

CHAPTER - 3

ANTI (T, S)-FUZZY IDEALS OF A HEMIRING AND TRANSLATIONS

3.1 Introduction: In this chapter, we introduce the concept of anti (T, S)-fuzzy ideals of a hemiring, translations of anti (T, S)-fuzzy subhemiring and establish some results on these.

3.1.1 Definition: A (T, S)-norm is a binary operations $T: [0, 1] \times [0, 1] \rightarrow [0, 1]$ and $S: [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following requirements;

- (i) $T(0, x) = 0, T(1, x) = x$ (boundary condition)
- (ii) $T(x, y) = T(y, x)$ (commutativity)
- (iii) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity)
- (iv) if $x \leq y$ and $w \leq z$, then $T(x, w) \leq T(y, z)$ (monotonicity).
- (v) $S(0, x) = x, S(1, x) = 1$ (boundary condition)
- (vi) $S(x, y) = S(y, x)$ (commutativity)
- (vii) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity)
- (viii) if $x \leq y$ and $w \leq z$, then $S(x, w) \leq S(y, z)$ (monotonicity).

3.1.2 Definition: Let $(R, +, \cdot)$ be a hemiring. A fuzzy subset A of R is said to be an anti (T, S)-fuzzy ideal (anti fuzzy ideal with respect to (T, S)-norm) of R if it satisfies the following conditions:

- (i) $\mu_A(x + y) \leq S(\mu_A(x), \mu_A(y))$,
- (ii) $\mu_A(xy) \leq T(\mu_A(x), \mu_A(y))$, for all x and $y \in R$.

3.1.3 Definition: Let $(R, +, \cdot)$ be a hemiring. An anti (T, S) -fuzzy ideal A of R is said to be an anti (T, S) -fuzzy normal ideal (ATSFNI) of R if $\mu_A(xy) = \mu_A(yx)$, for all x and $y \in R$.

3.1.4 Definition: Let A be an anti (T, S) -fuzzy ideal of a hemiring $(R, +, \cdot)$ and a in R . Then the pseudo anti (T, S) -fuzzy coset $(aA)^p$ is defined by $((a\mu_A)^p)(x) = p(a)\mu_A(x)$, for every $x \in R$ and for some $p \in P$.

3.1.5 Definition: Let A be a fuzzy subset of X and $\alpha \in [0, 1 - \text{Sup}\{A(x) : x \in X, 0 < A(x) < 1\}]$. Then $T = T_\alpha^A$ is called a **translation** of A if $T(x) = A(x) + \alpha$, for all $x \in X$.

3.2 - PROPERTIES OF ANTI (T, S) -FUZZY IDEALS OF A HEMIRING:

3.2.1 Theorem: Union of any two anti (T, S) -fuzzy ideal of a hemiring R is an anti (T, S) -fuzzy ideal of R .

Proof: Let A and B be any two anti (T, S) -fuzzy ideals of a hemiring R and x and $y \in R$. Let $A = \{(x, \mu_A(x)) / x \in R\}$ and $B = \{(x, \mu_B(x)) / x \in R\}$ and $C = A \cup B = \{(x, \mu_C(x)) / x \in R\}$, where $\max\{\mu_A(x), \mu_B(x)\} = \mu_C(x)$.

$$\begin{aligned} \text{Now, } \mu_C(x + y) &= \max\{\mu_A(x + y), \mu_B(x + y)\} \\ &\leq \max\{S(\mu_A(x), \mu_A(y)), S(\mu_B(x), \mu_B(y))\} \\ &\leq S(S(\mu_A(x), \mu_B(x)), S(\mu_A(y), \mu_B(y))) \\ &= S(\mu_C(x), \mu_C(y)). \end{aligned}$$

Therefore, $\mu_C(x + y) \leq S(\mu_C(x), \mu_C(y))$, for all x and $y \in R$.

$$\begin{aligned} \text{And, } \mu_C(xy) &= \max\{\mu_A(xy), \mu_B(xy)\} \\ &\leq \max\{T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y))\} \end{aligned}$$

$$\begin{aligned} &\leq T (T (\mu_A(x), \mu_B(x)), T (\mu_A(y), \mu_B(y))) \\ &= T (\mu_C(x), \mu_C(y)). \end{aligned}$$

Therefore, $\mu_C(xy) \leq T (\mu_C(x), \mu_C(y))$, for all x and $y \in R$.

Therefore C is an anti (T, S) -fuzzy ideal of a hemiring R .

Hence the union of any two anti (T, S) -fuzzy ideals of a hemiring R is an anti (T, S) -fuzzy ideal of R . □

3.2.2 Theorem: The union of a family of anti (T, S) -fuzzy ideals of hemiring R is an anti (T, S) -fuzzy ideal of R .

Proof: The argument is trivial.

3.2.3 Theorem: If A and B are any two anti (T, S) -fuzzy ideals of the hemirings R_1 and R_2 respectively, then anti-product $A \times B$ is an anti (T, S) -fuzzy ideal of $R_1 \times R_2$.

Proof: Let A and B be two anti (T, S) -fuzzy ideals of the hemirings R_1 and R_2 respectively. Let x_1 and $x_2 \in R_1$, y_1 and $y_2 \in R_2$.

Then (x_1, y_1) and $(x_2, y_2) \in R_1 \times R_2$.

Now, $\mu_{A \times B} [(x_1, y_1) + (x_2, y_2)] = \mu_{A \times B} (x_1 + x_2, y_1 + y_2)$

$$\begin{aligned} &= \max \{ \mu_A(x_1 + x_2), \mu_B(y_1 + y_2) \} \\ &\leq \max \{ S (\mu_A(x_1), \mu_A(x_2)), S (\mu_B(y_1), \mu_B(y_2)) \} \\ &\leq S (S (\mu_A(x_1), \mu_B(y_1)), S (\mu_A(x_2), \mu_B(y_2)))) \\ &= S (\mu_{A \times B} (x_1, y_1), \mu_{A \times B} (x_2, y_2)). \end{aligned}$$

Therefore, $\mu_{A \times B} [(x_1, y_1) + (x_2, y_2)] \leq S (\mu_{A \times B} (x_1, y_1), \mu_{A \times B} (x_2, y_2))$.

Also, $\mu_{A \times B} [(x_1, y_1)(x_2, y_2)] = \mu_{A \times B}(x_1x_2, y_1y_2)$

$$\begin{aligned}
&= \max \{ \mu_A(x_1x_2), \mu_B(y_1y_2) \} \\
&\leq \max \{ T(\mu_A(x_1), \mu_A(x_2)), T(\mu_B(y_1), \mu_B(y_2)) \} \\
&\leq T(T(\mu_A(x_1), \mu_B(y_1)), T(\mu_A(x_2), \mu_B(y_2))) \\
&= T(\mu_{A \times B}(x_1, y_1), \mu_{A \times B}(x_2, y_2)).
\end{aligned}$$

Therefore, $\mu_{A \times B}[(x_1, y_1)(x_2, y_2)] \leq T(\mu_{A \times B}(x_1, y_1), \mu_{A \times B}(x_2, y_2))$.

Hence $A \times B$ is an anti (T, S) -fuzzy ideal of hemiring of $R_1 \times R_2$. □

3.2.4 Theorem: If A is an anti (T, S) -fuzzy subhemiring of a hemiring $(R, +, \cdot)$, then $\mu_A(x) \geq \mu_A(0)$, for $x \in R$, the zero $0 \in R$.

Proof: For $x \in R$, and 0 is the zero element of R . Now, $\mu_A(x) = \mu_A(x+0) \leq S(\mu_A(x), \mu_A(0))$, for all $x \in R$. So, $\mu_A(x) \geq \mu_A(0)$ is only possible.

3.2.5 Theorem: Let A and B be anti (T, S) -fuzzy ideal of the hemirings R_1 and R_2 respectively. Suppose that 0_1 and 0_2 are the zero element of R_1 and R_2 respectively. If $A \times B$ is an anti (T, S) -fuzzy ideal of $R_1 \times R_2$, then at least one of the following two statements must hold.

- (i) $B(0_1) \leq A(x)$, for all $x \in R_1$,
- (ii) $A(0_1) \leq B(y)$, for all $y \in R_2$.

Proof: Let $A \times B$ be an anti (T, S) -fuzzy ideal of $R_1 \times R_2$.

By contraposition, suppose that none of the statements (i) and (ii) holds.

Then we can find a in R_1 and b in R_2 such that $A(a) < B(0_2)$ and $B(b) < A(0_1)$.

$$\begin{aligned}
\text{We have, } A \times B(a, b) &= \max\{ A(a), B(b) \} \\
&< \max\{ B(0_2), A(0_1) \} \\
&= \max\{ A(0_1), B(0_2) \}
\end{aligned}$$

$$= A \times B(0_1, 0_2).$$

Thus $A \times B$ is not an anti (T, S) -fuzzy ideal of $R_1 \times R_2$.

Hence either $B(0_2) \leq A(x)$, for all x in R_1 or $A(0_1) \leq B(y)$, for all $y \in R_2$.

3.2.6 Theorem: Let A and B be two fuzzy subsets of the hemirings R_1 and R_2 respectively and $A \times B$ is an anti (T, S) -fuzzy ideal of $R_1 \times R_2$. Then the following are true:

- (i) if $A(x) \geq B(0_2)$, then A is an anti (T, S) -fuzzy ideal of R_1 .
- (ii) if $B(x) \geq A(0_1)$, then B is an anti (T, S) -fuzzy ideal of R_2 .
- (iii) either A is an anti (T, S) -fuzzy ideal of R_1 or B is an anti (T, S) -fuzzy ideal of R_2 .

Proof: Let $A \times B$ be an anti (T, S) -fuzzy ideal of $R_1 \times R_2$ and x and $y \in R_1$ and $0_2 \in R_2$. Then $(x, 0_2)$ and $(y, 0_2) \in R_1 \times R_2$.

Now, using the property that $A(x) \geq B(0_2)$, for all $x \in R_1$.

We get, $A(x+y) = \max\{A(x+y), B(0_2+0_2)\}$

$$\begin{aligned} &= A \times B((x+y), (0_2+0_2)) \\ &= A \times B[(x, 0_2) + (y, 0_2)] \\ &\leq S(A \times B(x, 0_2), A \times B(y, 0_2)) \\ &= S(\max\{A(x), B(0_2)\}, \max\{A(y), B(0_2)\}) \\ &= S(A(x), A(y)). \end{aligned}$$

Therefore, $A(x+y) \leq S(A(x), A(y))$, for all x and $y \in R_1$.

Also, $A(xy) = \max\{A(xy), B(0_2 0_2)\}$

$$\begin{aligned} &= A \times B((xy), (0_2 0_2)) \\ &= A \times B[(x, 0_2)(y, 0_2)] \end{aligned}$$

$$\begin{aligned}
&\leq T (A \times B(x, 0_2), A \times B(y, 0_2)) \\
&= T (\max\{A(x), B(0_2)\}, \max\{A(y), B(0_2)\}) \\
&= T (A(x), A(y)).
\end{aligned}$$

Therefore, $A(xy) \leq T (A(x), A(y))$, for all x and $y \in R_1$.

Hence A is an anti (T, S) -fuzzy ideal of R_1 . Thus (i) is proved.

Now, $B(x) \geq A(0_2)$, for all $x \in R_2$,

let x and y in R_2 and $0_1 \in R_1$. Then $(0_1, x)$ and $(0_1, y) \in R_1 \times R_2$.

$$\begin{aligned}
\text{We get, } B(x+y) &= \max\{B(x+y), A(0_1+0_1)\} \\
&= \max\{A(0_1+0_1), B(x+y)\} \\
&= A \times B((0_1+0_1), (x+y)) \\
&= A \times B[(0_1, x) + (0_1, y)] \\
&\leq S (A \times B(0_1, x), A \times B(0_1, y)) \\
&= S (\max\{A(0_1), B(x)\}, \max\{A(0_1), B(y)\}) \\
&= S (B(x), B(y)).
\end{aligned}$$

Therefore, $B(x+y) \leq S (B(x), B(y))$, for all x and $y \in R_2$.

$$\begin{aligned}
\text{Also, } B(xy) &= \max\{B(xy), A(0_1 0_1)\} \\
&= \max\{A(0_1 0_1), B(xy)\} \\
&= A \times B((0_1 0_1), (xy)) \\
&= A \times B[(0_1, x)(0_1, y)] \\
&\leq T (A \times B(0_1, x), A \times B(0_1, y)) \\
&= T (\max\{A(0_1), B(x)\}, \max\{A(0_1), B(y)\}) \\
&= T (B(x), B(y)).
\end{aligned}$$

Therefore, $B(xy) \leq T (B(x), B(y))$, for all x and $y \in R_2$.

Hence B is an anti (T, S)-fuzzy ideal of a hemiring R_2 . Thus (ii) is proved.

(iii) is clear. □

3.2.7 Theorem: Let A be a fuzzy subset of a hemiring R and V be the anti-strongest fuzzy relation of R. Then A is an anti (T, S) fuzzy ideal of R if and only if V is an anti (T, S)-fuzzy ideal of $R \times R$.

Proof: Suppose that A is an anti (T, S)-fuzzy ideal of a hemiring R.

Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in R \times R$.

$$\begin{aligned}
 \text{We have, } \mu_V(x+y) &= \mu_V[(x_1, x_2) + (y_1, y_2)] \\
 &= \mu_V(x_1+y_1, x_2+y_2) \\
 &= \max \{ \mu_A(x_1+y_1), \mu_A(x_2+y_2) \} \\
 &\leq \max \{ S(\mu_A(x_1), \mu_A(y_1)), S(\mu_A(x_2), \mu_A(y_2)) \} \\
 &\leq S(\max \{ \mu_A(x_1), \mu_A(x_2) \}, \max \{ \mu_A(y_1), \mu_A(y_2) \}) \\
 &= S(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) \\
 &= S(\mu_V(x), \mu_V(y)).
 \end{aligned}$$

Therefore, $\mu_V(x+y) \leq S(\mu_V(x), \mu_V(y))$, for all x and $y \in R \times R$.

$$\begin{aligned}
 \text{And, } \mu_V(xy) &= \mu_V[(x_1, x_2)(y_1, y_2)] \\
 &= \mu_V(x_1y_1, x_2y_2) \\
 &= \max \{ \mu_A(x_1y_1), \mu_A(x_2y_2) \} \\
 &\leq \max \{ T(\mu_A(x_1), \mu_A(y_1)), T(\mu_A(x_2), \mu_A(y_2)) \} \\
 &\leq T(\max \{ \mu_A(x_1), \mu_A(x_2) \}, \max \{ \mu_A(y_1), \mu_A(y_2) \}) \\
 &= T(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) \\
 &= T(\mu_V(x), \mu_V(y)).
 \end{aligned}$$

Therefore, $\mu_V(xy) \leq T(\mu_V(x), \mu_V(y))$, for all x and $y \in R \times R$.

This proves that V is an anti (T, S) -fuzzy ideal of $R \times R$.

Conversely assume that V is an anti (T, S) -fuzzy ideal of $R \times R$, then for any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in R \times R$, we have

$$\begin{aligned} \max\{\mu_A(x_1+y_1), \mu_A(x_2+y_2)\} &= \mu_V(x_1+y_1, x_2+y_2) \\ &= \mu_V[(x_1, x_2) + (y_1, y_2)] \\ &= \mu_V(x+y) \\ &\leq S(\mu_V(x), \mu_V(y)) \\ &= S(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) \\ &= S(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\}). \end{aligned}$$

If $x_2 = 0, y_2 = 0$, we get, $\mu_A(x_1+y_1) \leq S(\mu_A(x_1), \mu_A(y_1))$, for all x_1 and $y_1 \in R$.

And, $\max\{\mu_A(x_1y_1), \mu_A(x_2y_2)\} = \mu_V(x_1y_1, x_2y_2)$

$$\begin{aligned} &= \mu_V[(x_1, x_2)(y_1, y_2)] \\ &= \mu_V(xy) \\ &\leq T(\mu_V(x), \mu_V(y)) \\ &= T(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) \\ &= T(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\}). \end{aligned}$$

If $x_2=0, y_2=0$, we get $\mu_A(x_1y_1) \leq T(\mu_A(x_1), \mu_A(y_1))$, for all x_1 and $y_1 \in R$.

Therefore A is an anti (T, S) -fuzzy ideal of R . □

3.2.8 Theorem: Let A be an anti (T, S) -fuzzy ideal of a hemiring $(R, +, \cdot)$.

If $\mu_A(x+y) = 1$, then either $\mu_A(x) = 1$ or $\mu_A(y) = 1$, for all x and $y \in R$.

Proof: The argument is trivial.

In the next theorem we introduce a new composition operation to get a S- fuzzy ideal of R

3.2.9 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring H and f is a isomorphism from hemiring R onto H. Then $A \circ f$ is an anti (T, S)-fuzzy ideal of R.

Proof: Let x and y in R and A be an anti (T, S)-fuzzy ideal of a hemiring H.

Then we have, $(\mu_{A \circ f})(x+y) = \mu_A (f(x+y))$

$$= \mu_A (f(x)+ f(y)), \text{ as } f \text{ is an isomorphism}$$

$$\leq S (\mu_A(f(x)), \mu_A(f(y))),$$

$$\leq S ((\mu_{A \circ f})(x), (\mu_{A \circ f})(y)),$$

which implies that $(\mu_{A \circ f})(x+y) \leq S ((\mu_{A \circ f})(x), (\mu_{A \circ f})(y))$.

And, $(\mu_{A \circ f})(xy) = \mu_A (f(xy))$

$$= \mu_A (f(x)f(y)), \text{ as } f \text{ is an isomorphism}$$

$$\leq T (\mu_A(f(x)), \mu_A(f(y))),$$

$$\leq T ((\mu_{A \circ f})(x), (\mu_{A \circ f})(y)),$$

which implies that $(\mu_{A \circ f})(xy) \leq T ((\mu_{A \circ f})(x), (\mu_{A \circ f})(y))$.

Therefore $(A \circ f)$ is an anti (T, S)-fuzzy ideal of a hemiring R. \square

3.2.10 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring H and f is an anti-isomorphism from a hemiring R onto H, then $A \circ f$ is an anti (T, S)-fuzzy ideal of R.

Proof: Let x and y \in R and A be an anti (T, S)-fuzzy ideal of a hemiring H.

Then we have,

$$(\mu_{A \circ f})(x+y) = \mu_A (f(x+y))$$

$$\begin{aligned}
&= \mu_A(f(y)+f(x)), \text{ as } f \text{ is an anti-isomorphism} \\
&\leq S (\mu_A(f(x)), \mu_A(f(y))), \\
&\leq S ((\mu_A \circ f)(x), (\mu_A \circ f)(y)),
\end{aligned}$$

which implies that $(\mu_A \circ f)(x+y) \leq S ((\mu_A \circ f)(x), (\mu_A \circ f)(y))$.

Now, $(\mu_A \circ f)(xy) = \mu_A(f(xy))$

$$\begin{aligned}
&= \mu_A(f(y)f(x)), \text{ as } f \text{ is an anti-isomorphism} \\
&\leq T (\mu_A(f(x)), \mu_A(f(y))), \\
&\leq T ((\mu_A \circ f)(x), (\mu_A \circ f)(y)),
\end{aligned}$$

which implies that $(\mu_A \circ f)(xy) \leq T ((\mu_A \circ f)(x), (\mu_A \circ f)(y))$.

Therefore $A \circ f$ is an anti (T, S) -fuzzy ideal of a hemiring R . □

3.2.11 Theorem: Let A be an anti (T, S) -fuzzy ideal of a hemiring $(R, +, \cdot)$, then the pseudo anti (T, S) -fuzzy coset $(aA)^p$ is an anti (T, S) -fuzzy ideal of a hemiring R , for every a in R .

Proof: Let A be an anti (T, S) -fuzzy ideal of a hemiring R .

For every x and $y \in R$, we have,

$$\begin{aligned}
((a\mu_A)^p)(x+y) &= p(a)\mu_A(x+y) \\
&\leq p(a) S (\mu_A(x), \mu_A(y)) \\
&= S (p(a)\mu_A(x), p(a)\mu_A(y)) \\
&= S (((a\mu_A)^p)(x), ((a\mu_A)^p)(y)).
\end{aligned}$$

Therefore, $((a\mu_A)^p)(x+ y) \leq S (((a\mu_A)^p)(x), ((a\mu_A)^p)(y))$.

Now, $((a\mu_A)^p)(xy) = p(a)\mu_A(xy)$

$$\leq p(a) T (\mu_A(x), \mu_A(y))$$

$$\begin{aligned}
&= T (p(a)\mu_A(x), p(a)\mu_A(y)) \\
&= T ((a\mu_A)^p(x), (a\mu_A)^p(y)).
\end{aligned}$$

Therefore, $(a\mu_A)^p(xy) \leq T ((a\mu_A)^p(x), (a\mu_A)^p(y))$.

Hence $(aA)^p$ is an anti (T, S) -fuzzy ideal of a hemiring R . \square

3.2.12 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings, then the homomorphic image of an anti (T, S) -fuzzy ideal of R is an anti (T, S) -fuzzy ideal of R^1 .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. Let $f : R \rightarrow R^1$ be a homomorphism. Then, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and $y \in R$. Let $V = f(A)$, where A is an anti (T, S) -fuzzy ideal of R . We have to prove that V is an anti (T, S) -fuzzy ideal of R^1 . Now, for $f(x), f(y) \in R^1$,

$$\begin{aligned}
\mu_v(f(x) + f(y)) &= \mu_v(f(x+y)), \text{ as } f \text{ is a homomorphism} \\
&\leq \mu_A(x+y) \\
&\leq S (\mu_A(x), \mu_A(y)),
\end{aligned}$$

which implies that $\mu_v(f(x) + f(y)) \leq S (\mu_v(f(x)), \mu_v(f(y)))$.

Again, $\mu_v(f(x)f(y)) = \mu_v(f(xy))$, as f is a homomorphism

$$\begin{aligned}
&\leq \mu_A(xy) \\
&\leq T (\mu_A(x), \mu_A(y)),
\end{aligned}$$

which implies that $\mu_v(f(x)f(y)) \leq T (\mu_v(f(x)), \mu_v(f(y)))$.

Hence V is an anti (T, S) -fuzzy ideal of R^1 . \square

3.2.13 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. The homomorphic preimage of an anti (T, S) -fuzzy ideal of R^1 is an anti (T, S) -fuzzy ideal of R .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. Let $f : R \rightarrow R^1$ be a homomorphism. Then, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and $y \in R$. Let $V = f(A)$, where V is an anti (T, S) -fuzzy ideal of R^1 . We have to prove that A is an anti (T, S) -fuzzy ideal of R . Let x and $y \in R$.

$$\begin{aligned} \text{Then, } \mu_A(x+y) &= \mu_v(f(x+y)), \text{ since } \mu_v(f(x)) = \mu_A(x) \\ &= \mu_v(f(x) + f(y)), \text{ as } f \text{ is a homomorphism} \\ &\leq S(\mu_v(f(x)), \mu_v(f(y))) \\ &= S(\mu_A(x), \mu_A(y)), \text{ since } \mu_v(f(x)) = \mu_A(x) \end{aligned}$$

which implies that $\mu_A(x+y) \leq S(\mu_A(x), \mu_A(y))$.

$$\begin{aligned} \text{Again, } \mu_A(xy) &= \mu_v(f(xy)), \text{ since } \mu_v(f(x)) = \mu_A(x) \\ &= \mu_v(f(x)f(y)), \text{ as } f \text{ is a homomorphism} \\ &\leq T(\mu_v(f(x)), \mu_v(f(y))) \\ &= T(\mu_A(x), \mu_A(y)), \text{ since } \mu_v(f(x)) = \mu_A(x) \end{aligned}$$

which implies that $\mu_A(xy) \leq T(\mu_A(x), \mu_A(y))$.

Hence A is an anti (T, S) -fuzzy ideal of R . □

3.2.14 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings, then the anti-homomorphic image of an anti (T, S) -fuzzy ideal of R is an anti (T, S) -fuzzy ideal of R^1 .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. Let $f : R \rightarrow R^1$ be an anti-homomorphism. Then, $f(x+y) = f(y) + f(x)$ and $f(xy) = f(y) f(x)$, for all x and y in R . Let $V = f(A)$, where A is an anti (T, S) -fuzzy ideal of R . We have to prove that V is an anti (T, S) -fuzzy ideal of R^1 .

Now, for $f(x), f(y)$ in R^1 ,

$$\begin{aligned} \mu_v(f(x) + f(y)) &= \mu_v(f(y + x)), \text{ as } f \text{ is an anti-homomorphism} \\ &\leq \mu_A(y + x) \\ &\leq S(\mu_A(y), \mu_A(x)) \\ &= S(\mu_A(x), \mu_A(y)), \end{aligned}$$

which implies that $\mu_v(f(x) + f(y)) \leq S(\mu_v(f(x)), \mu_v(f(y)))$.

Again, $\mu_v(f(x)f(y)) = \mu_v(f(yx))$, as f is an anti-homomorphism

$$\begin{aligned} &\leq \mu_A(yx) \\ &\leq T(\mu_A(y), \mu_A(x)) \\ &= T(\mu_A(x), \mu_A(y)), \end{aligned}$$

which implies that $\mu_v(f(x)f(y)) \leq T(\mu_v(f(x)), \mu_v(f(y)))$.

Hence V is an anti (T, S) -fuzzy ideal of R^1 . □

3.2.15 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings, then the anti-homomorphic preimage of an anti (T, S) -fuzzy ideal of R^1 is an anti (T, S) -fuzzy ideal of R .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. Let $f : R \rightarrow R^1$ be an anti-homomorphism. Then, $f(x+y) = f(y) + f(x)$ and $f(xy) = f(y) f(x)$, for all x and $y \in R$. Let $V = f(A)$, where V is an anti (T, S) -fuzzy ideal of R^1 . We have to prove that A is an anti (T, S) -fuzzy ideal of R . Let x and $y \in R$.

$$\begin{aligned}
\text{Then, } \mu_A(x+y) &= \mu_v(f(x+y)), \text{ since } \mu_v(f(x)) = \mu_A(x) \\
&= \mu_v(f(y) + f(x)), \text{ as } f \text{ is an anti-homomorphism} \\
&\leq S(\mu_v(f(y)), \mu_v(f(x))) \\
&= S(\mu_v(f(x)), \mu_v(f(y))) \\
&= S(\mu_A(x), \mu_A(y)), \text{ since } \mu_v(f(x)) = \mu_A(x)
\end{aligned}$$

which implies that $\mu_A(x+y) \leq S(\mu_A(x), \mu_A(y))$.

$$\begin{aligned}
\text{Again, } \mu_A(xy) &= \mu_v(f(xy)), \text{ since } \mu_v(f(x)) = \mu_A(x) \\
&= \mu_v(f(y)f(x)), \text{ as } f \text{ is an anti-homomorphism} \\
&\leq T(\mu_v(f(y)), \mu_v(f(x))) \\
&= T(\mu_v(f(x)), \mu_v(f(y))) \\
&= T(\mu_A(x), \mu_A(y)), \text{ since } \mu_v(f(x)) = \mu_A(x)
\end{aligned}$$

which implies that $\mu_A(xy) \leq T(\mu_A(x), \mu_A(y))$.

Hence A is an anti (T, S) -fuzzy ideal of R . □

3.2.16 Theorem: Let A be an anti (T, S) -fuzzy ideal of a hemiring R , A^+ be a fuzzy set in R defined by $A^+(x) = A(x)+1 - A(0)$, for all x in R , then A^+ is an anti (T, S) -fuzzy ideal of a hemiring R .

Proof : Let x and y in R . We have, $A^+(x+y) = A(x+y) +1- A(0)$

$$\leq S(A(x), A(y)) + 1- A(0) \leq S((A(x) +1- A(0)), (A(y) +1- A(0)))$$

$$= S(A^+(x), A^+(y)). \text{ Therefore, } A^+(x+y) \leq S(A^+(x), A^+(y)), \text{ for all } x, y \in R.$$

Similarly, $A^+(xy) = A(xy) +1- A(0) \leq T(A(x), A(y))+1- A(0)$

$$\leq T((A(x) +1- A(0)), (A(y) +1- A(0))) = T(A^+(x), A^+(y)).$$

Therefore, $A^+(xy) \leq T(A^+(x), A^+(y))$, for all $x, y \in R$.

Hence A^+ is an anti (T, S)-fuzzy ideal of a hemiring R. \square

3.2.17 Theorem : Let A be an anti (T, S)-fuzzy ideal of a hemiring R, A^+ be a fuzzy set in R defined by $A^+(x) = A(x) + 1 - A(0)$, for all $x \in R$, then there exists $0 \in R$ such that $A(0) = 1$ if and only if $A^+(x) = A(x)$.

Proof : The argument is trivial.

3.2.18 Theorem: Let A be an anti (T, S)-fuzzy ideal of a hemiring R, A^+ be a fuzzy set in R defined by $A^+(x) = A(x) + 1 - A(0)$, for all $x \in R$. Then there exists $x \in R$ such that $A^+(x) = 1$ if and only if $x = 0$.

Proof: The argument is trivial.

3.2.19 Theorem : Let A be an anti (T, S)-fuzzy ideal of a hemiring R, A^+ be a fuzzy set in R defined by $A^+(x) = A(x) + 1 - A(0)$, for all $x \in R$. Then $(A^+)^+ = A^+$.

Proof: Let x and y in R. We have, $(A^+)^+(x) = A^+(x) + 1 - A^+(0) = \{ A(x) + 1 - A(0) \} + 1 - \{ A(0) + 1 - A(0) \} = A(x) + 1 - A(0) = A^+(x)$. Hence $(A^+)^+ = A^+$.

3.2.20 Theorem : Let A be an anti (T, S)-fuzzy ideal of a hemiring R. Then A^0 is an anti (T, S)-fuzzy ideal of the hemiring R .

Proof: For any $x \in R$, we have $A^0(x+y) = A(x+y)A(0) \leq [A(0)] S (A(x), A(y)) \leq S ([A(x)A(0)], [A(y)A(0)]) = S (A^0(x), A^0(y))$. That is $A^0(x+y) \leq S(A^0(x), A^0(y))$, for all $x, y \in R$. Similarly, $A^0(xy) = A(xy)A(0) \leq [A(0)] T(A(x), A(y)) \leq T([A(x)A(0)], [A(y)A(0)]) = T(A^0(x), A^0(y))$. That is $A^0(xy) \leq T(A^0(x), A^0(y))$, for all $x, y \in R$. Hence A^0 is an anti (T, S)-fuzzy ideal of the hemiring R. \square

3.2.21 Theorem: Let A be an anti (T, S) -fuzzy ideal of a hemiring R . Then for α in $[0,1]$ such that $\mu_A(0) \leq \alpha$, A_α is a lower level ideal of R .

Proof: For all x and y in A_α , we have, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$. Now, $\mu_A(x-y) \leq S(\mu_A(x), \mu_A(y)) \leq S(\alpha, \alpha) = \alpha$, which implies that $\mu_A(x-y) \leq \alpha$.

And, $\mu_A(xy) \leq T(\mu_A(x), \mu_A(y)) \leq T(\alpha, \alpha) = \alpha$, which implies that $\mu_A(xy) \leq \alpha$.

Therefore, $\mu_A(x-y) \leq \alpha$ and $\mu_A(xy) \leq \alpha$. Therefore, $x-y$ and xy in A_α .

Hence A_α is a lower level ideal of a hemiring R . □

3.2.22 Theorem: Let A be an anti (T, S) -fuzzy ideal of a hemiring R . If any two lower level ideals of A belongs to R , then their intersection is also lower level ideal of $A \in R$.

Proof : Let $\alpha_1, \alpha_2 \in [0,1]$.

Case (i): If $\alpha_1 < \mu_A(x) < \alpha_2$, then $A_{\alpha_1} \subseteq A_{\alpha_2}$.

Therefore, $A_{\alpha_1} \cap A_{\alpha_2} = A_{\alpha_1}$, but A_{α_1} is a lower level ideal of A .

Case (ii): If $\alpha_1 > \mu_A(x) > \alpha_2$, then $A_{\alpha_2} \subseteq A_{\alpha_1}$.

Therefore, $A_{\alpha_1} \cap A_{\alpha_2} = A_{\alpha_2}$, but A_{α_2} is a lower level ideal of A .

Case (iii): If $\alpha_1 = \alpha_2$, then $A_{\alpha_1} = A_{\alpha_2}$.

Hence, intersection of any two lower level ideals is a lower level ideal of A .

3.2.23 Theorem: Let A be an anti (T, S) -fuzzy ideal of a hemiring R . If $\alpha_i \in [0,1]$ and $A_{\alpha_i}, i \in I$ is a collection of lower level ideals of A , then their intersection is also a lower level ideal of A .

Proof: The argument is trivial.

3.2.24 Theorem: Let A be an anti (T, S) -fuzzy ideal of a hemiring R . If any two lower level ideals of A belongs to R , then their union is also a lower level ideal of A in R .

Proof: Let $\alpha_1, \alpha_2 \in [0, 1]$.

Case (i): If $\alpha_1 < \mu_A(x) < \alpha_2$, then $A_{\alpha_1} \subseteq A_{\alpha_2}$.

Therefore, $A_{\alpha_1} \cup A_{\alpha_2} = A_{\alpha_2}$, but A_{α_2} is a lower level ideal of A .

Case (ii): If $\alpha_1 > \mu_A(x) > \alpha_2$, then $A_{\alpha_2} \subseteq A_{\alpha_1}$.

Therefore, $A_{\alpha_1} \cup A_{\alpha_2} = A_{\alpha_1}$, but A_{α_1} is a lower level ideal of A .

Case (iii): If $\alpha_1 = \alpha_2$, then $A_{\alpha_1} = A_{\alpha_2}$.

Hence, union of any two lower level ideal is also a lower level ideal of A . \square

3.2.25 Theorem: Let A be an anti (T, S) -fuzzy ideal of a hemiring R . If $\alpha_i \in [0, 1]$ and $A_{\alpha_i}, i \in I$ is a collection of lower level ideals of A , then their union is also a lower level ideal of A .

Proof: The argument is trivial.

3.2.26 Theorem: The homomorphic image of a lower level ideal of an anti (T, S) -fuzzy ideal of a hemiring R is a lower level ideal of an anti (T, S) -fuzzy ideal of a hemiring R^1 .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be a homomorphism. That is, $f(x+y)=f(x)+f(y)$ and $f(xy)=f(x)f(y)$, for all x and $y \in R$. Let $V = f(A)$, where A is an anti (T, S) -fuzzy ideal of a hemiring R . Clearly V is an anti (T, S) -fuzzy ideal of a hemiring R^1 . Let x and $y \in R$, implies $f(x)$ and $f(y) \in R^1$. Let A_α is a lower level ideal of A . That is, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$; $\mu_A(x+y) \leq \alpha$, $\mu_A(xy) \leq \alpha$. We have to prove that $f(A_\alpha)$ is a

lower level ideal of V . Now, $\mu_V(f(x)) \leq \mu_A(x) \leq \alpha$, which implies that $\mu_V(f(x)) \leq \alpha$; and $\mu_V(f(y)) \leq \mu_A(y) \leq \alpha$, which implies that $\mu_V(f(y)) \leq \alpha$ and $\mu_V(f(x) + f(y)) = \mu_V(f(x + y)) \leq \mu_A(x + y) \leq \alpha$, which implies that $\mu_V(f(x) + f(y)) \leq \alpha$. Also, $\mu_V(f(x)f(y)) = \mu_V(f(xy)) \leq \mu_A(xy) \leq \alpha$, which implies that $\mu_V(f(x)f(y)) \leq \alpha$. Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$, $\mu_V(f(x)f(y)) \leq \alpha$. Hence $f(A_\alpha)$ is a lower level ideal of an anti (T, S) -fuzzy ideal V of R^1 . \square

3.2.27 Theorem: The homomorphic pre-image of a lower level ideal of an anti (T, S) -fuzzy ideal of a hemiring R^1 is a lower level ideal of an anti (T, S) -fuzzy ideal of a hemiring R .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be a homomorphism. That is, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all x and $y \in R$. Let $V = f(A)$, where V is an anti (T, S) -fuzzy ideal of a hemiring R^1 . Clearly A is an anti (T, S) -fuzzy ideal of a hemiring R . Let $f(x)$ and $f(y) \in R^1$, implies x and $y \in R$. Let $f(A_\alpha)$ is a lower level ideal of V . That is, $\mu_V(f(x)) \leq \alpha$ and $\mu_V(f(y)) \leq \alpha$; $\mu_V(f(x) + f(y)) \leq \alpha$, $\mu_V(f(x)f(y)) \leq \alpha$. We have to prove that A_α is a lower level ideal of A . Now, $\mu_A(x) = \mu_V(f(x)) \leq \alpha$, implies that $\mu_A(x) \leq \alpha$; $\mu_A(y) = \mu_V(f(y)) \leq \alpha$, implies that $\mu_A(y) \leq \alpha$ and $\mu_A(x + y) = \mu_V(f(x + y)) = \mu_V(f(x) + f(y)) \leq \alpha$, which implies that $\mu_A(x+y) \leq \alpha$. Also, $\mu_A(xy) = \mu_V(f(xy)) = \mu_V(f(x)f(y)) \leq \alpha$, which implies that $\mu_A(xy) \leq \alpha$. Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$, $\mu_V(f(x)f(y)) \leq \alpha$. Hence, A_α is a lower level ideal of an anti (T, S) -fuzzy ideal A of R . \square

3.2.28 Theorem: The anti homomorphic image of a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R is a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R^1 .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be an anti-homomorphism. That is, $f(x+y)=f(y)+f(x)$ and $f(xy)=f(y)f(x)$, for all x and y in R . Let $V = f(A)$, where A is an anti (T, S)-fuzzy ideal of R . Clearly V is an anti (T, S)-fuzzy ideal of R^1 . Let x and $y \in R$, implies $f(x)$ and $f(y) \in R^1$. Let A_α is a lower level ideal of A . That is, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$, $\mu_A(y + x) \leq \alpha$, $\mu_A(yx) \leq \alpha$. We have to prove that $f(A_\alpha)$ is a lower level ideal of V . Now, $\mu_V(f(x)) \leq \mu_A(x) \leq \alpha$, which implies that $\mu_V(f(x)) \leq \alpha$; $\mu_V(f(y)) \leq \mu_A(y) \leq \alpha$, which implies that $\mu_V(f(y)) \leq \alpha$.

Now, $\mu_V(f(x)+f(y)) = \mu_V(f(y+x)) \leq \mu_A(y + x) \leq \alpha$, which implies that, $\mu_V(f(x) + f(y)) \leq \alpha$. Also, $\mu_V(f(x)f(y)) = \mu_V(f(yx)) \leq \mu_A(yx) \leq \alpha$, which implies that $\mu_V(f(x)f(y)) \leq \alpha$. Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x)f(y)) \leq \alpha$. Hence $f(A_\alpha)$ is a lower level ideal of an anti (T, S)-fuzzy ideal V of R^1 . □

3.2.29 Theorem: The anti-homomorphic pre-image of a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R^1 is a lower level ideal of an anti (T, S)-fuzzy ideal of a hemiring R .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be an anti-homomorphism. That is, $f(x + y) = f(y) + f(x)$ and $f(xy) = f(y)f(x)$, for all x and y in R . Let $V = f(A)$, where V is an anti (T, S)-fuzzy ideal of a hemiring R^1 . Clearly A is an anti (T, S)-fuzzy ideal of a hemiring R . Let $f(x)$

and $f(y)$ in R^1 , implies x and y in R . Let $f(A_\alpha)$ is a lower level ideal of V . That is, $\mu_V(f(x)) \leq \alpha$ and $\mu_V(f(y)) \leq \alpha$; $\mu_V(f(y)+f(x)) \leq \alpha$, $\mu_V(f(y)f(x)) \leq \alpha$.

We have to prove that A_α is a lower level ideal of A . Now, $\mu_A(x) = \mu_V(f(x)) \leq \alpha$, which implies that $\mu_A(x) \leq \alpha$; $\mu_A(y) = \mu_V(f(y)) \leq \alpha$, which implies that $\mu_A(y) \leq \alpha$.

Now, $\mu_A(x + y) = \mu_V(f(x + y)) = \mu_V(f(y) + f(x)) \leq \alpha$, which implies that $\mu_A(x + y) \leq \alpha$. Also, $\mu_A(xy) = \mu_V(f(xy)) = \mu_V(f(y)f(x)) \leq \alpha$, which implies that $\mu_A(xy) \leq \alpha$. Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x)f(y)) \leq \alpha$.

Hence A_α is a lower level ideal of an anti (T, S) -fuzzy ideal A of R . \square

3.3 ANTI (T, S) -FUZZY NORMAL IDEALS OF A HEMIRING

3.3.1 Theorem: Let $(R, +, \cdot)$ be a hemiring. If A and B are two anti (T, S) -fuzzy normal ideals of R , then $A \cup B$ is an anti (T, S) -fuzzy normal ideal of R .

Proof: Let $x, y \in R$. Let $A = \{ \langle x, \mu_A(x) \rangle / x \in R \}$ and $B = \{ \langle x, \mu_B(x) \rangle / x \in R \}$ be anti (T, S) -fuzzy normal ideals of a hemiring R . Let $C = A \cup B$ and $C = \{ \langle x, \mu_C(x) \rangle / x \in R \}$, where $\mu_C(x) = \max \{ \mu_A(x), \mu_B(x) \}$. Then, Clearly C is an anti (T, S) -fuzzy ideal of a hemiring R , since A and B are two anti (T, S) -fuzzy ideals of the hemiring R .

And, $\mu_C(xy) = \max \{ \mu_A(xy), \mu_B(xy) \}$,

$$= \max \{ \mu_A(yx), \mu_B(yx) \}$$

$$= \mu_C(yx), \text{ for all } x \text{ and } y \in R.$$

Therefore, $\mu_C(xy) = \mu_C(yx)$, for all x and $y \in R$.

Hence $A \cup B$ is an anti (T, S) -fuzzy normal ideal of the hemiring R . \square

3.3.2 Theorem: Let $(R, +, \cdot)$ be a hemiring. The union of a family of anti (T, S) -fuzzy normal ideals of R is an anti (T, S) -fuzzy normal ideal of R .

Proof: The argument is trivial.

3.3.3 Theorem: Let A and B be anti (T, S) -fuzzy ideals of the hemirings G and H , respectively. If A and B are anti (T, S) -fuzzy normal ideals, then $A \times B$ is an anti (T, S) -fuzzy normal ideal of $G \times H$.

Proof: Let A and B be anti (T, S) -fuzzy normal ideals of the hemirings G and H respectively. Clearly $A \times B$ is an anti (T, S) -fuzzy ideal of $G \times H$. Let x_1 and $x_2 \in G$, y_1 and $y_2 \in H$. Then (x_1, y_1) and (x_2, y_2) are $\in G \times H$.

$$= \max \{ \mu_A(x_1x_2), \mu_B(y_1y_2) \}$$

$$= \max \{ \mu_A(x_2x_1), \mu_B(y_2y_1) \},$$

$$= \mu_{A \times B}(x_2x_1, y_2y_1)$$

$$= \mu_{A \times B}[(x_2, y_2)(x_1, y_1)].$$

Therefore, $\mu_{A \times B}[(x_1, y_1)(x_2, y_2)] = \mu_{A \times B}[(x_2, y_2)(x_1, y_1)]$.

Hence $A \times B$ is an anti (T, S) -fuzzy normal ideal of $G \times H$. □

3.3.4 Theorem: Let A and B be anti (T, S) -fuzzy normal ideal of the hemirings R_1 and R_2 respectively. Suppose that 0_1 and 0_2 are the zero element of R_1 and R_2 respectively. If $A \times B$ is an anti (T, S) -fuzzy normal ideal of $R_1 \times R_2$, then at least one of the following two statements must hold.

$$(i) B(0_2) \leq A(x), \text{ for all } x \in R_1,$$

$$(ii) A(0_1) \leq B(y), \text{ for all } y \in R_2.$$

Proof: The argument is trivial.

3.3.5 Theorem: Let A and B be two fuzzy subsets of the hemirings R_1 and R_2 respectively and $A \times B$ is an anti (T, S) -fuzzy normal ideal of $R_1 \times R_2$. Then the following are true:

- (i) if $A(x) \geq B(0_2)$, then A is an anti (T, S) -fuzzy normal ideal of R_1 .
- (ii) if $B(x) \geq A(0_1)$, then B is an anti (T, S) -fuzzy normal ideal of R_2 .
- (iii) either A is an anti (T, S) -fuzzy normal ideal of R_1 or B is an anti (T, S) -fuzzy normal ideal of R_2 .

Proof: The argument is trivial.

3.3.6 Theorem: Let A be a fuzzy subset in a hemiring R and V be the anti-strongest fuzzy relation on R , then A is an anti (T, S) -fuzzy normal ideal of R if and only if V is an anti (T, S) -fuzzy normal ideal of $R \times R$.

Proof: The argument is trivial.

3.3.7 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. The homomorphic image of an anti (T, S) -fuzzy normal ideal of R is an anti (T, S) -fuzzy normal ideal of R^1 .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be a homomorphism. Then, $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$, for all x and y in R . Let $V = f(A)$, where A is an anti (T, S) -fuzzy normal ideal of a hemiring R . We have to prove that V is an anti (T, S) -fuzzy normal ideal of a hemiring R^1 . Now, for $f(x), f(y)$ in R^1 , clearly V is an anti (T, S) -fuzzy ideal of a hemiring R^1 , since A is an anti (T, S) -fuzzy ideal of a hemiring R .

Now, $\mu_v(f(x)f(y)) = \mu_v(f(xy))$, as f is a homomorphism

$$\leq \mu_A(xy)$$

$$\begin{aligned}
&= \mu_A(yx) \\
&\geq \mu_V(f(yx)) \\
&= \mu_V(f(y) f(x)), \text{ as } f \text{ is a homomorphism}
\end{aligned}$$

which implies that $\mu_V(f(x)f(y)) = \mu_V(f(y) f(x))$, for all $f(x)$ and $f(y) \in R^1$.

Hence V is an anti (T, S) -fuzzy normal ideal of a hemiring R^1 . \square

3.3.8 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings, then the homomorphic preimage of an anti (T, S) -fuzzy normal ideal of R^1 is an anti (T, S) -fuzzy normal ideal of R .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be a homomorphism. Then, $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x) f(y)$, for all x and y in R . Let $V = f(A)$, where V is an anti (T, S) -fuzzy normal ideal of a hemiring R^1 . We have to prove that A is an anti (T, S) -fuzzy normal ideal of a hemiring R . Let x and y in R . Then, clearly A is an anti (T, S) -fuzzy ideal of a hemiring R , since V is an anti (T, S) -fuzzy ideal of a hemiring R^1 .

$$\begin{aligned}
\text{Now, } \mu_A(xy) &= \mu_V(f(xy)), \text{ since } \mu_A(x) = \mu_V(f(x)) \\
&= \mu_V(f(x)f(y)), \text{ as } f \text{ is a homomorphism} \\
&= \mu_V(f(y)f(x)) \\
&= \mu_V(f(yx)), \text{ as } f \text{ is a homomorphism} \\
&= \mu_A(yx), \text{ since } \mu_A(x) = \mu_V(f(x))
\end{aligned}$$

which implies that $\mu_A(xy) = \mu_A(yx)$, for all x and $y \in R$.

Hence A is an anti (T, S) -fuzzy normal ideal of a hemiring R . \square

3.3.9 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings, then the anti-homomorphic image of an anti (T, S) -fuzzy normal ideal of R is an anti (T, S) -fuzzy normal ideal of R^1 .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be an anti-homomorphism. Then, $f(x+y) = f(y) + f(x)$ and $f(xy) = f(y) f(x)$, for all x and y in R . Let $V = f(A)$, where A is an anti (T, S) -fuzzy normal ideal of a hemiring R . We have to prove that V is an anti (T, S) -fuzzy normal ideal of a hemiring R^1 . Now, for $f(x)$ and $f(y)$ in R^1 , clearly V is an anti (T, S) -fuzzy ideal of a hemiring R^1 , since A is an anti (T, S) -fuzzy ideal of a hemiring R .

$$\begin{aligned}
 \text{Now, } \mu_V(f(x)f(y)) &= \mu_V(f(yx)), \text{ as } f \text{ is an anti-homomorphism} \\
 &\leq \mu_A(yx) \\
 &= \mu_A(xy) \\
 &\geq \mu_V(f(xy)) \\
 &= \mu_V(f(y) f(x)), \text{ as } f \text{ is an anti-homomorphism}
 \end{aligned}$$

which implies that $\mu_V(f(x)f(y)) = \mu_V(f(y)f(x))$, for all $f(x)$ and $f(y) \in R^1$.

Hence V is an anti (T, S) -fuzzy normal ideal of a hemiring R^1 . □

3.3.10 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings, then the anti-homomorphic preimage of an anti (T, S) -fuzzy normal ideal of R^1 is an anti (T, S) -fuzzy normal ideal of R .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be an anti-homomorphism. Then, $f(x+y) = f(y) + f(x)$ and $f(xy) = f(y) f(x)$, for all x and y in R . Let $V = f(A)$, where V is an anti (T, S) -fuzzy normal ideal of

a hemiring R^1 . We have to prove that A is an anti (T, S) -fuzzy normal ideal of a hemiring R .

Let x and $y \in R$, then, clearly A is an anti (T, S) -fuzzy ideal of a hemiring R , since V is an anti (T, S) -fuzzy ideal of a hemiring R^1 .

$$\begin{aligned} \text{Now, } \mu_A(xy) &= \mu_v(f(xy)), \text{ since } \mu_A(x) = \mu_v(f(x)) \\ &= \mu_v(f(y)f(x)), \text{ as } f \text{ is an anti-homomorphism} \\ &= \mu_v(f(x)f(y)) \\ &= \mu_v(f(yx)), \text{ as } f \text{ is an anti-homomorphism} \\ &= \mu_A(yx), \text{ since } \mu_A(x) = \mu_v(f(x)) \end{aligned}$$

which implies that $\mu_A(xy) = \mu_A(yx)$, for all x and $y \in R$.

Hence A is an anti (T, S) -fuzzy normal ideal of a hemiring R . □

In the next theorem we introduce a composition operation to get a S -fuzzy normal ideal.

3.3.11 Theorem: Let A be an anti (T, S) -fuzzy ideal of a hemiring H and f is an isomorphism from a hemiring R onto H . If A is an anti (T, S) -fuzzy normal ideal of the hemiring H , then $A \circ f$ is an anti (T, S) -fuzzy normal ideal of the hemiring R .

Proof: Let $x, y \in R$ and A be an anti (T, S) -fuzzy normal ideal of a hemiring H . Then, clearly $A \circ f$ is an anti (T, S) -fuzzy ideal of a hemiring R .

$$\begin{aligned} \text{Now, } (\mu_{A \circ f})(xy) &= \mu_A(f(xy)) \\ &= \mu_A(f(x)f(y)), \text{ as } f \text{ is an isomorphism} \\ &= \mu_A(f(y)f(x)) \\ &= \mu_A(f(yx)), \text{ as } f \text{ is an isomorphism} \end{aligned}$$

$$= (\mu_{A \circ f})(yx),$$

which implies that $(\mu_{A \circ f})(xy) = (\mu_{A \circ f})(yx)$, for all x and $y \in R$.

Hence $A \circ f$ is an anti (T, S) -fuzzy normal ideal of a hemiring R . \square

3.3.12 Theorem: Let A be an anti (T, S) -fuzzy ideal of a hemiring H and f is an anti-isomorphism from a hemiring R onto H . If A is an anti (T, S) -fuzzy normal ideal of the hemiring H , then $A \circ f$ is an anti (T, S) -fuzzy normal ideal of the hemiring R .

Proof: Let x, y in R and A be an anti (T, S) -fuzzy normal ideal of a hemiring H . Then, clearly $A \circ f$ is anti (T, S) -fuzzy ideal of the hemiring R .

$$\begin{aligned} \text{Now, } (\mu_{A \circ f})(xy) &= \mu_A(f(xy)) \\ &= \mu_A(f(y)f(x)), \text{ as } f \text{ is an anti-isomorphism} \\ &= \mu_A(f(x)f(y)) \\ &= \mu_A(f(yx)), \text{ as } f \text{ is an anti-isomorphism} \\ &= (\mu_{A \circ f})(yx), \end{aligned}$$

which implies that $(\mu_{A \circ f})(xy) = (\mu_{A \circ f})(yx)$, for all x and $y \in R$.

Hence $A \circ f$ is an anti (T, S) -fuzzy normal ideal of the hemiring R . \square

3.3.13 Theorem: The homomorphic image of a lower level ideal of an anti (T, S) -fuzzy normal ideal of a hemiring R is a lower level ideal of an anti (T, S) -fuzzy normal ideal of a hemiring R^1 .

Proof: The argument is trivial.

3.3.14 Theorem: The homomorphic pre-image of a lower level ideal of an anti (T, S) -fuzzy normal ideal of a hemiring R^1 is a lower level ideal of an anti (T, S) -fuzzy normal ideal of a hemiring R .

Proof: The argument is trivial.

3.3.15 Theorem: The anti-homomorphic image of a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R is a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R^l .

Proof: The argument is trivial.

3.3.16 Theorem: The anti-homomorphic pre-image of a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R^l is a lower level ideal of an anti (T, S)-fuzzy normal ideal of a hemiring R.

Proof: The argument is trivial.

3.4 TRANSLATIONS OF ANTI S-FUZZY SUBHEMIRINGS OF A HEMIRING

3.4.1 Theorem: If M and N are two translations of anti S-fuzzy subhemiring A of a hemiring $(R, +, \cdot)$, then their intersection $M \cap N$ is translation of anti S-fuzzy subhemiring A.

Proof: Let x and $y \in R$. Let $M = T_\alpha^A = \{ \langle x, \mu_A(x) + \alpha \rangle / x \in R \}$ and $N = T_\gamma^A = \{ \langle x, \mu_A(x) + \gamma \rangle / x \in R \}$ be two translations of anti S-fuzzy subhemiring A of a hemiring $(R, +, \cdot)$. Let $C = M \cap N$ and $C = \{ \langle x, \mu_C(x) \rangle / x \in R \}$, where $\mu_C(x) = \min\{\mu_A(x) + \alpha, \mu_A(x) + \gamma\}$.

Case (i): $\alpha \leq \gamma$.

$$\begin{aligned} \text{Now, } \mu_C(x+y) &= \min \{ \mu_M(x+y), \mu_N(x+y) \} \\ &= \min \{ \mu_A(x+y) + \alpha, \mu_A(x+y) + \gamma \} \\ &= \mu_A(x+y) + \alpha \\ &= \mu_M(x+y), \text{ for all } x \text{ and } y \in R. \end{aligned}$$

$$\text{And, } \mu_C(xy) = \min \{ \mu_M(xy), \mu_N(xy) \}$$

$$\begin{aligned}
&= \min \{ \mu_A(xy) + \alpha, \mu_A(xy) + \gamma \} \\
&= \mu_A(xy) + \alpha \\
&= \mu_M(xy), \text{ for all } x \text{ and } y \in R.
\end{aligned}$$

Therefore $C = T_\alpha^A = \{ \langle x, \mu_A(x) + \alpha \rangle / x \in R \}$ is a translation of anti S-fuzzy subhemiring A of the hemiring $(R, +, \cdot)$.

Case (ii): $\alpha \geq \gamma$.

$$\begin{aligned}
\text{Now, } \mu_C(x+y) &= \min \{ \mu_M(x+y), \mu_N(x+y) \} \\
&= \min \{ \mu_A(x+y) + \alpha, \mu_A(x+y) + \gamma \} \\
&= \mu_A(x+y) + \gamma \\
&= \mu_N(x+y), \text{ for all } x \text{ and } y \in R.
\end{aligned}$$

$$\begin{aligned}
\text{And } \mu_C(xy) &= \min \{ \mu_M(xy), \mu_N(xy) \} \\
&= \min \{ \mu_A(xy) + \alpha, \mu_A(xy) + \gamma \} \\
&= \mu_A(xy) + \gamma \\
&= \mu_N(xy), \text{ for all } x \text{ and } y \in R.
\end{aligned}$$

Therefore $C = T_\gamma^A = \{ \langle x, \mu_A(x) + \gamma \rangle / x \in R \}$ is a translation of anti S-fuzzy subhemiring A of the hemiring $(R, +, \cdot)$. Thus in all cases, intersection of any two translations of anti S-fuzzy subhemiring A of a hemiring $(R, +, \cdot)$ is also a translation of anti S-fuzzy subhemiring A. \square

3.4.2 Theorem: The intersection of a family of translations of anti S-fuzzy subhemiring A of a hemiring $(R, +, \cdot)$ is also a translation of anti S-fuzzy subhemiring A.

Proof: The argument is trivial.

3.4.3 Theorem: If M and N are two translations of anti S -fuzzy subhemiring A of a hemiring $(R, +, \cdot)$, then their union $M \cup N$ is also a translation of anti S -fuzzy subhemiring A .

Proof: Let x and y belong to R . Let $M = T_\alpha^A = \{ \langle x, \mu_A(x) + \alpha \rangle / x \text{ in } R \}$ and $N = T_\gamma^A = \{ \langle x, \mu_A(x) + \gamma \rangle / x \text{ in } R \}$ be two translations of anti S -fuzzy subhemiring A of a hemiring $(R, +, \cdot)$. Let $C = M \cup N$ and $C = \{ \langle x, \mu_C(x) \rangle / x \text{ in } R \}$, where $\mu_C(x) = \max \{ \mu_A(x) + \alpha, \mu_A(x) + \gamma \}$.

Case (i): $\alpha \leq \gamma$.

$$\begin{aligned} \text{Now, } \mu_C(x+y) &= \max \{ \mu_M(x+y), \mu_N(x+y) \} \\ &= \max \{ \mu_A(x+y) + \alpha, \mu_A(x+y) + \gamma \} \\ &= \mu_A(x+y) + \gamma \\ &= \mu_N(x+y), \text{ for all } x \text{ and } y \in R. \end{aligned}$$

$$\begin{aligned} \text{And, } \mu_C(xy) &= \max \{ \mu_M(xy), \mu_N(xy) \} \\ &= \max \{ \mu_A(xy) + \alpha, \mu_A(xy) + \gamma \} \\ &= \mu_A(xy) + \gamma \\ &= \mu_N(xy), \text{ for all } x \text{ and } y \in R. \end{aligned}$$

Therefore $C = T_\gamma^A = \{ \langle x, \mu_A(x) + \gamma \rangle / x \in R \}$ is a translation of anti S -fuzzy subhemiring A of a hemiring $(R, +, \cdot)$.

Case (ii): $\alpha \geq \gamma$.

$$\begin{aligned} \text{Now, } \mu_C(x+y) &= \max \{ \mu_M(x+y), \mu_N(x+y) \} \\ &= \max \{ \mu_A(x+y) + \alpha, \mu_A(x+y) + \gamma \} \\ &= \mu_A(x+y) + \alpha \end{aligned}$$

$$= \mu_M(x+y), \text{ for all } x \text{ and } y \in R.$$

And $\mu_C(xy) = \max \{ \mu_M(xy), \mu_N(xy) \}$

$$= \max \{ \mu_A(xy) + \alpha, \mu_A(xy) + \gamma \}$$

$$= \mu_A(xy) + \alpha$$

$$= \mu_M(xy), \text{ for all } x \text{ and } y \in R.$$

Therefore $C = T_\alpha^A = \{ \langle x, \mu_A(x) + \alpha \rangle / x \text{ in } R \}$ is a translation of anti S-fuzzy subhemiring A of the hemiring $(R, +, \cdot)$. Hence all cases, union of any two translations of anti S-fuzzy subhemiring A of a hemiring $(R, +, \cdot)$ is also a translation of anti S-fuzzy subhemiring A. \square

3.4.4 Theorem: The union of a family of translations of anti S-fuzzy subhemiring A of a hemiring $(R, +, \cdot)$ is also a translation of anti S-fuzzy subhemiring A.

Proof: The argument is trivial.

3.4.5 Theorem: If T_α^A is a translation of anti S-fuzzy subhemiring A of a hemiring R, then T_α^A is anti S-fuzzy subhemiring of R.

Proof: Assume that T_α^A is a translation of anti S-fuzzy subhemiring A of a hemiring R. Let x and y in R.

We have, $T_\alpha^A(x+y) = A(x+y) + \alpha$

$$\leq S(A(x), A(y)) + \alpha$$

$$\leq S(A(x) + \alpha, A(y) + \alpha)$$

$$= S(T_\alpha^A(x), T_\alpha^A(y)).$$

Therefore, $T_\alpha^A(x+y) \leq S(T_\alpha^A(x), T_\alpha^A(y))$, for all x and y $\in R$.

$$\begin{aligned}
\text{And, } T_{\alpha}^A(xy) &= A(xy) + \alpha \\
&\leq S(A(x), A(y)) + \alpha \\
&\leq S(A(x) + \alpha, A(y) + \alpha) \\
&= S(T_{\alpha}^A(x), T_{\alpha}^A(y)).
\end{aligned}$$

Therefore, $T_{\alpha}^A(xy) \geq S(T_{\alpha}^A(x), T_{\alpha}^A(y))$, for all x and y in R . Hence T_{α}^A is anti S -fuzzy subhemiring of R □

3.4.6 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. If $f: R \rightarrow R^1$ is a homomorphism, then the translation of anti S -fuzzy subhemiring A of R under the homomorphic image is anti S -fuzzy subhemiring of $f(R) = R^1$.

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f: R \rightarrow R^1$ be a homomorphism. That is $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$, for all x and y in R . Let T_{α}^A be a translation of anti S -fuzzy subhemiring A of R . Let V be the homomorphic image of T_{α}^A under f . We have to prove that V is anti S -fuzzy subhemiring of $f(R) = R^1$. Now, for $f(x)$ and $f(y)$ in R^1 , we have

$$\begin{aligned}
V[f(x)+f(y)] &= V[f(x+y)] \leq T_{\alpha}^A(x+y) \\
&= A(x+y) + \alpha \\
&\leq S(A(x), A(y)) + \alpha \\
&\leq S(A(x) + \alpha, A(y) + \alpha) \\
&= S(T_{\alpha}^A(x), T_{\alpha}^A(y)),
\end{aligned}$$

which implies that $V[f(x)+f(y)] \leq S(V(f(x)), V(f(y)))$, for all $f(x), f(y) \in R^1$.

$$\begin{aligned}
\text{And } V[f(x)f(y)] &= V[f(xy)] \leq T_{\alpha}^A(xy) \\
&= A(xy) + \alpha \\
&\leq S(A(x), A(y)) + \alpha \\
&\leq S(A(x) + \alpha, A(y) + \alpha) \\
&= S(T_{\alpha}^A(x), T_{\alpha}^A(y)),
\end{aligned}$$

which implies that $V[f(x)f(y)] \leq S(V(f(x)), V(f(y)))$, for all $f(x)$ and $f(y) \in R^1$. Therefore, V is an anti S-fuzzy subhemiring of R^1 .

Hence the homomorphic image of translation of A of R is an anti S-fuzzy subhemiring of R^1 . \square

3.4.7 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. If $f : R \rightarrow R^1$ is a homomorphism, then the translation of an anti S-fuzzy subhemiring V of $f(R) = R^1$ under the homomorphic pre-image is an anti S-fuzzy subhemiring of R .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be a homomorphism. That is $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and y in R . Let T_{α}^V be the translation of anti S-fuzzy subhemiring V of R^1 and A be the homomorphic pre-image of T_{α}^V under f . We have to prove that A is an anti S-fuzzy subhemiring of R . Let x and y be in R .

$$\begin{aligned}
\text{Then, } A(x+y) &= T_{\alpha}^V(f(x+y)) \\
&= T_{\alpha}^V(f(x)+f(y)) \\
&= V[f(x)+f(y)] + \alpha \\
&\leq S(V(f(x)), V(f(y))) + \alpha
\end{aligned}$$

$$\begin{aligned}
&\leq S ((V(f(x)) + \alpha, V(f(y)) + \alpha) \\
&= S (T_{\alpha}^V (f(x)), T_{\alpha}^V (f(y))) \\
&= S (A(x), A(y)),
\end{aligned}$$

which implies that $A(x+y) \leq S (A(x), A(y))$, for all x, y in R .

$$\text{And, } A(xy) = T_{\alpha}^V (f(xy)) = T_{\alpha}^V (f(x)f(y))$$

$$\begin{aligned}
&= V[f(x)f(y)] + \alpha \\
&\leq S (V(f(x)), V(f(y))) + \alpha \\
&\leq S ((V(f(x)) + \alpha, V(f(y)) + \alpha)) \\
&= S (T_{\alpha}^V (f(x)), T_{\alpha}^V (f(y))) \\
&= S (A(x), A(y)),
\end{aligned}$$

which implies that, $A(xy) \leq S (A(x), A(y))$, for all x and $y \in R$.

Therefore, A is anti S -fuzzy subhemiring of R . □

3.4.8 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. If $f : R \rightarrow R^1$ is an anti-homomorphism, then the translation of an anti S -fuzzy subhemiring A of R under the anti-homomorphic image is an anti S -fuzzy subhemiring of $f(R) = R^1$.

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be an anti-homomorphism. That is $f(x+y) = f(y)+ f(x)$ and $f(xy) = f(y)f(x)$, for all x and $y \in R$. Let T_{α}^A be the translation of an anti S -fuzzy subhemiring A of R and V be the anti-homomorphic image of T_{α}^A under f . We have to prove that V is an anti S -fuzzy subhemiring of $f(R) = R^1$. Now, for $f(x)$ and $f(y)$ in R^1 and q in Q , we have, $V[f(x)+f(y)] = V[f(y+x)]$

$$\begin{aligned}
&\leq T_{\alpha}^A (y+x) \\
&= A (y+x) + \alpha \\
&\leq S (A(x), A(y)) + \alpha \\
&\leq S (A(x) + \alpha, A(y)+ \alpha) \\
&= S (T_{\alpha}^A (x), T_{\alpha}^A (y))
\end{aligned}$$

which implies that $V[f(x) + f(y)] \leq S (V(f(x)), V(f(y)),$ for all $f(x), f(y) \in R^1$.

And, $V[f(x)f(y)] = V[f(yx)]$

$$\begin{aligned}
&\leq T_{\alpha}^A (yx) \\
&= A (yx) + \alpha \\
&\leq S (A(x), A(y)) + \alpha \\
&\leq S (A(x)+\alpha, A(y)+\alpha) \\
&= S (T_{\alpha}^A (x), T_{\alpha}^A (y))
\end{aligned}$$

which implies that $V[f(x)f(y)] \leq S (V(f(x)), V(f(y)))$, for all $f(x)$ and $f(y)$ in R^1 . Therefore, V is an anti S -fuzzy subhemiring of the hemiring R^1 . Hence the anti-homomorphic image of translation of A of R is an anti S -fuzzy subhemiring of R^1 . □

3.4.9 Theorem: Let $(R, +, .)$ and $(R^1, +, .)$ be any two hemirings. If $f: R \rightarrow R^1$ is an anti-homomorphism, then the translation of an anti S -fuzzy subhemiring V of $f(R) = R^1$ under the anti-homomorphic pre-image is an anti S -fuzzy subhemiring of R .

Proof: Let $(R, +, .)$ and $(R^1, +, .)$ be any two hemirings and $f: R \rightarrow R^1$ be an anti-homomorphism. That is $f(x+y) = f(y)+f(x)$ and $f(xy) = f(y)f(x)$, for all x

and $y \in R$. Let T_α^V be the translation of an anti S-fuzzy subhemiring V of $f(R) = R^1$ and A be the anti-homomorphic pre-image of T_α^V under f . We have to prove that A is an anti S-fuzzy subhemiring of R . Let x and $y \in R$. Then,

$$\begin{aligned}
A(x+y) &= T_\alpha^V (f(x+y)) \\
&= T_\alpha^V [f(y)+f(x)] \\
&= V[f (y)+f(x)] + \alpha \\
&\leq S (V(f(x)), V(f(y))) + \alpha \\
&\leq S (V(f(x))+\alpha, V(f(y))+\alpha) \\
&= S (T_\alpha^V (f(x))), T_\alpha^V (f(y))) \\
&= S (A(x), A(y)),
\end{aligned}$$

which implies that $A(x+y) \leq S (A(x), A(y)$, for all x and $y \in R$.

And, $A(xy) = T_\alpha^V (f(xy))$

$$\begin{aligned}
&= T_\alpha^V (f(y)f(x)) \\
&= V[f (y)f(x)] + \alpha \\
&\leq S (V(f(x)), V(f(y))) + \alpha \\
&\leq S (V(f(x))+\alpha, V(f(y))+\alpha) \\
&= S (T_\alpha^V (f(x))), T_\alpha^V (f(y))) \\
&= S (A(x), A(y)),
\end{aligned}$$

which implies that $A(xy) \leq S (A(x), A(y)$, for all x and $y \in R$.

Therefore, A is an anti S-fuzzy subhemiring of R . □

3.4.10 Theorem: If M and N are two translations of an anti S -fuzzy normal subhemiring A of a hemiring $(R, +, \cdot)$, then their intersection $M \cap N$ is also a translation of A .

Proof: The argument is trivial.

3.4.11 Theorem: The intersection of a family of translations of an anti S -fuzzy normal subhemiring A of a hemiring $(R, +, \cdot)$ is a translation of A .

Proof: The argument is trivial.

3.4.12 Theorem: If M and N are two translations of an anti S -fuzzy normal subhemiring A of a hemiring $(R, +, \cdot)$, then their union $M \cup N$ is also a translation of A .

Proof: The argument is trivial.

3.4.13 Theorem: The union of a family of translations of an anti S -fuzzy normal subhemiring A of a hemiring $(R, +, \cdot)$ is also a translation of A .

Proof: The argument is trivial.

3.4.14 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. If $f : R \rightarrow R^1$ is a homomorphism, then the translation of an anti S -fuzzy normal subhemiring A of R under the homomorphic image is an anti S -fuzzy normal subhemiring of $f(R) = R^1$.

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be a homomorphism. That is $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$, for all x and $y \in R$. Let T_α^A be the translation of an anti S -fuzzy normal subhemiring A of R and V be the homomorphic image of T_α^A under f . We have to prove that V

is an anti S-fuzzy normal subhemiring of R^1 . Now, for $f(x)$ and $f(y) \in R^1$, clearly V is an anti S-fuzzy subhemiring of R^1 .

$$\begin{aligned} \text{We have } V(f(x)f(y)) &= V(f(xy)) \leq T_{\alpha}^A(xy) = A(xy) + \alpha = A(yx) + \alpha \\ &= T_{\alpha}^A(yx) \geq V(f(yx)) = V(f(y) f(x)), \end{aligned}$$

which implies that $V(f(x)f(y)) = V(f(y)f(x))$, for all $f(x)$ and $f(y) \in R^1$.

Therefore, V is an anti S-fuzzy normal subhemiring of the hemiring R^1 . \square

3.4.15 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. If $f : R \rightarrow R^1$ is a homomorphism, then translation of an anti S-fuzzy normal subhemiring V of $f(R) = R^1$ under the homomorphic pre-image is an anti S-fuzzy normal subhemiring of R .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be a homomorphism. That is $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$, for all x and $y \in R$. Let T_{α}^V be the translation of an anti S-fuzzy normal subhemiring V of R^1 and A be the homomorphic pre-image of T_{α}^V under f . We have to prove that A is an anti S-fuzzy normal subhemiring of R . Let x and $y \in R$. Then, clearly A is an anti S-fuzzy subhemiring of R , $A(xy) = T_{\alpha}^V(f(xy)) = V(f(xy)) + \alpha = V(f(x)f(y)) + \alpha = V(f(y)f(x)) + \alpha = V(f(yx)) + \alpha = T_{\alpha}^V(f(yx)) = A(yx)$, which implies that $A(xy) = A(yx)$, for all x and y in R . Therefore, A is an anti S-fuzzy normal subhemiring of R . \square

3.4.16 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. If $f : R \rightarrow R^1$ is an anti-homomorphism, then the translation of an anti S-fuzzy

normal subhemiring A of R under the anti-homomorphic image is an anti S -fuzzy normal subhemiring of R^1 .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be an anti-homomorphism. That is $f(x+y) = f(y)+f(x)$ and $f(xy) = f(y)f(x)$, for all x and $y \in R$. Let T_α^A be the translation of an anti S -fuzzy normal subhemiring A of R and V be the anti-homomorphic image of T_α^A under f . We have to prove that V is an anti S -fuzzy normal subhemiring of $f(R) = R^1$. Now, for $f(x)$ and $f(y)$ in R^1 , clearly V is an anti S -fuzzy subhemiring of R^1 . We have, $V(f(x)f(y)) = V(f(yx)) \leq T_\alpha^A(yx) = A(yx) + \alpha = A(xy) + \alpha = T_\alpha^A(xy) \geq V(f(xy)) = V(f(y)f(x))$, which implies that $V(f(x)f(y)) = V(f(y)f(x))$, for $f(x)$ and $f(y)$ in R^1 . Therefore, V is an anti S -fuzzy normal subhemiring of the hemiring R^1 . \square

3.4.17 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. If $f : R \rightarrow R^1$ is an anti-homomorphism, then the translation of an anti S -fuzzy normal subhemiring V of $f(R) = R^1$ under the anti-homomorphic pre-image is an anti S -fuzzy normal subhemiring of R .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^1$ be an anti-homomorphism. That is $f(x + y) = f(y) + f(x)$ and $f(xy) = f(y)f(x)$, for all x and $y \in R$. Let T_α^V be the translation of an anti S -fuzzy normal subhemiring V of R^1 and A be the anti-homomorphic pre-image of T_α^V under f . We have to prove that A is an anti S -fuzzy normal

subhemiring of R . Let x and $y \in R$. Then, clearly A is an anti S -fuzzy

subhemiring of R , $A(xy) = T_{\alpha}^V (f(xy)) = V(f(xy)) + \alpha$

$$= V(f(y)f(x)) + \alpha$$

$$= V(f(x)f(y)) + \alpha$$

$$= V(f(yx)) + \alpha = T_{\alpha}^V (f(yx)) = A(yx),$$

which implies that $A(xy) = A(yx)$, for all x and $y \in R$.

Therefore, A is an anti S -fuzzy normal subhemiring of R . \square