$ and $\mu_V(f(x) + f(y)) = \mu_V(f(x + y)) \leq \mu_A(x + y) \leq \alpha$, which implies that $\mu_V(f(x) + f(y)) \leq \alpha$. Also, $\mu_V(f(x) f(y)) = \mu_V(f(xy)) \leq \mu_A(xy) \leq \alpha$, which implies that $\mu_V(f(x)f(y)) \leq \alpha$. Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$, $\mu_V(f(x) f(y)) \leq \alpha$. Hence f (A_α) is a lower level ideal of an anti (T, S)-fuzzy ideal V of R^I. □

3.2.27 Theorem: The homomorphic pre-image of a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R^I is a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R.

Proof: Let (R, +, .) and (R^I , +, .) be any two hemirings and f : R → R^I be a homomorphism.That is, f(x+y)=f(x)+f(y) and f(xy) = f(x)f(y) for all x and y ∈ R. Let V = f(A), where V is an anti (T, S)-fuzzy ideal of a hemiring R^I. Clearly A is an anti (T, S)-fuzzy ideal of a hemiring R. Let f(x) and f(y) ∈ R^I, implies x and y ∈ R. Let $f(A_\alpha)$ is a lower level ideal of V. That is, $\mu_V(f(x)) \leq \alpha$ and $\mu_V(f(y)) \leq \alpha$; $\mu_V(f(x) + f(y)) \leq \alpha$, $\mu_V(f(x) f(y)) \leq \alpha$. We have to prove that A_α is a lower level ideal of A. Now, $\mu_A(x) = \mu_V(f(x)) \leq \alpha$, implies that $\mu_A(x) \leq \alpha$; $\mu_A(y) = \mu_V(f(y)) \leq \alpha$, implies that $\mu_A(y) \leq \alpha$ and $\mu_A(x + y) = \mu_V(f(x + y)) = \mu_V(f(x) + f(y)) \leq \alpha$, which implies that $\mu_A(x+y) \leq \alpha$. Also, $\mu_A(xy) = \mu_V(f(xy)) = \mu_V(f(x)f(y)) \leq \alpha$, which implies that $\mu_A(xy) \leq \alpha$. Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$, $\mu_V(f(x)f(y)) \leq \alpha$. Hence, A_α is a lower level ideal of an anti (T, S)-fuzzy ideal A of R. □

3.2.28 Theorem: The anti homomorphic image of a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R is a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R^l.

Proof: Let (R, +, .) and (R^l , +, .) be any two hemirings and f : $R \rightarrow R^l$ be an anti-homomorphism. That is, f(x+y)=f(y)+f(x) and f(xy)=f(y)f(x), for all x and y in R. Let V = f(A), where A is an anti (T, S)-fuzzy ideal of R. Clearly V is an anti (T, S)-fuzzy ideal of R^l. Let x and y $\in$ R, implies f(x) and f(y) $\in R^l$. Let A_α is a lower level ideal of A. That is, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$, $\mu_A(y + x) \leq \alpha$, $\mu_A(yx) \leq \alpha$. We have to prove that f (A_α) is a lower level ideal of V. Now, $\mu_V(f(x)) \leq \mu_A(x) \leq \alpha$, which implies that $\mu_V(f(x)) \leq \alpha$; $\mu_V(f(y)) \leq \mu_A(y) \leq\alpha$, which implies that $\mu_V(f(y)) \leq \alpha$.

Now, $\mu_V(f(x)+f(y)) = \mu_V(f(y+x))$

$\leq \mu_A(y + x) \leq \alpha$, which implies that, $\mu_V(f(x) + f(y)) \leq \alpha$. Also, $\mu_V(f(x)f(y)) = \mu_V(f(yx)) \leq \mu_A(yx) \leq \alpha$, which implies that $\mu_V(f(x)f(y)) \leq \alpha$.Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x)f(y)) \leq \alpha$. Hence $f(A_\alpha)$ is a lower level ideal of an anti (T, S)-fuzzy ideal V of R^l. □

3.2.29 Theorem: The anti-homomorphic pre-image of a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R^l is a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R.

Proof: Let (R, +, .) and (R^l , +, .) be any two hemirings and f : $R \rightarrow R^l$ be an anti-homomorphism. That is, f(x + y) = f(y) + f(x) and f(xy) = f(y)f(x), for all x and y in R. Let V = f(A), where V is an anti (T, S)-fuzzy ideal of a hemiring R^l. Clearly A is an anti (T, S)-fuzzy ideal of a hemiring R. Let f(x) and f(y) in R^{l}, implies x and y in R. Let $f(A_\alpha)$ is a lower level ideal of V. That is, $\mu_V(f(x)) \leq \alpha$ and $\mu_V(f(y)) \leq \alpha$; $\mu_V(f(y)+f(x)) \leq \alpha$, $\mu_V(f(y)\, f(x)) \leq \alpha$. We have to prove that A_α is a lower level ideal of A. Now, $\mu_A(x) = \mu_V(f(x)) \leq \alpha$, which implies that $\mu_A(x) \leq \alpha$; $\mu_A(y) = \mu_V(f(y)) \leq \alpha$, which implies that $\mu_A(y) \leq \alpha$.

Now, $\mu_A(x + y) = \mu_V(f(x + y)) = \mu_V(f(y) + f(x)) \leq \alpha$, which implies that $\mu_A(x + y) \leq \alpha$. Also, $\mu_A(xy) = \mu_V(f(xy)) = \mu_V(f(y)f(x)) \leq \alpha$, which implies that $\mu_A(xy) \leq \alpha$. Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x)\, f(y)) \leq \alpha$. Hence A_α is a lower level ideal of an anti (T, S)-fuzzy ideal A of R. □

3.3 ANTI (T, S)-FUZZY NORMAL IDEALS OF A HEMIRING

3.3.1 Theorem: Let (R, +, .) be a hemiring. If A and B are two anti (T, S)-fuzzy normal ideals of R, then $A \cup B$ is an anti (T, S)-fuzzy normal ideal of R.

Proof: Let x, $y \in R$. Let $A = \{\langle x, \mu_A(x) \rangle / x \in R\}$ and $B = \{\langle x, \mu_B(x) \rangle / x \in R\}$ be anti (T, S)-fuzzy normal ideals of a hemiring R. Let $C = A \cup B$ and $C = \{\langle x, \mu_C(x) \rangle / x \in R\}$, where $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$. Then, Clearly C is an anti (T, S)-fuzzy ideal of a hemiring R, since A and B are two anti (T, S)-fuzzy ideals of the hemiring R.

And, $\mu_C(xy) = \max\{\mu_A(xy), \mu_B(xy)\}$,

$= \max\{\mu_A(yx), \mu_B(yx)\}$

$= \mu_C(yx)$, for all x and $y \in R$.

Therefore, $\mu_C(xy) = \mu_C(yx)$, for all x and $y \in R$.

Hence $A \cup B$ is an anti (T, S)-fuzzy normal ideal of the hemiring R. □

3.3.2 Theorem: Let (R, +, .) be a hemiring. The union of a family of anti (T, S)-fuzzy normal ideals of R is an anti (T, S)-fuzzy normal ideal of R.

Proof: The argument is trivial.

3.3.3 Theorem: Let A and B be anti (T, S)-fuzzy ideals of the hemirings G and H, respectively. If A and B are anti (T, S)-fuzzy normal ideals, then A×B is an anti (T, S)-fuzzy normal ideal of G×H.

Proof: Let A and B be anti (T, S)-fuzzy normal ideals of the hemirings G and H respectively. Clearly A×B is an anti (T, S)-fuzzy ideal of G×H. Let x_1 and $x_2 \in$ G, y_1 and $y_2 \in$ H. Then (x_1, y_1) and (x_2, y_2) are $\in$ G×H.

$$= \max \{ \mu_A(x_1x_2), \mu_B(y_1y_2) \}$$

$$= \max \{ \mu_A(x_2x_1), \mu_B(y_2y_1) \},$$

$$= \mu_{A\times B} (x_2x_1, y_2y_1)$$

$$= \mu_{A\times B} [(x_2, y_2)(x_1, y_1)].$$

Therefore, $\mu_{A\times B} [(x_1, y_1)(x_2, y_2)] = \mu_{A\times B} [(x_2, y_2)(x_1, y_1)]$.

Hence A×B is an anti (T, S)-fuzzy normal ideal of G×H. □

3.3.4 Theorem: Let A and B be anti (T, S)-fuzzy normal ideal of the hemirings R_1 and R_2 respectively. Suppose that 0_1 and 0_2 are the zero element of R_1 and R_2 respectively. If A×B is an anti (T, S)-fuzzy normal ideal of $R_1 \times R_2$, then at least one of the following two statements must hold.

(i) $B(0_2) \leq A(x)$, for all $x \in R_1$,

(ii) $A(0_1) \leq B(y)$, for all $y \in R_2$.

Proof: The argument is trivial.

3.3.5 Theorem: Let A and B be two fuzzy subsets of the hemirings R_1 and R_2 respectively and A×B is an anti (T, S)-fuzzy normal ideal of $R_1 \times R_2$. Then the following are true:

(i) if $A(x) \geq B(0_2)$, then A is an anti (T, S)-fuzzy normal ideal of R_1.

(ii) if $B(x) \geq A(0_1)$, then B is an anti (T, S)-fuzzy normal ideal of R_2.

(iii) either A is an anti (T, S)-fuzzy normal ideal of R_1 or B is an anti (T, S)-fuzzy normal ideal of R_2.

Proof: The argument is trivial.

3.3.6 Theorem: Let A be a fuzzy subset in a hemiring R and V be the anti-strongest fuzzy relation on R, then A is an anti (T, S)-fuzzy normal ideal of R if and only if V is an anti (T, S)-fuzzy normal ideal of R×R.

Proof: The argument is trivial.

3.3.7 Theorem: Let (R, +, .) and (R^I, +, .) be any two hemirings. The homomorphic image of an anti (T, S)-fuzzy normal ideal of R is an anti (T, S)-fuzzy normal ideal of R^I.

Proof: Let (R, +, .) and (R^I, +, .) be any two hemirings and $f : R \to R^I$ be a homomorphism. Then, $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$, for all x and y in R. Let V = f(A), where A is an anti (T, S)-fuzzy normal ideal of a hemiring R. We have to prove that V is an anti (T, S)-fuzzy normal ideal of a hemiring R^I. Now, for f(x), f(y) in R^I, clearly V is an anti (T, S)-fuzzy ideal of a hemiring R^I, since A is an anti (T, S)-fuzzy ideal of a hemiring R.

Now , $\mu_v(f(x)f(y)) = \mu_v(f(xy))$, as f is a homomorphism

$$\leq \mu_A(xy)$$

$= \mu_A(yx)$

$\geq \mu_v(f(yx))$

$= \mu_v(f(y) f(x))$, as f is a homomorphism

which implies that $\mu_v(f(x)f(y)) = \mu_v(f(y) f(x))$, for all f(x) and f(y) $\in R^l$.

Hence V is an anti (T, S)-fuzzy normal ideal of a hemiring R^l. □

3.3.8 Theorem: Let (R, +, .) and (R^l, +, .) be any two hemirings, then the homomorphic preimage of an anti (T, S)-fuzzy normal ideal of R^l is an anti (T, S)-fuzzy normal ideal of R.

Proof: Let (R, +, .) and (R^l, +, .) be any two hemirings and $f : R \rightarrow R^l$ be a homomorphism. Then, f(x+y) = f(x)+f(y) and f(xy) = f(x) f(y), for all x and y in R. Let V = f(A), where V is an anti (T, S)-fuzzy normal ideal of a hemiring R^l. We have to prove that A is an anti (T, S)-fuzzy normal ideal of a hemiring R. Let x and y in R. Then, clearly A is an anti (T, S)-fuzzy ideal of a hemiring R, since V is an anti (T, S)-fuzzy ideal of a hemiring R^l.

Now, $\mu_A(xy) = \mu_v(f(xy))$, since $\mu_A(x) = \mu_v(f(x))$

$= \mu_v(f(x)f(y))$, as f is a homomorphism

$= \mu_v(f(y)f(x))$

$= \mu_v(f(yx))$, as f is a homomorphism

$= \mu_A(yx)$, since $\mu_A(x) = \mu_v(f(x))$

which implies that $\mu_A(xy) = \mu_A(yx)$, for all x and y $\in$ R.

Hence A is an anti (T, S)-fuzzy normal ideal of a hemiring R. □

3.3.9 Theorem: Let (R, +, .) and (R^l, +, .) be any two hemirings, then the anti-homomorphic image of an anti (T, S)-fuzzy normal ideal of R is an anti (T, S)-fuzzy normal ideal of R^l.

Proof: Let (R, +, .) and (R^l, +, .) be any two hemirings and $f : R \to R^l$ be an anti-homomorphism. Then, $f(x+y) = f(y) + f(x)$ and $f(xy) = f(y) f(x)$, for all x and y in R. Let V = f(A), where A is an anti (T, S)-fuzzy normal ideal of a hemiring R. We have to prove that V is an anti (T, S)-fuzzy normal ideal of a hemiring R^l. Now, for f(x) and f(y) in R^l, clearly V is an anti (T, S)-fuzzy ideal of a hemiring R^l, since A is an anti (T, S)-fuzzy ideal of a hemiring R.

Now, $\mu_v(f(x)f(y))$ $= \mu_v(f(yx))$, as f is an anti-homomorphism

$\leq \mu_A(yx)$

$= \mu_A(xy)$

$\geq \mu_v(f(xy))$

$= \mu_v(f(y) f(x))$, as f is an anti-homomorphism

which implies that $\mu_v(f(x)f(y) = \mu_v(f(y)f(x))$, for all f(x) and f(y) $\in R^l$.

Hence V is an anti (T, S)-fuzzy normal ideal of a hemiring R^l. □

3.3.10 Theorem: Let (R, +, .) and (R^l, +, .) be any two hemirings, then the anti-homomorphic preimage of an anti (T, S)-fuzzy normal ideal of R^l is an anti (T, S)-fuzzy normal ideal of R.

Proof: Let (R, +, .) and (R^l, +, .) be any two hemirings and $f : R \to R^l$ be an anti-homomorphism. Then, f (x+y) = f(y) + f(x) and f(xy) = f(y) f(x), for all x and y in R. Let V = f(A), where V is an anti (T, S)-fuzzy normal ideal of a hemiring R^{l}. We have to prove that A is an anti (T, S)-fuzzy normal ideal of a hemiring R.

Let x and y $\in$ R, then, clearly A is an anti (T, S)-fuzzy ideal of a hemiring R, since V is an anti (T, S)-fuzzy ideal of a hemiring R^{l}.

Now, $\mu_A(xy) = \mu_v(f(xy))$, since $\mu_A(x) = \mu_v(f(x))$

$= \mu_v(f(y)f(x))$, as f is an anti-homomorphism

$= \mu_v(f(x)f(y))$

$= \mu_v(f(yx))$, as f is an anti-homomorphism

$= \mu_A(yx)$, since $\mu_A(x) = \mu_v(f(x))$

which implies that $\mu_A(xy) = \mu_A(yx)$, for all x and y $\in$ R.

Hence A is an anti (T, S)-fuzzy normal ideal of a hemiring R. □

In the next theorem we introduce a composition operation to get s S-fuzzy normal ideal.

3.3.11 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring H and f is an isomorphism from a hemiring R onto H. If A is an anti (T, S)-fuzzy normal ideal of the hemiring H, then A$\circ$f is an anti (T, S)-fuzzy normal ideal of the hemiring R.

Proof: Let x, y $\in$ R and A be an anti (T, S)-fuzzy normal ideal of a hemiring H. Then, clearly A$\circ$f is an anti (T, S)-fuzzy ideal of a hemiring R.

Now, $(\mu_A \circ f)(xy) = \mu_A(f(xy))$

$= \mu_A(f(x)f(y))$, as f is an isomorphism

$= \mu_A(f(y)f(x))$

$= \mu_A(f(yx))$, as f is an isomorphism

$= (\mu_A \circ f)(yx)$,

which implies that $(\mu_A \circ f)(xy) = (\mu_A \circ f)(yx)$, for all x and $y \in R$.

Hence $A \circ f$ is an anti (T, S)-fuzzy normal ideal of a hemiring R. □

3.3.12 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring H and f is an anti-isomorphism from a hemiring R onto H. If A is an anti (T, S)-fuzzy normal ideal of the hemiring H, then $A \circ f$ is an anti (T, S)-fuzzy normal ideal of the hemiring R.

Proof: Let x, y in R and A be an anti (T, S)-fuzzy normal ideal of a hemiring H. Then, clearly $A \circ f$ is anti (T, S)-fuzzy ideal of the hemiring R.

Now, $(\mu_A \circ f)(xy) = \mu_A(f(xy))$

$= \mu_A(f(y)f(x))$, as f is an anti-isomorphism

$= \mu_A(f(x)f(y))$

$= \mu_A(f(yx))$, as f is an anti-isomorphism

$= (\mu_A \circ f) (yx)$,

which implies that $(\mu_A \circ f)(xy) = (\mu_A \circ f)(yx)$, for all x and $y \in R$.

Hence $A \circ f$ is an anti (T, S)-fuzzy normal ideal of the hemiring R. □

3.3.13 Theorem: The homomorphic image of a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R is a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R^{l}.

Proof: The argument is trivial.

3.3.14 Theorem: The homomorphic pre-image of a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R^{l} is a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R.

Proof: The argument is trivial.

3.3.15 Theorem: The anti-homomorphic image of a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R is a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R^{l}.

Proof: The argument is trivial.

3.3.16 Theorem: The anti-homomorphic pre-image of a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R^{l} is a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R.

Proof: The argument is trivial.

3.4 TRANSLATIONS OFANTI S-FUZZY SUBHEMIRINGS OF A HEMIRING

3.4.1 Theorem: If M and N are two translations of anti S-fuzzy subhemiring A of a hemiring (R, +, .), then their intersection $M \cap N$ is translation of anti S-fuzzy subhemiring A.

Proof: Let x and y $\in$R. Let M $= T_{\alpha}^{A} = \{ \langle x, \mu_A(x)+ \alpha \rangle / x \in R\}$ and N $= T_{\gamma}^{A} = \{ \langle x, \mu_A(x) + \gamma \rangle / x \in R\}$ be two translations of anti S-fuzzy subhemiring A of a hemiring (R, +, .). Let C = $M \cap N$ and C = $\{ \langle x, \mu_C(x)\rangle / x \text{ in } R\}$, where $\mu_C(x) = \min\{\mu_A(x)+\alpha, \mu_A(x)+ \gamma \}$.

Case (i): $\alpha \leq \gamma$.

Now, $\mu_C(x+y) = \min \{ \mu_M(x+y), \mu_N(x+y) \}$

$= \min \{ \mu_A(x+y) + \alpha, \mu_A(x+y) + \gamma \}$

$= \mu_A(x+y) + \alpha$

$= \mu_M(x+y)$, for all x and y $\in$ R.

And, $\mu_C(xy) = \min \{ \mu_M(xy), \mu_N(xy) \}$

$= \min \{ \mu_A(xy) + \alpha, \mu_A(xy) + \gamma \}$

$= \mu_A(xy) + \alpha$

$= \mu_M(xy)$, for all x and y $\in$ R.

Therefore C = $T_\alpha^A = \{ \langle x, \mu_A(x)+\alpha \rangle / x \in R\}$ is a translation of anti S-fuzzy subhemiring A of the hemiring (R, +, .).

Case (ii): $\alpha \geq \gamma$.

Now, $\mu_C(x+y) = \min \{ \mu_M(x+y), \mu_N(x+y) \}$

$= \min \{ \mu_A(x+y) + \alpha, \mu_A(x+y) + \gamma \}$

$= \mu_A(x+y) + \gamma$

$= \mu_N(x+y)$, for all x and y $\in$ R.

And $\mu_C(xy) = \min \{ \mu_M(xy), \mu_N(xy) \}$

$= \min \{ \mu_A(xy) + \alpha, \mu_A(xy) + \gamma \}$

$= \mu_A(xy) + \gamma$

$= \mu_N(xy)$, for all x and y $\in$ R.

Therefore C $= T_\gamma^A = \{ \langle x, \mu_A(x) + \gamma \rangle / x$ in R$\}$ is a translation of anti S-fuzzy subhemiring A of the hemiring (R, +, .). Thus in all cases, intersection of any two translations of anti S-fuzzy subhemiring A of a hemiring (R, +, ·) is also a translation of anti S-fuzzy subhemiring A. □

3.4.2 Theorem: The intersection of a family of translations of anti S-fuzzy subhemiring A of a hemiring (R, +, ·) is also a translation of anti S-fuzzy subhemiring A.

Proof: The argument is trivial.

3.4.3 Theorem: If M and N are two translations of anti S-fuzzy subhemiring A of a hemiring (R, +, ·), then their union M ∪ N is also a translation of anti S-fuzzy subhemiring A.

Proof: Let x and y belong to R. Let M $= T_{\alpha}^{A} = \{ \langle x, \mu_A(x)+ \alpha\rangle / x \text{ in } R\}$ and N $= T_{\gamma}^{A} = \{ \langle x, \mu_A(x)+ \gamma \rangle / x \text{ in } R\}$ be two translations of anti S-fuzzy subhemiring A of a hemiring (R, +, .). Let C = M ∪ N and $C = \{ \langle x, \mu_C(x) \rangle / x \text{ in } R\}$, where $\mu_C(x) = \max\{ \mu_A(x)+\alpha, \mu_A(x) + \gamma \}$.

Case (i): $\alpha \leq \gamma$.

Now, $\mu_C(x+y) = \max \{ \mu_M(x+y), \mu_N(x+y) \}$

$= \max \{ \mu_A(x+y) + \alpha, \mu_A(x+y) + \gamma \}$

$= \mu_A(x+y) + \gamma$

$= \mu_N(x+y)$, for all x and y ∈ R.

And, $\mu_C(xy) = \max \{ \mu_M(xy), \mu_N(xy) \}$

$= \max\{ \mu_A(xy) + \alpha, \mu_A(xy) + \gamma \}$

$= \mu_A(xy) + \gamma$

$= \mu_N(xy)$, for all x and y ∈ R.

Therefore C $= T_{\gamma}^{A} = \{ \langle x, \mu_A(x) + \gamma\rangle / x \in R\}$ is a translation of anti S-fuzzy subhemiring A of a hemiring (R, +, .).

Case (ii): $\alpha \geq \gamma$.

Now, $\mu_C(x+y) = \max \{ \mu_M(x+y), \mu_N(x+y) \}$

$= \max \{ \mu_A(x+y) + \alpha, \mu_A(x+y) + \gamma \}$

$= \mu_A(x+y) + \alpha$

$= \mu_M(x+y)$, for all x and y $\in$ R.

And $\mu_C(xy) = \max \{ \mu_M(xy), \mu_N(xy) \}$

$= \max \{ \mu_A(xy) + \alpha, \mu_A(xy) + \gamma \}$

$= \mu_A(xy) + \alpha$

$= \mu_M(xy)$, for all x and y $\in$ R.

Therefore C $= T_\alpha^A = \{ \langle x, \mu_A(x)+\alpha \rangle / x$ in R$\}$ is a translation of anti S-fuzzy subhemiring A of the hemiring (R, +, .). Hence all cases, union of any two translations of anti S-fuzzy subhemiring A of a hemiring (R, +, ·) is also a translation of anti S-fuzzy subhemiring A. □

3.4.4 Theorem: The union of a family of translations of anti S-fuzzy subhemiring A of a hemiring (R, +, ·) is also a translation of anti S-fuzzy subhemiring A.

Proof: The argument is trivial.

3.4.5 Theorem: If T_α^A is a translation of anti S-fuzzy subhemiring A of a hemiring R, then T_α^A is anti S-fuzzy subhemiring of R.

Proof: Assume that T_α^A is a translation of anti S-fuzzy subhemiring A of a hemiring R. Let x and y in R.

We have, $T_\alpha^A(x+y) = A(x+y) + \alpha$

$\leq S(A(x), A(y)) + \alpha$

$\leq S(A(x) + \alpha, A(y) + \alpha)$

$= S(T_\alpha^A(x), T_\alpha^A(y)\}$.

Therefore, $T_\alpha^A(x+y) \leq S(T_\alpha^A(x), T_\alpha^A(y))$, for all x and y $\in$ R.

And, T_{α}^{A} (xy) = A(xy) + α

$\leq$ S (A(x), A(y)) + α

$\leq$ S (A(x) + α , A(y) + α)

= S (T_{α}^{A} (x), T_{α}^{A} (y)).

Therefore, T_{α}^{A} (xy) $\geq$ S (T_{α}^{A} (x), T_{α}^{A} (y)), for all x and y in R. Hence T_{α}^{A} is anti S-fuzzy subhemiring of R □

3.4.6 Theorem: Let (R, +, ,.) and (R^{l}, +, .) be any two hemirings. If f: R→ R^{l} is a homomorphism, then the translation of anti S-fuzzy subhemiring A of R under the homomorphic image is anti S-fuzzy subhemiring of f (R) = R^{l}.

Proof: Let (R, +, .) and (R^{l}, +, .) be any two hemirings and f : R→ R^{l} be a homomorphism. That is f(x+y) = f(x)+f(y) and f(xy) = f(x)f(y), for all x and y in R. Let T_{α}^{A} be a translation of anti S-fuzzy subhemiring A of R. Let V be the homomorphic image of T_{α}^{A} under f. We have to prove that V is anti S-fuzzy subhemiring of f(R) = R^{l}. Now, for f(x) and f(y) in R^{l}, we have

V[f(x)+ f(y)] = V [f(x+y)] $\leq$ T_{α}^{A} (x+y)

= A (x+y) + α

$\leq$ S (A(x), A(y)) + α

$\leq$ S (A(x) + α, A(y) + α)

= S (T_{α}^{A} (x), T_{α}^{A} (y)),

which implies that V[f(x)+f(y)] $\leq$ S (V(f(x)), V(f(y))), for all f(x), f(y)$\in$ R^{l}.

And V[f(x)f(y)] = V [f(xy)] $\leq T_{\alpha}^{A}$ (xy)

$= A(xy) + \alpha$

$\leq S(A(x), A(y)) + \alpha$

$\leq S(A(x) + \alpha, A(y) + \alpha)$

$= S(T_{\alpha}^{A}(x), T_{\alpha}^{A}(y))$,

which implies that V[f(x)f(y)] $\leq$ S (V(f(x)), V(f(y))), for all

f(x) and f(y) $\in R^{l}$. Therefore, V is an anti S-fuzzy subhemiring of R^{l}.

Hence the homomorphic image of translation of A of R is an anti S-fuzzy subhemiring of R^{l}. □

3.4.7 Theorem: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. If f : R→ R^{l} is a homomorphism, then the translation of an anti S-fuzzy subhemiring V of f (R) = R^{l} under the homomorphic pre-image is an anti S-fuzzy subhemiring of R.

Proof: Let (R, +, .) and (R^{l}, +, .) be any two hemirings and f : R→ R^{l} be a homomorphism. That is f(x+y) = f(x) + f(y) and f(xy) = f(x)f(y), for all x and y in R. Let T_{α}^{V} be the translation of anti S-fuzzy subhemiring V of R^{l} and A be the homomorphic pre-image of T_{α}^{V} under f. We have to prove that A is an anti S-fuzzy subhemiring of R. Let x and y be in R.

Then, A(x+y) = T_{α}^{V} (f(x+y))

$= T_{\alpha}^{V}(f(x)+f(y))$

$= V[f(x)+f(y)] + \alpha$

$\leq S(V(f(x)), V(f(y))) + \alpha$

$\leq S ((V(f(x)) + \alpha, \; V(f(y)) + \alpha)$

$= S (T_{\alpha}^{V} (f(x)), \; T_{\alpha}^{V} (f(y)))$

$= S (A(x), A(y)),$

which implies that $A(x+y) \leq S (A(x), A(y))$, for all x, y in R.

And, $A(xy) = T_{\alpha}^{V} (f(xy)) = T_{\alpha}^{V} (f(x)f(y))$

$= V[f(x)f(y)] + \alpha$

$\leq S (V(f(x)), V(f(y))) + \alpha$

$\leq S ((V(f(x)) + \alpha, \; V(f(y)) + \alpha))$

$= S (T_{\alpha}^{V} (f(x)), \; T_{\alpha}^{V} (f(y)))$

$= S (A(x), A(y)),$

which implies that, $A(xy) \leq S (A(x), A(y))$, for all x and y $\in$ R.

Therefore, A is anti S-fuzzy subhemiring of R. □

3.4.8 Theorem: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. If f : R→ R^{l} is an anti-homomorphism, then the translation of an anti S-fuzzy subhemiring A of R under the anti-homomorphic image is an anti S-fuzzy subhemiring of f (R) = R^{l}.

Proof: Let (R, +, .) and (R^{l}, +, .) be any two hemirings and f : R→ R^{l} be an anti-homomorphism. That is $f(x+y) = f(y)+ f(x)$ and $f(xy) = f(y)f(x)$, for all x and y $\in$ R. Let T_{α}^{A} be the translation of an anti S-fuzzy subhemiring A of R and V be the anti-homomorphic image of T_{α}^{A} under f. We have to prove that V is an anti S-fuzzy subhemiring of f (R) = R^{l}. Now, for f(x) and f(y) in R^{l} and q in Q, we have, $V[f(x)+f(y)] = V[f(y+x)]$

$\leq T_{\alpha}^{A}$ (y+x)

= A (y+x) + α

≤ S (A(x), A(y)) + α

≤ S (A(x) + α, A(y)+ α)

= S (T_{α}^{A}(x), T_{α}^{A}(y))

which implies that V[f(x) + f(y)] ≤ S (V(f(x)), V(f(y)), for all f(x), f(y)∈ R^{l}.

And, V[f(x)f(y)] = V[f(yx)]

$\leq T_{\alpha}^{A}$ (yx)

= A (yx) + α

≤ S (A(x), A(y)) + α

≤ S (A(x)+α, A(y)+α)

= S (T_{α}^{A}(x), T_{α}^{A}(y))

which implies that V[f(x)f(y)] ≤ S (V(f(x)), V(f(y))), for all f(x) and f(y) in R^{l}. Therefore, V is an anti S-fuzzy subhemiring of the hemiring R^{l}. Hence the anti-homomorphic image of translation of A of R is an anti S-fuzzy subhemiring of R^{l}. □

3.4.9 Theorem: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. If f: R → R^{l} is an anti-homomorphism, then the translation of an anti S-fuzzy subhemiring V of f (R) = R^{l} under the anti-homomorphic pre-image is an anti S-fuzzy subhemiring of R.

Proof: Let (R, +, .) and (R^{l}, +, .) be any two hemirings and f: R → R^{l} be an anti-homomorphism. That is f(x+y) = f(y)+f(x) and f(xy) = f(y)f(x), for all x and y $\in$ R. Let T_α^V be the translation of an anti S-fuzzy subhemiring V of f(R) = R^l and A be the anti-homomorphic pre-image of T_α^V under f. We have to prove that A is an anti S-fuzzy subhemiring of R. Let x and y $\in$ R. Then,

$$
\begin{aligned}
A(x+y) &= T_\alpha^V(\, f(x+y)\,) \\
&= T_\alpha^V[\, f(y)+f(x)] \\
&= V[\, f(y)+f(x)] + \alpha \\
&\leq S(\, V(f(x)), V(f(y))\,) + \alpha \\
&\leq S(\, V(f(x))+\alpha, V(f(y))+\alpha\,) \\
&= S(\, T_\alpha^V(f(x))\,), T_\alpha^V(f(y))\,) \\
&= S(\, A(x), A(y)\,),
\end{aligned}
$$

which implies that A(x+y) $\leq$ S (A(x), A(y), for all x and y $\in$ R.

$$
\begin{aligned}
\text{And, } A(xy) &= T_\alpha^V(\, f(xy)\,) \\
&= T_\alpha^V(\, f(y)f(x)) \\
&= V[\, f(y)f(x)] + \alpha \\
&\leq S(\, V(f(x)), V(f(y))\,) + \alpha \\
&\leq S(\, V(f(x))+\alpha, V(f(y))+\alpha\,) \\
&= S(\, T_\alpha^V(\, f(x))\,), T_\alpha^V(f(y))\,) \\
&= S(\, A(x), A(y)\,),
\end{aligned}
$$

which implies that A(xy) $\leq$ S (A(x), A(y), for all x and y $\in$ R.

Therefore, A is an anti S-fuzzy subhemiring of R. □

3.4.10 Theorem: If M and N are two translations of an anti S-fuzzy normal subhemiring A of a hemiring (R, +, .), then their intersection $M \cap N$ is also a translation of A.

Proof: The argument is trivial.

3.4.11 Theorem: The intersection of a family of translations of an anti S-fuzzy normal subhemiring A of a hemiring (R, +, .) is a translation of A.

Proof: The argument is trivial.

3.4.12 Theorem: If M and N are two translations of an anti S-fuzzy normal subhemiring A of a hemiring (R, +, .), then their union $M \cup N$ is also a translation of A.

Proof: The argument is trivial.

3.4.13 Theorem: The union of a family of translations of an anti S-fuzzy normal subhemiring A of a hemiring (R, +, .) is also a translation of A.

Proof: The argument is trivial.

3.4.14 Theorem: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. If $f : R \rightarrow R^{l}$ is a homomorphism, then the translation of an anti S-fuzzy normal subhemiring A of R under the homomorphic image is an anti S-fuzzy normal subhemiring of $f(R) = R^{l}$.

Proof: Let (R, +, .) and (R^{l}, +, .) be any two hemirings and $f : R \rightarrow R^{l}$ be a homomorphism. That is $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$, for all x and y $\in$ R. Let T_{α}^{A} be the translation of an anti S-fuzzy normal subhemiring A of R and V be the homomorphic image of T_{α}^{A} under f. We have to prove that V is an anti S-fuzzy normal subhemiring of R^{l}. Now, for f(x) and f(y) $\in$ R^{l}, clearly V is an anti S-fuzzy subhemiring of R^{l}.

We have V(f(x)f(y)) = V(f(xy)) $\leq$ T_{α}^{A}(xy) = A(xy) + α = A(yx) + α

= T_{α}^{A}(yx) $\geq$ V(f(yx)) = V(f(y) f(x)),

which implies that V(f(x)f(y)) = V(f(y)f(x)), for all f(x) and f(y) $\in$ R^{l}. Therefore, V is an anti S-fuzzy normal subhemiring of the hemiring R^{l}. □

3.4.15 Theorem: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. If f : R $\rightarrow$ R^{l} is a homomorphism, then translation of an anti S-fuzzy normal subhemiring V of f(R) = R^{l} under the homomorphic pre-image is an anti S-fuzzy normal subhemiring of R.

Proof: Let (R, +, .) and (R^{l}, +, .) be any two hemirings and f : R $\rightarrow$ R^{l} be a homomorphism. That is f (x+y) = f(x)+f(y) and f(xy) = f(x)f(y), for all x and y $\in$ R. Let T_{α}^{V} be the translation of an anti S-fuzzy normal subhemiring V of R^{l} and A be the homomorphic pre-image of T_{α}^{V} under f. We have to prove that A is an anti S-fuzzy normal subhemiring of R. Let x and y $\in$ R. Then, clearly A is an anti S-fuzzy subhemiring of R, A(xy) = T_{α}^{V}(f(xy)) = V(f(xy)) + α = V(f(x)f(y)) + α = V(f(y)f(x)) + α = V(f(yx)) + α = T_{α}^{V}(f(yx)) = A(yx), which implies that A(xy) = A(yx), for all x and y in R. Therefore, A is an anti S-fuzzy normal subhemiring of R. □

3.4.16 Theorem: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. If f : R $\rightarrow$ R^{l} is an anti-homomorphism, then the translation of an anti S-fuzzy normal subhemiring A of R under the anti-homomorphic image is an anti S-fuzzy normal subhemiring of R^{l}.

Proof: Let (R, +, .) and (R^{l}, +, .) be any two hemirings and $f : R \rightarrow R^{l}$ be an anti-homomorphism. That is f(x+y) = f(y)+f(x) and f(xy) = f(y)f(x), for all x and y $\in$ R. Let T_{α}^{A} be the translation of an anti S-fuzzy normal subhemiring A of R and V be the anti-homomorphic image of T_{α}^{A} under f. We have to prove that V is an anti S-fuzzy normal subhemiring of f(R) = R^{l}. Now, for f(x) and f(y) in R^{l}, clearly V is an anti S-fuzzy subhemiring of R^{l}. We have, V(f(x)f(y)) = V(f(yx)) $\leq T_{\alpha}^{A}$(yx) = A(yx) + α = A(xy) + α = T_{α}^{A}(xy) $\geq$ V(f(xy)) = V(f(y)f(x)), which implies that V(f(x)f(y)) = V(f(y)f(x)), for f(x) and f(y) in R^{l}. Therefore, V is an anti S-fuzzy normal subhemiring of the hemiring R^{l}. □

3.4.17 Theorem: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. If $f : R \rightarrow R^{l}$ is an anti-homomorphism, then the translation of an anti S-fuzzy normal subhemiring V of f(R) = R^{l} under the anti-homomorphic pre-image is an anti S-fuzzy normal subhemiring of R.

Proof: Let (R, +, .) and (R^{l}, +, .) be any two hemirings and $f : R \rightarrow R^{l}$ be an anti-homomorphism. That is f(x + y) = f(y) + f(x) and f(xy) = f(y)f(x), for all x and y $\in$ R. Let T_{α}^{V} be the translation of an anti S-fuzzy normal subhemiring V of R^{l} and A be the anti-homomorphic pre-image of T_{α}^{V} under f. We have to prove that A is an anti S-fuzzy normal subhemiring of R. Let x and y $\in$ R. Then, clearly A is an anti S-fuzzy subhemiring of R, $A(xy) = T_{\alpha}^{V}(\,f(xy)\,) = V(\,f(xy)\,) + \alpha$

$$= V(\,f(y)f(x)) + \alpha$$

$$= V(\,f(x)f(y)) + \alpha$$

$$= V(\,f(yx)) + \alpha = T_{\alpha}^{V}(\,f(yx)) = A(yx),$$

which implies that $A(xy) = A(yx)$, for all x and y $\in$ R.

Therefore, A is an anti S-fuzzy normal subhemiring of R. □