CHAPTER - 2

ANTI S-FUZZY SUBHEMIRINGS OF A HEMIRING

2.1 Introduction: In this chapter, we introduce the concept of anti S-fuzzy subhemirings of a hemiring and establish some results on these. We also made an attempt to study the properties of anti S-fuzzy subhemirings of hemiring under homomorphism and anti-homomorphism.

2.1.1 Definition: A S-norm is a binary operation $S: [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following requirements;

(i) $S(0, x) = x$, $S(1, x) = 1$ (boundary condition)

(ii) $S(x, y) = S(y, x)$ (commutativity)

(iii) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity)

(iv) if $x \leq y$ and $w \leq z$, then $S(x, w) \leq S(y, z)$ (monotonicity).

2.1.2 Definition: Let $(R, +, .)$ be a hemiring. A fuzzy subset $A$ of $R$ is said to be an anti S-fuzzy subhemiring (anti fuzzy subhemiring with respect to S-norm) of $R$ if it satisfies the following conditions:

(i) $\mu_A(x + y) \leq S(\mu_A(x), \mu_A(y))$,

(ii) $\mu_A(xy) \leq S(\mu_A(x), \mu_A(y))$, for all $x$ and $y \in R$.

2.1.3 Definition: Let $(R, +, .)$ be a hemiring. An anti S-fuzzy subhemiring $A$ of $R$ is said to be an anti S-fuzzy normal subhemiring (ASFNSHR) of $R$ if $\mu_A(xy) = \mu_A(yx)$, for all $x$ and $y \in R$. 
2.1.4 Definition: Let A and B be fuzzy subsets of sets G and H, respectively. The anti-product of A and B, denoted by $A \times B$, is defined as $A \times B = \{ (x, y), \mu_{A \times B}(x,y) \}$ / for all $x \in G$ and $y \in H \}$, where $\mu_{A \times B}(x, y) = \max \{ \mu_A(x), \mu_B(y) \}$.

2.1.5 Definition: Let A be a fuzzy subset in a set S, the anti-strongest fuzzy relation on S, that is a fuzzy relation on A is V given by $\mu_V(x, y) = \max \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y \in S$.

2.1.6 Definition: An anti S-fuzzy subhemiring A of a hemiring R is called an anti S-fuzzy characteristic subhemiring of R if $\mu_A(x) = \mu_A(f(x))$, for all $x \in R$ and f in Aut (R).

2.1.7 Definition: Let R and $R^l$ be any two hemirings. Let $f : R \rightarrow R^l$ be any function and A be an anti S-fuzzy subhemiring in R, V be an anti S-fuzzy subhemiring in $f(R) = R^l$, defined by $\mu_V(y) = \inf_{x \in f^{-1}(y)} \mu_A(x)$, for all $x \in R$, $y \in R^l$.

Then A is called a preimage of V under f and is denoted by $f^{-1}(V)$.

Note: This definition is used throughout this chapter for image and preimage in functions.

2.1.8 Definition: Let A be an anti S-fuzzy subhemiring of a hemiring $(R, +, .)$ and a in R. Then the pseudo anti S-fuzzy coset $(aA)^p$ is defined by $(a \mu_A)^p(x) = p(a) \mu_A(x)$, for every $x$ in R and for some $p \in P$.

2.1.9 Definition: Let A be a fuzzy subset of X. For $\alpha$ in $[0, 1]$, the lower level subset of A is the set $A_\alpha = \{ x \in X : \mu_A(x) \leq \alpha \}$.

2.1.10 Definition: Let A be an anti S-fuzzy subhemiring of a hemiring R. Then $A^0$ is defined as $A^0(x) = A(x)A(0)$, for all $x$ in R, where $A(0) \neq 0$. 

2.2 - PROPERTIES OF ANTI S-FUZZY SUBHEMIRING OF A HEMIRING

2.2.1 Theorem: Union of any two anti S-fuzzy subhemiring of a hemiring R is an anti S-fuzzy subhemiring of R.

Proof: Let A and B be any two anti S-fuzzy subhemirings of a hemiring R and x and y in R. Let \(A = \{(x, \mu_A(x)) / x \in R\}\) and \(B = \{(x, \mu_B(x)) / x \in R\}\) and also let \(C = A \cup B = \{(x, \mu_C(x)) / x \in R\}\), where \(\max\{\mu_A(x), \mu_B(x)\} = \mu_C(x)\).

Now, \(\mu_C(x+y) = \max\{\mu_A(x+y), \mu_B(x+y)\}\)

\[\leq \max\{S(\mu_A(x), \mu_A(y)), S(\mu_B(x), \mu_B(y))\}\]

\[\leq S\{S(\mu_A(x), \mu_B(x)), S(\mu_A(y), \mu_B(y))\}\]

\[= S(\mu_C(x), \mu_C(y))\].

Therefore, \(\mu_C(x+y) \leq S(\mu_C(x), \mu_C(y))\), for all \(x, y \in R\).

And, \(\mu_C(xy) = \max\{\mu_A(xy), \mu_B(xy)\}\)

\[\leq \max\{S(\mu_A(x), \mu_A(y)), S(\mu_B(x), \mu_B(y))\}\]

\[\leq S\{S(\mu_A(x), \mu_B(x)), S(\mu_A(y), \mu_B(y))\}\]

\[= S(\mu_C(x), \mu_C(y))\].

Therefore, \(\mu_C(xy) \leq S(\mu_C(x), \mu_C(y))\), for all \(x, y \in R\).

Therefore C is an anti S-fuzzy subhemiring of a hemiring R.

Hence the union of any two anti S-fuzzy subhemirings of a hemiring R is an anti S-fuzzy subhemiring of R. \[\Box\]

2.2.2 Theorem: The union of a family of anti S-fuzzy subhemirings of hemiring R is an anti S-fuzzy subhemiring of R.

Proof: The argument is trivial.
2.2.3 **Theorem:** If A and B are any two anti S-fuzzy subhemirings of the hemirings $R_1$ and $R_2$ respectively, then anti-product $A \times B$ is an anti S-fuzzy subhemiring of $R_1 \times R_2$.

**Proof:** Let A and B be two anti S-fuzzy subhemirings of the hemirings $R_1$ and $R_2$ respectively. Let $x_1$ and $x_2$ be in $R_1$, $y_1$ and $y_2$ be in $R_2$.

Then $(x_1, y_1)$ and $(x_2, y_2)$ are in $R_1 \times R_2$.

Now, $\mu_{A \times B} ( (x_1, y_1) + (x_2, y_2) ) = \mu_{A \times B} ( x_1 + x_2, y_1 + y_2 )$

\[= \max \{ \mu_A(x_1 + x_2), \mu_B( y_1 + y_2 ) \} \]

\[\leq \max \{ S (\mu_A(x_1), \mu_A(x_2) ), S (\mu_B(y_1), \mu_B(y_2) ) \} \]

\[\leq S ( S (\mu_A(x_1), \mu_B(y_1) ), S (\mu_A(x_2), \mu_B(y_2) ) ) \]

\[= S (\mu_{A \times B} (x_1, y_1), \mu_{A \times B} (x_2, y_2) ). \]

Therefore, $\mu_{A \times B} ( (x_1, y_1) + (x_2, y_2) ) \leq S (\mu_{A \times B} (x_1, y_1), \mu_{A \times B} (x_2, y_2) )$.

Also, $\mu_{A \times B} ( (x_1, y_1)(x_2, y_2) ) = \mu_{A \times B} (x_1x_2, y_1y_2 )$

\[= \max \{ \mu_A( x_1x_2 ), \mu_B( y_1y_2 ) \} \]

\[\leq \max \{ S (\mu_A(x_1), \mu_A(x_2) ), S (\mu_B(y_1), \mu_B(y_2) ) \} \]

\[\leq S ( S (\mu_A(x_1), \mu_B(y_1) ), S (\mu_A(x_2), \mu_B(y_2) ) ) \]

\[= S (\mu_{A \times B} (x_1, y_1), \mu_{A \times B} (x_2, y_2) ). \]

Therefore, $\mu_{A \times B} ( (x_1, y_1)(x_2, y_2) ) \leq S (\mu_{A \times B} (x_1, y_1), \mu_{A \times B} (x_2, y_2) )$.

Hence $A \times B$ is an anti S-fuzzy subhemiring of hemiring of $R_1 \times R_2$.

2.2.4 **Theorem:** If A is an anti S-fuzzy subhemiring of a hemiring $(R, +, \cdot)$, then $\mu_A(x) \geq \mu_A(0)$, for $x \in R$, the zero $0 \in R$.

**Proof:** For $x \in R$, and 0 is the zero element of R.
Now, \( \mu_A(x) = \mu_A(x+0) \leq S(\mu_A(x), \mu_A(0)) \), for all \( x \in \mathbb{R} \).

So, \( \mu_A(x) \geq \mu_A(0) \) is only possible.

**2.2.5 Theorem:** Let \( A \) and \( B \) be anti S-fuzzy subhemiring of the hemirings \( R_1 \) and \( R_2 \) respectively. Suppose that \( 0_1 \) and \( 0_2 \) are the zero elements of \( R_1 \) and \( R_2 \) respectively. If \( A \times B \) is an anti S-fuzzy subhemiring of \( R_1 \times R_2 \), then at least one of the following two statements must hold.

(i) \( \mu_B(0_2) \leq \mu_A(x) \), for all \( x \in R_1 \),

(ii) \( \mu_A(0_1) \leq \mu_B(y) \), for all \( y \in R_2 \).

**Proof:** Let \( A \times B \) be an anti S-fuzzy subhemiring of \( R_1 \times R_2 \).

By contraposition, suppose that none of the statements (i) and (ii) holds.

Then we can find an element \( a \in R_1 \) and \( b \in R_2 \) such that \( \mu_A(a) < \mu_B(0_2) \) and \( \mu_B(b) < \mu_A(0_1) \).

We have, \( \mu_{A \times B}(a, b) = \max\{\mu_A(a), \mu_B(b)\} \)

\[< \max\{\mu_B(0_2), \mu_A(0_1)\}\]

\[= \max\{\mu_A(0_1), \mu_B(0_2)\}\]

\[= \mu_{A \times B}(0_1, 0_2).\]

Thus \( A \times B \) is not an anti S-fuzzy subhemiring of \( R_1 \times R_2 \).

Hence either \( \mu_B(0_2) \leq \mu_A(x) \), \( x \in R_1 \) or \( \mu_A(0_1) \leq \mu_B(y) \), for all \( y \in R_2 \).

**2.2.6 Theorem:** Let \( A \) and \( B \) be two fuzzy subsets of the hemirings \( R_1 \) and \( R_2 \) respectively. If \( A \times B \) is an anti S-fuzzy subhemiring of \( R_1 \times R_2 \). Then the following are true:

i. if \( \mu_A(x) \geq \mu_B(0_2) \), then \( A \) is an anti S-fuzzy subhemiring of \( R_1 \).
ii. if \( \mu_B(x) \geq \mu_A(0) \), then B is an anti S-fuzzy subhemiring of \( R_2 \).

iii. either A is an anti S-fuzzy subhemiring of \( R_1 \) or B is an anti S-fuzzy subhemiring of \( R_2 \).

**Proof:** Let \( A \times B \) be an anti S-fuzzy subhemiring of \( R_1 \times R_2 \) and 

\( x \) and \( y \in R_1 \) and \( 0_2 \in R_2 \). Then \( (x, 0_2) \) and \( (y, 0_2) \in R_1 \times R_2 \).

Now, using the property that \( \mu_A(x) \geq \mu_B(0_2) \), for all \( x \in R_1 \).

We get, 

\[
\mu_A(x+y) = \max\{ \mu_A(x+y), \mu_B(0_2+0_2) \}
\]

\[
= \mu_{A \times B}( (x+y), (0_2+0_2) )
\]

\[
= \mu_{A \times B}( (x, 0_2)+(y, 0_2) )
\]

\[
\leq S( \mu_{A \times B}(x, 0_2), \mu_{A \times B}(y, 0_2) )
\]

\[
= S( \max\{ \mu_A(x), \mu_B(0_2) \}, \max\{ \mu_A(y), \mu_B(0_2) \} )
\]

\[
= S( \mu_A(x), \mu_A(y) ).
\]

Therefore, \( \mu_A(x+y) \leq S( \mu_A(x), \mu_A(y) ) \), for all \( x \) and \( y \in R_1 \).

Also, \( \mu_A(xy) = \max\{ \mu_A(xy), \mu_B(0_20_2) \} \)

\[
= \mu_{A \times B}( (xy), (0_20_2) )
\]

\[
= \mu_{A \times B}( (x, 0_2)(y, 0_2) )
\]

\[
\leq S( \mu_{A \times B}(x, 0_2), \mu_{A \times B}(y, 0_2) )
\]

\[
= S( \max\{ \mu_A(x), \mu_B(0_2) \}, \max\{ \mu_A(y), \mu_B(0_2) \} )
\]

\[
= S( \mu_A(x), \mu_A(y) ).
\]

Therefore, \( \mu_A(xy) \leq S( \mu_A(x), \mu_A(y) ) \), for all \( x \) and \( y \in R_1 \).

Hence A is an anti S-fuzzy subhemiring of \( R_1 \). Thus (i) is proved.

Now, \( \mu_B(x) \geq \mu_A(0) \), for all \( x \in R_2 \).
let x and y in $R_2$ and $0_1$ in $R_1$. Then $(0_1, x)$ and $(0_1, y)$ are $\in R_1 \times R_2$.

We get, $\mu_B(x+y) = \max\{ \mu_B(x+y), \mu_A(0_1 + 0_1) \}$

$$= \max\{ \mu_A(0_1 + 0_1), \mu_B(x+y) \}$$

$$= \mu_{A \times B}(0_1 + 0_1, (x+y))$$

$$= \mu_{A \times B}[(0_1, x) + (0_1, y)]$$

$$\leq S(\mu_{A \times B}(0_1, x), \mu_{A \times B}(0_1, y))$$

$$= S(\max\{\mu_A(0_1), \mu_B(x)\}, \max\{\mu_A(0_1), \mu_B(y)\})$$

$$= S(\mu_B(x), \mu_B(y)).$$

Therefore, $\mu_B(x+y) \leq S(\mu_B(x), \mu_B(y))$, for all $x$ and $y \in R_2$.

Also, $\mu_B(xy) = \max\{ \mu_B(xy), \mu_A(0_1 0_1) \}$

$$= \max\{\mu_A(0_1 0_1), \mu_B(xy)\}$$

$$= \mu_{A \times B}(0_1 0_1, (xy))$$

$$= \mu_{A \times B}[(0_1, x)(0_1, y)]$$

$$\leq S(\mu_{A \times B}(0_1, x), \mu_{A \times B}(0_1, y))$$

$$= S(\max\{\mu_A(0_1), \mu_B(x)\}, \max\{\mu_A(0_1), \mu_B(y)\})$$

$$= S(\mu_B(x), \mu_B(y)).$$

Therefore, $\mu_B(xy) \leq S(\mu_B(x), \mu_B(y))$, for all $x$ and $y \in R_2$.

Hence B is an anti S-fuzzy subhemiring of a hemiring $R_2$.

Thus (ii) is proved. (iii) is clear.

\[ \square \]

**2.2.7 Theorem:** Let A be a fuzzy subset of a hemiring $R$ and $V$ be the anti-strongest fuzzy relation of $R$. Then A is an anti S-fuzzy subhemiring of $R$ if and only if $V$ is an anti S-fuzzy subhemiring of $R \times R$. 


Proof: Suppose that $A$ is an anti $S$-fuzzy subhemiring of a hemiring $R$.

Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in R \times R$, we have,

$$\mu_V(x+y) = \mu_V[(x_1, x_2) + (y_1, y_2)]$$

$$= \mu_V(x_1 + y_1, x_2 + y_2)$$

$$= \max \{ \mu_A(x_1 + y_1), \mu_A(x_2 + y_2) \}$$

$$\leq \max \{ S(\mu_A(x_1), \mu_A(y_1)), S(\mu_A(x_2), \mu_A(y_2)) \}$$

$$\leq S(\max \{ \mu_A(x_1), \mu_A(x_2) \}, \max \{ \mu_A(y_1), \mu_A(y_2) \})$$

$$= S(\mu_V(x_1, x_2), \mu_V(y_1, y_2))$$

$$= S(\mu_V(x), \mu_V(y)).$$

Therefore, $\mu_V(x+y) \leq S(\mu_V(x), \mu_V(y))$, for all $x$ and $y \in R \times R$.

Also, $\mu_V(xy) = \mu_V[(x_1, x_2)(y_1, y_2)]$

$$= \mu_V(x_1 y_1, x_2 y_2)$$

$$= \max \{ \mu_A(x_1 y_1), \mu_A(x_2 y_2) \}$$

$$\leq \max \{ S(\mu_A(x_1), \mu_A(y_1)), S(\mu_A(x_2), \mu_A(y_2)) \}$$

$$\leq S(\max \{ \mu_A(x_1), \mu_A(x_2) \}, \max \{ \mu_A(y_1), \mu_A(y_2) \})$$

$$= S(\mu_V(x_1, x_2), \mu_V(y_1, y_2))$$

$$= S(\mu_V(x), \mu_V(y)).$$

Therefore, $\mu_V(xy) \leq S(\mu_V(x), \mu_V(y))$, for all $x$ and $y \in R \times R$.

This proves that $V$ is an anti $S$-fuzzy subhemiring of $R \times R$.

Conversely assume that $V$ is an anti $S$-fuzzy subhemiring of $R \times R$, then for any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in R \times R$, we have

$$\max \{ \mu_A(x_1 + y_1), \mu_A(x_2 + y_2) \} = \mu_V(x_1 + y_1, x_2 + y_2)$$

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= \mu_V [(x_1, x_2) + (y_1, y_2)] = \mu_V (x + y)

\leq S (\mu_V (x), \mu_V (y)) = S (\mu_V (x_1, x_2), \mu_V (y_1, y_2))

= S (\max \{ \mu_A(x_1), \mu_A(x_2) \}, \max \{ \mu_A(y_1), \mu_A(y_2) \} ).

If x_2 = 0, y_2 = 0, we get, \mu_A(x_1+y_1) \leq S (\mu_A(x_1), \mu_A(y_1) ), \text{ for all } x_1 \text{ and } y_1 \in R.

and, \max \{ \mu_A(x_1y_1), \mu_A(x_2y_2) \} = \mu_V (x_1y_1, x_2y_2)

= \mu_V [(x_1, x_2) (y_1, y_2)] = \mu_V (x y)

\leq S (\mu_V (x), \mu_V (y)) = S (\mu_V (x_1, x_2), \mu_V (y_1, y_2))

= S (\max \{ \mu_A(x_1), \mu_A(x_2) \}, \max \{ \mu_A(y_1), \mu_A(y_2) \} ).

Therefore A is an anti S-fuzzy subhemiring of R.

\[\square\]

2.2.8 Theorem: If A is an anti S-fuzzy subhemiring of a hemiring (R, +, .),
then H = \{ x / x \in R: \mu_A(x) = 0 \} is either empty or is a subhemiring of R.

Proof: The argument is trivial.

2.2.9 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring (R, +, .).

If \mu_A(x+y) = 1, then either \mu_A(x) = 1 or \mu_A(y) = 1, \text{ for all } x \text{ and } y \in R.

Proof: The argument is trivial.

In the next theorem, we use composition operation in S- fuzzy subhemiring

2.2.10 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring H and
f is an isomorphism from a hemiring R onto H. Then A \circ f is an anti S-fuzzy subhemiring of R.

Proof: Let x, y in R and A be an anti S-fuzzy subhemiring of a hemiring H.

Then we have, ( \mu_A \circ f)(x+y) = \mu_A ( f(x+y) ) = \mu_A( f(x) + f(y) )

\leq S (\mu_A(f(x) ), \mu_A( f(y) ) ) \leq S ( (\mu_A \circ f)(x) ), (\mu_A \circ f ) (y) ),
which implies that \((\mu_{A^f})(x+y) \leq S ( (\mu_{A^f})(x), (\mu_{A^f})(y) )\).

And, \((\mu_{A^f})(xy) = \mu_A(f(xy)) = \mu_A(f(x)f(y)) \)

\[ \leq S ( \mu_A( f(x) ), \mu_A( f(y) ) ), \]

which implies that \((\mu_{A^f})(xy) \leq S ((\mu_{A^f})(x), (\mu_{A^f})(y) )\).

Therefore \((A^f)\) is an anti S-fuzzy subhemiring of a hemiring \(R\).

2.2.11 Theorem: Let \(A\) be an anti S-fuzzy subhemiring of a hemiring \(H\) and \(f\) is an anti S-isomorphism from a hemiring \(R\) onto \(H\). Then \(A^f\) is an anti S-fuzzy subhemiring of \(R\).

Proof: Let \(x, y\) in \(R\) and \(A\) be an anti S-fuzzy subhemiring of a hemiring \(H\). Then we have, \((\mu_{A^f})(x+y) = \mu_A(f(x+y))\)

\[= \mu_A(f(y)+f(x)), \text{ as } f \text{ is an anti-isomorphism} \]

\[\leq S ( \mu_A( f(x) ), \mu_A( f(y) ) ), \]

\[\leq S ( (\mu_{A^f})(x), (\mu_{A^f})(y) ), \]

which implies that \((\mu_{A^f})(x+y) \leq S ( (\mu_{A^f})(x), (\mu_{A^f})(y) )\).

Now, \((\mu_{A^f})(xy) = \mu_A(f(xy)) = \mu_A(f(y)f(x))\)

\[\leq S ( \mu_A( f(x) ), \mu_A( f(y) ) ), \]

\[\leq S ( (\mu_{A^f})(x), (\mu_{A^f})(y) ), \]

which implies that \((\mu_{A^f})(xy) \leq S ( (\mu_{A^f})(x), (\mu_{A^f})(y) )\).

Therefore \(A^f\) is an anti S-fuzzy subhemiring of a hemiring \(R\).

2.2.12 Theorem: Let \(A\) be an anti S-fuzzy subhemiring of a hemiring \((R, +, \cdot)\), then the pseudo anti S-fuzzy coset \((aA)^p\) is an anti S-fuzzy subhemiring of a hemiring \(R\), for every \(a\) in \(R\).
Proof: Let $A$ be an anti $S$-fuzzy subhemiring of a hemiring $R$.

For every $x$ and $y$ in $R$, we have, 
\[
( (a\mu_A)^p)( x+y ) = p(a)\mu_A( x+y ) \leq p(a) S ( (a\mu_A)(x), (a\mu_A)(y) ) = S ( ( (a\mu_A)^p )(x), ( (a\mu_A)^p )(y) ).
\]
Therefore, 
\[
( (a\mu_A)^p)( x+ y ) \leq S ( ( (a\mu_A)^p )(x), ( (a\mu_A)^p )(y) ).
\]

Now, 
\[
( (a\mu_A)^p)( xy ) = p(a)\mu_A(xy) \leq p(a) S ( \mu_A(x), \mu_A(y) ) = S ( p(a)\mu_A(x), p(a)\mu_A(y) ) = S ( ( (a\mu_A)^p )(x), ( (a\mu_A)^p )(y) ).
\]
Therefore, 
\[
( (a\mu_A)^p)( xy ) \leq S ( ( (a\mu_A)^p )(x), ( (a\mu_A)^p )(y) ).
\]

Hence $(aA)^p$ is an anti $S$-fuzzy subhemiring of a hemiring $R$. 

2.2.13 Theorem: Let $( R, +, \cdot )$ and $( R^l, +, \cdot )$ be any two hemirings. The homomorphic image of an anti $S$-fuzzy subhemiring of $R$ is an anti $S$-fuzzy subhemiring of $R^l$.

Proof: Let $( R, +, \cdot )$ and $( R^l, +, \cdot )$ be any two hemirings. Let $f : R \to R^l$ be a homomorphism. Then, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all $x$ and $y$ in $R$. Let $V = f(A)$, where $A$ is an anti $S$-fuzzy subhemiring of $R$. We have to prove that $V$ is an anti $S$-fuzzy subhemiring of $R^l$. Now, for $f(x)$, $f(y)$ in $R^l$, $\mu_v( f(x) + f(y) ) = \mu_v( f(x+y) )$, as $f$ is a homomorphism
\[
\leq \mu_A(x+y) \leq S ( \mu_A(x), \mu_A(y) ),
\]
which implies that $\mu_v( f(x) + f(y) ) \leq S ( \mu_v( f(x) ), \mu_v( f(y) ) )$.

Again, $\mu_v( f(x)f(y) ) = \mu_v( f(xy) )$, as $f$ is a homomorphism
\[
\leq \mu_A(xy) \leq S ( \mu_A(x), \mu_A(y) ),
\]
which implies that \( \mu_v( f(x)f(y) ) \leq S (\mu_v( f(x) ), \mu_v( f(y) ) ) \).

Hence \( V \) is an anti S-fuzzy subhemiring of \( R^l \). \( \square \)

2.2.14 Theorem: Let \( ( R, +, .) \) and \( ( R^l, +, .) \) be any two hemirings. The homomorphic preimage of an anti S-fuzzy subhemiring of \( R^l \) is an anti S-fuzzy subhemiring of \( R \).

**Proof:** Let \( ( R, +, .) \) and \( ( R^l, +, .) \) be any two hemirings. Let \( f : R \to R^l \) be a homomorphism. Then, \( f(x+y) = f(x) + f(y) \) and \( f(xy) = f(x)f(y) \), for all \( x \) and \( y \in R \). Let \( V = f(A) \), where \( V \) is an anti S-fuzzy subhemiring of \( R^l \). We have to prove that \( A \) is an anti S-fuzzy subhemiring of \( R \). Let \( x \) and \( y \in R \).

Then, \( \mu_A(x + y) = \mu_v( f(x + y) ) \), since \( \mu_v(f(x)) = \mu_A(x) \)

\[
= \mu_v( f(x) + f(y) ), \text{ as } f \text{ is a homomorphism }
\]

\[
\leq S (\mu_v( f(x) ), \mu_v( f(y) ) )
\]

\[
= S (\mu_A(x), \mu_A(y) ), \text{ since } \mu_v(f(x)) = \mu_A(x)
\]

which implies that \( \mu_A(x + y) \leq S (\mu_A(x), \mu_A(y) ) \).

Again, \( \mu_A(xy) = \mu_v(f(xy) ) \), since \( \mu_v(f(x)) = \mu_A(x) \)

\[
= \mu_v( f(x)f(y) ), \text{ as } f \text{ is a homomorphism }
\]

\[
\leq S (\mu_v(f(x)), \mu_v(f(y)) )
\]

\[
= S (\mu_A(x), \mu_A(y) ), \text{ since } \mu_v(f(x)) = \mu_A(x)
\]

which implies that \( \mu_A(xy) \leq S (\mu_A(x), \mu_A(y) ) \).

Hence \( A \) is an anti S-fuzzy subhemiring of \( R \). \( \square \)
2.2.15 Theorem: Let \((R, +, .)\) and \((R^l, +, .)\) be any two hemirings. The anti-homomorphic image of an anti S-fuzzy subhemiring of \(R\) is an anti S-fuzzy subhemiring of \(R^l\).

**Proof:** Let \((R, +, .)\) and \((R^l, +, .)\) be any two hemirings. Let \(f : R \to R^l\) be an anti-homomorphism. Then, \(f(x+y) = f(y) + f(x)\) and \(f(xy) = f(y) f(x)\), for all \(x\) and \(y\) in \(R\). Let \(V = f(A)\), where \(A\) is an anti S-fuzzy subhemiring of \(R\). We have to prove that \(V\) is an anti S-fuzzy subhemiring of \(R^l\).

Now, for \(f(x), f(y)\) in \(R^l\), 
\[
\mu_V(f(x) + f(y)) \leq S(\mu_A(y+x), \mu_A(x)) 
\]
\[
= S(\mu_A(x), \mu_A(y)),
\]
which implies that 
\[
\mu_V(f(x) + f(y)) \leq S(\mu_A(f(x)), \mu_A(f(y))).
\]
Again, \(\mu_V(f(x)f(y)) = \mu_V(f(xy))\), as \(f\) is an anti-homomorphism
\[
\leq \mu_A(yx) \leq S(\mu_A(y), \mu_A(x))
\]
\[
= S(\mu_A(x), \mu_A(y)),
\]
which implies that 
\[
\mu_V(f(x)f(y)) \leq S(\mu_V(f(x)), \mu_V(f(y))).
\]
Hence \(V\) is an anti S-fuzzy subhemiring of \(R^l\).

2.2.16 Theorem: Let \((R, +, .)\) and \((R^l, +, .)\) be any two hemirings. The anti-homomorphic preimage of an anti S-fuzzy subhemiring of \(R^l\) is an anti S-fuzzy subhemiring of \(R\).

**Proof:** Let \((R, +, .)\) and \((R^l, +, .)\) be any two hemirings. Let \(f : R \to R^l\) be an anti-homomorphism. Then, \(f(x+y) = f(y) + f(x)\) and \(f(xy) = f(y) f(x)\), for all \(x\) and \(y\) in \(R\). Let \(V = f(A)\), where \(V\) is an anti S-fuzzy subhemiring of \(R^l\). We have to prove that \(A\) is an anti S-fuzzy subhemiring of \(R\). Let \(x\) and \(y\) in \(R\).
Then, $\mu_A(x+y) = \mu_A(f(x+y))$, since $\mu_A(f(x)) = \mu_A(x)$

\[= \mu_A(f(y) + f(x)), \text{ as } f \text{ is an anti-homomorphism} \]

\[\leq S (\mu_A(f(y)) , \mu_A(f(x))) \]

\[= S (\mu_A(f(x)) , \mu_A(f(y))) \]

\[= S (\mu_A(x), \mu_A(y)), \text{ since } \mu_A(f(x)) = \mu_A(x) \]

which implies that $\mu_A(x+y) \leq S (\mu_A(x), \mu_A(y))$.

Again, $\mu_A(xy) = \mu_A(f(xy))$, since $\mu_A(f(x)) = \mu_A(x)$

\[= \mu_A(f(y)f(x)), \text{ as } f \text{ is an anti-homomorphism} \]

\[\leq S (\mu_A(f(y)) , \mu_A(f(x))) \]

\[= S (\mu_A(f(x)) , \mu_A(f(y))) \]

\[= S (\mu_A(x), \mu_A(y)), \text{ since } \mu_A(f(x)) = \mu_A(x) \]

which implies that $\mu_A(xy) \leq S (\mu_A(x), \mu_A(y))$.

Hence $A$ is an anti $S$-fuzzy subhemiring of $R$. \hfill \Box

2.2.17 **Theorem:** Let $A$ be an anti $S$-fuzzy subhemiring of a hemiring $R$, $A^+$ be a fuzzy set in $R$ defined by $A^+(x) = A(x)+1 - A(0)$, for all $x \in R$. Then $A^+$ is an anti $S$-fuzzy subhemiring of a hemiring $R$.

**Proof:** Let $x$ and $y$ in $R$. We have,

\[A^+(x+y) = A(x+y) + 1 - A(0) \leq S (A(x), A(y)) + 1 - A(0) \]

\[\leq S( (A(x) + 1 - A(0)) , (A(y) + 1 - A(0))) \]

\[= S (A^+(x), A^+(y)). \]

Therefore, $A^+(x+y) \leq S (A^+(x), A^+(y))$, for all $x, y \in R$.

Similarly, $A^+(xy) = A(xy) + 1 - A(0) \leq S (A(x), A(y)) + 1 - A(0)$
\[ S( (A(x) +1- A(0) ), (A(y) +1- A(0) ) ) \]
\[ = S( A^+(x), A^+(y) ). \]

Therefore, \( A^+(xy) \leq S( A^+(x), A^+(y) ) \), for all \( x, y \in R \).

Hence \( A^+ \) is an anti S-fuzzy subhemiring of a hemiring \( R \). \( \square \)

2.2.18 Theorem: Let \( A \) be an anti S-fuzzy subhemiring of a hemiring \( R \), \( A^+ \) be a fuzzy set in \( R \) defined by \( A^+(x) = A(x) +1 - A(0) \), for all \( x \in R \). Then there exists \( 0 \) in \( R \) such that \( A(0) = 1 \) if and only if \( A^+(x) = A(x) \).

Proof: The argument is trivial.

2.2.19 Theorem: Let \( A \) be an anti S-fuzzy subhemiring of a hemiring \( R \), \( A^+ \) be a fuzzy set in \( R \) defined by \( A^+(x) = A(x) +1 - A(0) \), for all \( x \in R \). Then there exists \( x \in R \) such that \( A^+(x) = 1 \) if and only if \( x = 0 \).

Proof: The argument is trivial.

2.2.20 Theorem: Let \( A \) be an anti S-fuzzy subhemiring of a hemiring \( R \), \( A^+ \) be a fuzzy set in \( R \) defined by \( A^+(x) = A(x) +1 - A(0) \), for all \( x \in R \). Then \( (A^+)^+ = A^+ \).

Proof: Let \( x \) and \( y \in R \). We have, \( (A^+)^+(x) = A^+(x) +1 - A^+(0) = \{ A(x) +1 - A(0) \} +1 - \{ A(0) +1 - A(0) \} = A(x) +1 - A(0) = A^+(x). \) Hence \( (A^+)^+ = A^+ \).

2.2.21 Theorem: Let \( A \) be an anti S-fuzzy subhemiring of a hemiring \( R \). Then \( A^0 \) is an anti S-fuzzy subhemiring of the hemiring \( R \).

Proof: For any \( x \in R \), we have
\[ A^0(x+y) = A(x+y)A(0) \leq [A(0)] S( A(x), A(y) ) \]
\[ \leq S( [A(x)A(0)], [A(y)A(0)] ) = S( A^0(x), A^0(y) ). \]
That is \( A^0(x+y) \leq S( A^0(x), A^0(y) ), \) for all \( x, y \in R \).
Similarly, \( A^0(xy) = A(xy)A(0) \leq [A(0)] S (A(x), A(y)) \)
\[ \leq S ([A(x)A(0)], [A(y)A(0)]) = S (A^0(x), A^0(y)). \]
That is \( A^0(xy) \leq S (A^0(x), A^0(y)), \) for all \( x, y \in R. \)

Hence \( A^0 \) is an anti S-fuzzy subhemiring of the hemiring \( R. \)

2.3 LOWER LEVEL SUBHEMIRINGS OF ANTI S-FUZZY SUBHEMIRING OF A HEMIRING

2.3.1 Theorem: Let \( A \) be an anti S-fuzzy subhemiring of a hemiring \( R. \) Then for \( \alpha \) in \([0,1]\) such that \( \mu_A(0) \leq \alpha, A_\alpha \) is a lower level subhemiring of \( R. \)

Proof: For all \( x \) and \( y \) in \( A_\alpha \), we have, \( \mu_A(x) \leq \alpha \) and \( \mu_A(y) \leq \alpha. \)

Now, \( \mu_A(x+y) \leq S(\mu_A(x), \mu_A(y)) \leq S(\alpha, \alpha) = \alpha, \) which implies that \( \mu_A(x+y) \leq \alpha. \)

And, \( \mu_A(xy) \leq S (\mu_A(x), \mu_A(y)) \leq S(\alpha, \alpha) = \alpha, \) which implies that \( \mu_A(xy) \leq \alpha. \)

Therefore, \( \mu_A(x+y) \leq \alpha \) and \( \mu_A(xy) \leq \alpha. \) Therefore, \( x + y \) and \( xy \in A_\alpha. \)

Hence \( A_\alpha \) is a lower level subhemiring of a hemiring \( R. \)

2.3.2 Theorem: Let \( A \) be an anti S-fuzzy subhemiring of a hemiring \( R. \) Then two lower level subhemiring \( A_{\alpha_1}, A_{\alpha_2} \) and \( \alpha_1, \alpha_2 \) are in \([0,1]\) such that \( \mu_A(0) \leq \alpha_1, \mu_A(0) \leq \alpha_2 \) with \( \alpha_1 < \alpha_2 \) of \( A \) are equal if and only if there is no \( x \in R \) such that \( \alpha_2 > \mu_A(x) > \alpha_1. \)

Proof: Assume that \( A_{\alpha_1} = A_{\alpha_2}. \) Suppose there exists \( x \) in \( R \) such that \( \alpha_2 > \mu_A(x) > \alpha_1. \) Then \( A_{\alpha_1} \subseteq A_{\alpha_2} \) implies \( x \) belongs to \( A_{\alpha_2}, \) but not in \( A_{\alpha_1}. \) This is contradiction to \( A_{\alpha_1} = A_{\alpha_2}. \) Therefore there is no \( x \in R \) such that \( \alpha_2 > \mu_A(x) > \alpha_1. \) Conversely if there is no \( x \in R \) such that \( \alpha_2 > \mu_A(x) > \alpha_1. \) Then \( A_{\alpha_1} = A_{\alpha_2}. \) (by the definition of lower level set ).
2.3.3 Theorem: Let R be a hemiring and A be a fuzzy subset of R such that $A_\alpha$ be a subhemiring of R. If $\alpha$ in $[0,1]$, then A is an anti S-fuzzy subhemiring of R.

Proof: The argument is trivial.

2.3.4 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R. If any two lower level subhemirings of A belongs to R, then their intersection is also lower level subhemiring of $A \in R$.

Proof: Let $\alpha_1, \alpha_2 \in [0,1]$.

Case (i): If $\alpha_1 < \mu_A(x) < \alpha_2$, then $A_{\alpha_1} \subseteq A_{\alpha_2}$.

Therefore, $A_{\alpha_1} \cap A_{\alpha_2} = A_{\alpha_1}$, but $A_{\alpha_1}$ is a lower level subhemiring of A.

Case (ii): If $\alpha_1 > \mu_A(x) > \alpha_2$, then $A_{\alpha_2} \subseteq A_{\alpha_1}$.

Therefore, $A_{\alpha_1} \cap A_{\alpha_2} = A_{\alpha_2}$, but $A_{\alpha_2}$ is a lower level subhemiring of A.

Case (iii): If $\alpha_1 = \alpha_2$, then $A_{\alpha_1} = A_{\alpha_2}$.

In all cases, intersection of any two lower level subhemirings is a lower level subhemiring of A. \hspace{1cm} \Box

2.3.5 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R. If $\alpha_i \in [0,1]$ and $A_{\alpha_i}, i \in I$ is a collection of lower level subhemirings of A, then their intersection is also a lower level subhemiring of A.

Proof: The argument is trivial.

2.3.6 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R. If any two lower level subhemirings of A belongs to R, then their union is also a lower level subhemiring of $A \in R$.

Proof: Let $\alpha_1, \alpha_2 \in [0,1]$.
Case (i): If $\alpha_1 < \mu_A(x) < \alpha_2$, then $A_{\alpha_1} \subseteq A_{\alpha_2}$.

Therefore, $A_{\alpha_1} \cup A_{\alpha_2} = A_{\alpha_2}$, but $A_{\alpha_2}$ is a lower level subhemiring of $A$.

Case (ii): If $\alpha_1 > \mu_A(x) > \alpha_2$, then $A_{\alpha_2} \subseteq A_{\alpha_1}$.

Therefore, $A_{\alpha_1} \cup A_{\alpha_2} = A_{\alpha_1}$, but $A_{\alpha_1}$ is a lower level subhemiring of $A$.

Case (iii): If $\alpha_1 = \alpha_2$, then $A_{\alpha_1} = A_{\alpha_2}$.

In all cases, union of any two lower level subhemiring is also a lower level subhemiring of $A$.  

2.3.7 Theorem: Let $A$ be an anti $S$-fuzzy subhemiring of a hemiring $R$. If $\alpha_i \in [0,1]$ and $A_{\alpha_i}$, $i \in I$ is a collection of lower level subhemirings of $A$, then their union is also a lower level subhemiring of $A$.

Proof: The argument is trivial.

2.3.8 Theorem: The homomorphic image of a lower level subhemiring of an anti $S$-fuzzy subhemiring of a hemiring $R$ is a lower level subhemiring of an anti $S$-fuzzy subhemiring of a hemiring $R^l$.

Proof: Let $(R, +, \cdot)$ and $(R^l, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^l$ be a homomorphism. That is, $f(x+y)=f(x)+f(y)$ and $f(xy)=f(x)f(y)$, for all $x$ and $y \in R$. Let $V = f(A)$, where $A$ is an anti $S$-fuzzy subhemiring of a hemiring $R$. Clearly $V$ is an anti $S$-fuzzy subhemiring of a hemiring $R^l$. Let $x$ and $y \in R$, implies $f(x)$ and $f(y) \in R^l$. Let $A_{\alpha}$ is a lower level subhemiring of $A$.

That is, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$; $\mu_A(x+ y) \leq \alpha$, $\mu_A(xy) \leq \alpha$.

We have to prove that $f (A_{\alpha})$ is a lower level subhemiring of $V$.

Now, $\mu_V(f(x)) \leq \mu_A(x) \leq \alpha$, which implies that $\mu_V(f(x)) \leq \alpha$; and $\mu_V( f(y) ) \leq \mu_A(y) \leq \alpha$, which implies that $\mu_V( f(y) ) \leq \alpha$ and
\( \mu_V(f(x) + f(y)) = \mu_V(f(x + y)), \) as \( f \) is a homomorphism
\[ \leq \mu_A(x+y) \leq \alpha, \] which implies that \( \mu_V(f(x) + f(y)) \leq \alpha. \)

Also, \( \mu_V(f(x)f(y)) = \mu_V(f(xy)), \) as \( f \) is a homomorphism
\[ \leq \mu_A(xy) \leq \alpha, \] which implies that \( \mu_V(f(x)f(y)) \leq \alpha. \)

Therefore, \( \mu_V(f(x) + f(y)) \leq \alpha, \mu_V(f(x)f(y)) \leq \alpha. \)

Hence \( f(A_\alpha) \) is a lower level subhemiring of an anti S-fuzzy subhemiring \( V \) of a hemiring \( R^l. \)

**2.3.9 Theorem:** The homomorphic pre-image of a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring \( R^l \) is a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring \( R. \)

**Proof:** Let \( (R, +, \cdot) \) and \( (R^l, +, \cdot) \) be any two hemirings and \( f : R \to R^l \) be a homomorphism. That is, \( f(x+y) = f(x) + f(y) \) and \( f(xy) = f(x)f(y) \) for all \( x \) and \( y \) in \( R. \) Let \( V = f(A), \) where \( V \) is an anti S-fuzzy subhemiring of a hemiring \( R^l. \) Clearly \( A \) is an anti S-fuzzy subhemiring of a hemiring \( R. \) Let \( f(x) \) and \( f(y) \) in \( R^l, \) implies \( x \) and \( y \) in \( R. \) Let \( f(A_\alpha) \) is a lower level subhemiring of \( V. \)

That is, \( \mu_V(f(x)) \leq \alpha \) and \( \mu_V(f(y)) \leq \alpha; \)
\[ \mu_V(f(x) + f(y)) \leq \alpha, \mu_V(f(x)f(y)) \leq \alpha. \]

We have to prove that \( A_\alpha \) is a lower level subhemiring of \( A. \)

Now, \( \mu_A(x) = \mu_V(f(x)) \leq \alpha, \) implies that \( \mu_A(x) \leq \alpha; \)
\[ \mu_A(y) = \mu_V(f(y)) \leq \alpha, \] implies that \( \mu_A(y) \leq \alpha \)
and \( \mu_A(x + y) = \mu_V(f(x + y)) = \mu_V(f(x) + f(y)) \leq \alpha, \)
which implies that \( \mu_A(x + y) \leq \alpha. \)
Also, \( \mu_A(xy) = \mu_V(f(xy)) = \mu_V(f(x)f(y)) \leq \alpha \), which implies that \( \mu_A(xy) \leq \alpha \).

Therefore, \( \mu_V(f(x) + f(y)) \leq \alpha \), \( \mu_V(f(x)f(y)) \leq \alpha \). Hence, \( A_\alpha \) is a lower level subhemiring of an anti S-fuzzy subhemiring \( A \) of \( R \).

2.3.10 Theorem: The anti S-homomorphic image of a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring \( R \) is a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring \( R^l \).

Proof: Let \( (R, +, \cdot) \) and \( (R^l, +, \cdot) \) be any two hemirings and \( f : R \rightarrow R^l \) be an anti-homomorphism. That is, \( f(x+y) = f(y) + f(x) \) and \( f(xy) = f(y)f(x) \), for all \( x \) and \( y \in R \). Let \( V = f(A) \), where \( A \) is an anti S-fuzzy subhemiring of \( R \).

Clearly \( V \) is an anti S-fuzzy subhemiring of \( R^l \). Let \( x \) and \( y \in R \), implies \( f(x) \) and \( f(y) \) in \( R^l \). Let \( A_\alpha \) is a lower level subhemiring of \( A \).

That is, \( \mu_A(x) \leq \alpha \) and \( \mu_A(y) \leq \alpha \), \( \mu_A(y + x) \leq \alpha \), \( \mu_A(yx) \leq \alpha \).

We have to prove that \( f(A_\alpha) \) is a lower level subhemiring of \( V \).

Now, \( \mu_V(f(x)) \leq \mu_A(x) \leq \alpha \), which implies that \( \mu_V(f(x)) \leq \alpha \);

\[ \mu_V(f(y)) \leq \mu_A(y) \leq \alpha \], which implies that \( \mu_V(f(y)) \leq \alpha \).

Now, \( \mu_V(f(x) + f(y)) = \mu_V(f(y + x)) \), as \( f \) is an anti-homomorphism

\[ \leq \mu_A(y + x) \leq \alpha \], which implies that \( \mu_V(f(x) + f(y)) \leq \alpha \).

Also, \( \mu_V(f(x)f(y)) = \mu_V(f(yx)) \), as \( f \) is an anti-homomorphism

\[ \leq \mu_A(yx) \leq \alpha \], which implies that \( \mu_V(f(x)f(y)) \leq \alpha \).

Therefore, \( \mu_V(f(x) + f(y)) \leq \alpha \) and \( \mu_V(f(x)f(y)) \leq \alpha \). Hence \( f(A_\alpha) \) is a lower level subhemiring of an anti S-fuzzy subhemiring \( V \) of \( R^l \).
2.3.11 **Theorem:** The anti-homomorphic pre-image of a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring $R^l$ is a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring $R$.

**Proof:** Let $(R, +, \cdot)$ and $(R^l, +, \cdot)$ be any two hemirings and $f : R \rightarrow R^l$ be an anti-homomorphism. That is, $f(x+y) = f(y)+ f(x)$ and $f(xy) = f(y)f(x)$, for all $x$ and $y \in R$. Let $V = f(A)$, where $V$ is an anti S-fuzzy subhemiring of a hemiring $R^l$. Clearly $A$ is an anti S-fuzzy subhemiring of a hemiring $R$. Let $f(x)$ and $f(y)$ in $R^l$, implies $x$ and $y$ in $R$. Let $f(A_\alpha)$ is a lower level subhemiring of $V$. That is, $\mu_V(f(x)) \leq \alpha$ and $\mu_V(f(y)) \leq \alpha$; $\mu_V(f(y)+f(x)) \leq \alpha$, $\mu_V(f(y)f(x)) \leq \alpha$. We have to prove that $A_\alpha$ is a lower level subhemiring of $A$. Now, $\mu_A(x) = \mu_V(f(x)) \leq \alpha$, which implies that $\mu_A(x) \leq \alpha$;

$$\mu_A(y) = \mu_V(f(y)) \leq \alpha,$$

which implies that $\mu_A(y) \leq \alpha$.

Now, $\mu_A(x+y) = \mu_V(f(x+y)) = \mu_V(f(y)+f(x)) \leq \alpha$,

which implies that $\mu_A(x+y) \leq \alpha$.

Also, $\mu_A(xy) = \mu_V(f(xy)) = \mu_V(f(y)f(x)) \leq \alpha$,

which implies that $\mu_A(xy) \leq \alpha$.

Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x)f(y)) \leq \alpha$. Hence $A_\alpha$ is a lower level subhemiring of an anti S-fuzzy subhemiring $A$ of $R$. \(\square\)

2.3.12 **Theorem:** Let $(R, +, \cdot)$ be a hemiring and $A$ be a non empty subset of $R$. Then $A$ is a subhemiring of $R$ if and only if $B = <\chi_A>$ is an anti S-fuzzy subhemiring of $R$, where $\chi_A$ is the characteristic function.

**Proof:** The argument is trivial.
2.4 ANTI S-FUZZY NORMAL SUBHEMIRINGS OF A HEMIRING

2.4.1 Theorem: Let \((R, +, .)\) be a hemiring. If \(A\) and \(B\) are two anti S-fuzzy normal subhemirings of \(R\), then their union \(A \cup B\) is an anti S-fuzzy normal subhemiring of \(R\).

**Proof:** Let \(x\) and \(y \in R\). Let \(A = \{ \langle x, \mu_A(x) \rangle / x \in R \} \) and \(B = \{ \langle x, \mu_B(x) \rangle / x \in R \} \) be anti S-fuzzy normal subhemirings of a hemiring \(R\). Let \(C = A \cup B \) and \(C = \{ \langle x, \mu_C(x) \rangle / x \in R \} \), where \(\mu_C(x) = \max \{ \mu_A(x), \mu_B(x) \} \). Then, clearly \(C\) is an anti S-fuzzy subhemiring of a hemiring \(R\), since \(A\) and \(B\) are two anti S-fuzzy subhemirings of the hemiring \(R\).

And, \(\mu_C(xy) = \max \{ \mu_A(xy), \mu_B(xy) \} = \max \{ \mu_A(yx), \mu_B(yx) \} = \mu_C(yx)\), for all \(x\) and \(y \in R\). Therefore, \(\mu_C(xy) = \mu_C(yx)\), for all \(x\) and \(y \in R\).

Hence \(A \cup B\) is an anti S-fuzzy normal subhemiring of the hemiring \(R\). \(\square\)

2.4.2 Theorem: Let \((R, +, .)\) be a hemiring. The union of a family of anti S-fuzzy normal subhemirings of \(R\) is an anti S-fuzzy normal subhemiring of \(R\).

**Proof:** The argument is trivial.

2.4.3 Theorem: Let \(A\) and \(B\) be anti S-fuzzy subhemirings of the hemirings \(G\) and \(H\), respectively. If \(A\) and \(B\) are anti S-fuzzy normal subhemirings, then \(A \times B\) is an anti S-fuzzy normal subhemiring of \(G \times H\).
**Proof:** Let $A$ and $B$ be anti $S$-fuzzy normal subhemirings of the hemirings $G$ and $H$ respectively. Clearly $A \times B$ is an anti $S$-fuzzy subhemiring of $G \times H$. Let $x_1$ and $x_2 \in G$, $y_1$ and $y_2 \in H$. Then $(x_1, y_1)$ and $(x_2, y_2) \in G \times H$.

Now, $\mu_{A \times B}[ (x_1, y_1)(x_2, y_2) ] = \mu_{A \times B}( x_1x_2, y_1y_2 )$

$$= \max \{ \mu_A(x_1x_2), \mu_B(y_1y_2) \}$$

$$= \max \{ \mu_A(x_2x_1), \mu_B(y_2y_1) \},$$

$$= \mu_{A \times B}( x_2x_1, y_2y_1 )$$

$$= \mu_{A \times B}[ (x_2, y_2)(x_1, y_1) ].$$

Therefore, $\mu_{A \times B}[ (x_1, y_1)(x_2, y_2) ] = \mu_{A \times B}[ (x_2, y_2)(x_1, y_1) ].$

Hence $A \times B$ is an anti $S$-fuzzy normal subhemiring of $G \times H$. 

2.4.4 Theorem: Let $A$ and $B$ be anti $S$-fuzzy normal subhemiring of the hemirings $R_1$ and $R_2$ respectively. Suppose that $0_1$ and $0_2$ are the zero element of $R_1$ and $R_2$ respectively. If $A \times B$ is an anti $S$-fuzzy normal subhemiring of $R_1 \times R_2$, then at least one of the following two statements must hold.

(i) $\mu_B(0_1) \leq \mu_A(x)$, for all $x \in R_1$,

(ii) $\mu_A(0) \leq \mu_B(y)$, for all $y \in R_2$.

**Proof:** The argument is trivial.

2.4.5 Theorem: Let $A$ and $B$ be two fuzzy subsets of the hemirings $R_1$ and $R_2$ respectively and $A \times B$ is an anti $S$-fuzzy normal subhemiring of $R_1 \times R_2$. Then the following are true:

(i) if $\mu_A(x) \geq \mu_B(0_1)$, then $A$ is an anti $S$-fuzzy normal subhemiring of $R_1$.

(ii) if $\mu_B(x) \geq \mu_A(0)$, then $B$ is an anti $S$-fuzzy normal subhemiring of $R_2$. 

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(iii) either A is an anti S-fuzzy normal subhemiring of R₁ or B is an anti S-fuzzy normal subhemiring of R₂.

**Proof:** The argument is trivial.

**2.4.6 Theorem:** Let A be a fuzzy subset in a hemiring R and V be the anti-strongest fuzzy relation on R. Then A is an anti S-fuzzy normal subhemiring of R if and only if V is an anti S-fuzzy normal subhemiring of R×R.

**Proof:** The argument is trivial.

**2.4.7 Theorem:** Let (R, +, .) and (R¹, +, .) be any two hemirings. The homomorphic image of an anti S-fuzzy normal subhemiring of R is an anti S-fuzzy normal subhemiring of R¹.

**Proof:** Let (R, +, .) and (R¹, +, .) be any two hemirings and f : R → R¹ be a homomorphism. Then, f(x+y) = f(x)+f(y) and f(xy) = f(x)f(y), for all x and y ∈ R. Let V = f(A), where A is an anti S-fuzzy normal subhemiring of a hemiring R. We have to prove that V is an anti S-fuzzy normal subhemiring of a hemiring R¹. Now, for f(x), f(y) in R¹, clearly V is an anti S-fuzzy subhemiring of a hemiring R¹, since A is an anti S-fuzzy subhemiring of a hemiring R. Now, μᵥ(f(x)f(y)) = μᵥ(f(xy)) ≤ μₐ(xy) = μᵥ(xy) ≥ μᵥ(f(yx))

= μᵥ(f(y)f(x)), which implies that μᵥ(f(x)f(y)) = μᵥ(f(y)f(x)), for all f(x) and f(y) in R¹. Hence V is an anti S-fuzzy normal subhemiring of a hemiring R¹. □

**2.4.8 Theorem:** Let (R, +, .) and (R¹, +, .) be any two hemirings. The homomorphic preimage of an anti S-fuzzy normal subhemiring of R¹ is an anti S-fuzzy normal subhemiring of R.
Proof: Let \(( R, +, \cdot \) and \(( R^l, +, \cdot \) be any two hemirings and \( f : R \rightarrow R^l \) be a homomorphism. Then, \( f(x+y) = f(x)+f(y) \) and \( f(xy) = f(x) f(y) \), for all \( x \) and \( y \in R \). Let \( V = f(A) \), where \( V \) is an anti S-fuzzy normal subhemiring of a hemiring \( R^l \). We have to prove that \( A \) is an anti S-fuzzy normal subhemiring of a hemiring \( R \). Let \( x \) and \( y \in R \). Then, clearly \( A \) is an anti S-fuzzy subhemiring of a hemiring \( R \), since \( V \) is an anti S-fuzzy subhemiring of a hemiring \( R \). Now, \( \mu_A(xy) = \mu_v( f(xy) ) = \mu_v( f(x) f(y)) = \mu_v( f(y) f(x)) \\
= \mu_v(f(yx)) = \mu_A(yx) \), which implies that \( \mu_A(xy) = \mu_A(yx) \), for all \( x \) and \( y \) in \( R \). Hence \( A \) is an anti S-fuzzy normal subhemiring of a hemiring \( R \). \( \Box \)

2.4.9 Theorem: Let \(( R, +, \cdot \) and \(( R^l, +, \cdot \) be any two hemirings. The anti-homomorphic image of an anti S-fuzzy normal subhemiring of \( R \) is an anti S-fuzzy normal subhemiring of \( R^l \).

Proof: Let \(( R, +, \cdot \) and \(( R^l, +, \cdot \) be any two hemirings and \( f : R \rightarrow R^l \) be an anti-homomorphism. Then, \( f(x+y) = f(y) + f(x) \) and \( f(xy) = f(y) f(x) \), for all \( x \) and \( y \) in \( R \). Let \( V = f(A) \), where \( A \) is an anti S-fuzzy normal subhemiring of a hemiring \( R \). We have to prove that \( V \) is an anti S-fuzzy normal subhemiring of a hemiring \( R^l \). Now, for \( f(x) \) and \( f(y) \) in \( R^l \), clearly \( V \) is an anti S-fuzzy subhemiring of a hemiring \( R^l \), since \( A \) is an anti S-fuzzy subhemiring of a hemiring \( R \).

Now, \( \mu_v(f(x)f(y)) = \mu_v(f(yx)) \), as \( f \) is an anti-homomorphism

\[
\begin{align*}
\leq \mu_A(yx) \\
= \mu_A(xy) \\
\geq \mu_v(f(xy))
\end{align*}
\]
= \mu_v( f(y) f(x) ), as f is an anti-homomorphism

which implies that \mu_v( f(x)f(y) = \mu_v(f(y)f(x) ), for all f(x) and f(y) \in R^1.

Hence V is an anti S-fuzzy normal subhemiring of a hemiring R^1.

\[
\square
\]

2.4.10 Theorem: Let (R, +, \cdot) and (R^1, +, \cdot) be any two hemirings. The anti-homomorphic preimage of an anti S-fuzzy normal subhemiring of R^1 is an anti S-fuzzy normal subhemiring of R.

Proof: Let (R, +, \cdot) and (R^1, +, \cdot) be any two hemirings and f : R \rightarrow R^1 be an anti-homomorphism. Then, f(x+y) = f(y) + f(x) and f(xy) = f(y)f(x), for all x and y in R. Let V = f(A), where V is an anti S-fuzzy normal subhemiring of a hemiring R^1. We have to prove that A is an anti S-fuzzy normal subhemiring of a hemiring R. Let x and y in R, then, clearly A is an anti S-fuzzy subhemiring of a hemiring R, since V is an anti S-fuzzy subhemiring of a hemiring R^1.

Now, \mu_A(xy) = \mu_v( f(xy) ), since \mu_A(x) = \mu_v( f(x) )

= \mu_v( f(y)f(x) ), as f is an anti-homomorphism

= \mu_v( f(x)f(y) )

= \mu_v( f(yx) ), as f is an anti-homomorphism

= \mu_A(yx), since \mu_A(x) = \mu_v( f(x) )

which implies that \mu_A(xy) = \mu_A(yx), for all x and y in R.

Hence A is an anti S-fuzzy normal subhemiring of a hemiring R.

\[
\square
\]
In the next theorem we introduce a new composition operation in
S- fuzzy normal subhemiring

2.4.11 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring H and
f is an isomorphism from a hemiring R onto H. If A is an anti S-fuzzy normal
subhemiring of the hemiring H, then \( A^o f \) is an anti S-fuzzy normal
subhemiring of the hemiring R.

Proof: Let \( x \) and \( y \in R \) and A be an anti S-fuzzy normal subhemiring of a
hemiring H. Then clearly \( A^o f \) is an anti S-fuzzy subhemiring of a hemiring R.
Now since f is anti-isomorphism, \( (\mu_A^o f)(xy) = \mu_A( f(xy) ) \)
\[
= \mu_A( f(x)f(y) ), \quad \text{as f is an isomorphism}
\]
\[
= \mu_A( f(y)f(x) )
\]
\[
= \mu_A( f(yx) ), \quad \text{as f is an isomorphism}
\]
\[
= (\mu_A^o f)(yx),
\] which implies that \( (\mu_A^o f)(xy) = (\mu_A^o f)(yx) \), for all \( x \) and \( y \in R \).
Hence \( A^o f \) is an anti S-fuzzy normal subhemiring of a hemiring R.

2.4.12 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring H and
f is an anti-isomorphism from a hemiring R onto H. If A is an anti S-fuzzy
normal subhemiring of the hemiring H, then \( A^o f \) is an anti S-fuzzy normal
subhemiring of the hemiring R.

Proof: Let \( x \) and \( y \) in R and A be an anti S-fuzzy normal subhemiring of a
hemiring H. Then clearly \( A^o f \) is an anti S-fuzzy subhemiring of the hemiring R.
Now since f is anti-isomorphism, \( (\mu_A^o f)(xy) = \mu_A( f(xy) ) \)
\[
= \mu_A( f(y)f(x) ),
\]
\[
\mu_A(f(x)f(y)) = \mu_A(f(yx)) = (\mu_A \circ f)(yx),
\]
which implies that \((\mu_A \circ f)(xy) = (\mu_A \circ f)(yx)\), for all \(x, y \in R\).

Hence \(A \circ f\) is an anti S-fuzzy normal subhemiring of the hemiring \(R\).

\textbf{2.4.13 Theorem:} Let \(A\) be an anti S-fuzzy normal subhemiring of a hemiring \(R\). Then for \(\alpha \in [0, 1]\) such that \(\mu_A(0) \leq \alpha\), \(A_\alpha\) is a lower level subhemiring of \(R\).

\textbf{Proof:} The argument is trivial.

\textbf{2.4.14 Theorem:} Let \(A\) be an anti S-fuzzy normal subhemiring of a hemiring \(R\), then two lower level subhemiring \(A_{\alpha_1}, A_{\alpha_2}\) and \(\alpha_1, \alpha_2\) are in \([0, 1]\) such that \(\mu_A(0) \leq \alpha_1, \mu_A(0) \leq \alpha_2\) with \(\alpha_1 < \alpha_2\) of \(A\) are equal if and only if there is no \(x\) in \(R\) such that \(\alpha_2 > \mu_A(x) > \alpha_1\).

\textbf{Proof:} The argument is trivial.

\textbf{2.4.15 Theorem:} Let \(A\) be an anti S-fuzzy normal subhemiring of a hemiring \(R\). If any two lower level subhemirings of \(A\) belongs to \(R\), then their intersection is also lower level subhemiring of \(A \in R\).

\textbf{Proof :} The argument is trivial.

\textbf{2.4.16 Theorem:} Let \(A\) be an anti S-fuzzy normal subhemiring of a hemiring \(R\). If \(\alpha \in [0, 1]\), and \(A_{\alpha_i}, i \in I\) is a collection of lower level subhemirings of \(A\), then their intersection is also a lower level subhemiring of \(A\).

\textbf{Proof:} The argument is trivial.
2.4.17 Theorem: Let $A$ be an anti $S$-fuzzy normal subhemiring of a hemiring $R$. If any two lower level subhemirings of $A$ belongs to $R$, then their union is also a lower level subhemiring of $A \in R$.

**Proof:** The argument is trivial.

2.4.18 Theorem: Let $A$ be an anti $S$-fuzzy normal subhemiring of a hemiring $R$. If $\alpha_i \in [0,1]$ and $A_{\alpha_i}, i \in I$ is a collection of lower level subhemirings of $A$, then their union is also a lower level subhemiring of $A$.

**Proof:** The argument is trivial.

2.4.19 Theorem: The homomorphic image of a lower level subhemiring of an anti $S$-fuzzy normal subhemiring of a hemiring $R$ is a lower level subhemiring of an anti $S$-fuzzy normal subhemiring of $R^l$.

**Proof:** The argument is trivial.

2.4.20 Theorem: The homomorphic pre-image of a lower level subhemiring of an anti $S$-fuzzy normal subhemiring of a hemiring $R^l$ is a lower level subhemiring of an anti $S$-fuzzy normal subhemiring of $R$.

**Proof:** The argument is trivial.

2.4.21 Theorem: The anti-homomorphic image of a lower level subhemiring of an anti $S$-fuzzy normal subhemiring of a hemiring $R$ is a lower level subhemiring of an anti $S$-fuzzy normal subhemiring of $R^l$.

**Proof:** The argument is trivial.

2.4.22 Theorem: The anti-homomorphic pre-image of a lower level subhemiring of an anti $S$-fuzzy normal subhemiring of a hemiring $R^l$ is a lower level subhemiring of an anti $S$-fuzzy normal subhemiring of $R$.

**Proof:** The argument is trivial.