CHAPTER - 2

ANTI S-FUZZY SUBHEMIRINGS OF A HEMIRING

2.1 Introduction: In this chapter, we introduce the concept of anti S-fuzzy subhemirings of a hemiring and establish some results on these. We also made an attempt to study the properties of anti S-fuzzy subhemirings of hemiring under homomorphism and anti-homomorphism.

2.1.1 Definition: A S-norm is a binary operation S: $[0, 1]\times[0, 1] \rightarrow [0, 1]$ satisfying the following requirements;

(i) $S(0, x) = x$, $S(1, x) = 1$ (boundary condition)

(ii) $S(x, y) = S(y, x)$ (commutativity)

(iii) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity)

(iv) if $x \leq y$ and $w \leq z$, then $S(x, w) \leq S(y, z)$ (monotonicity).

2.1.2 Definition: Let (R , + , .) be a hemiring. A fuzzy subset A of R is said to be an anti S-fuzzy subhemiring (anti fuzzy subhemiring with respect to S-norm) of R if it satisfies the following conditions:

(i) $\mu_A(x + y) \leq S(\mu_A(x), \mu_A(y))$,

(ii) $\mu_A(xy) \leq S(\mu_A(x), \mu_A(y))$, for all x and $y \in R$.

2.1.3 Definition: Let (R , + , .) be a hemiring. An anti S-fuzzy subhemiring A of R is said to be an anti S-fuzzy normal subhemiring (ASFNSHR) of R if $\mu_A(xy) = \mu_A(yx)$, for all x and $y \in R$.

2.1.4 Definition: Let A and B be fuzzy subsets of sets G and H, respectively. The anti-product of A and B, denoted by A×B, is defined as A×B ={⟨ (x, y), $\mu_{A\times B}$(x,y) ⟩ / for all x∈G and y∈H }, where $\mu_{A\times B}(x, y) = \max\{ \mu_A(x), \mu_B(y) \}$.

2.1.5 Definition: Let A be a fuzzy subset in a set S, the anti-strongest fuzzy relation on S, that is a fuzzy relation on A is V given by $\mu_V(x, y) = \max\{ \mu_A(x), \mu_A(y) \}$, for all x and y ∈ S.

2.1.6 Definition: An anti S-fuzzy subhemiring A of a hemiring R is called an anti S-fuzzy characteristic subhemiring of R if $\mu_A(x) = \mu_A(f(x))$, for all x ∈ R and f in Aut (R).

2.1.7 Definition: Let R and R^{l} be any two hemirings. Let f : R→R^{l} be any function and A be an anti S-fuzzy subhemiring in R, V be an anti S-fuzzy subhemiring in f (R) = R^{l}, defined by $\mu_V(y) = \inf_{x\in f^{-1}(y)} \mu_A(x)$, for all x∈R, y∈$R^{l}$.

Then A is called a preimage of V under f and is denoted by $f^{-1}(V)$.

Note: This definition is used throughout this chapter for image and preimage in functions.

2.1.8 Definition: Let A be an anti S-fuzzy subhemiring of a hemiring (R, +, .) and a in R. Then the pseudo anti S-fuzzy coset $(aA)^p$ is defined by $((a\mu_A)^p)(x) = p(a)\mu_A(x)$, for every x in R and for some p ∈ P.

2.1.9 Definition: Let A be a fuzzy subset of X. For α in [0, 1], the lower level subset of A is the set $A_\alpha = \{ x\in X : \mu_A(x) \le \alpha\}$.

2.1.10 Definition: Let A be an anti S-fuzzy subhemiring of a hemiring R. Then A^0 is defined as $A^0(x) = A(x)A(0)$, for all x in R, where $A(0) \ne 0$.

2.2 - PROPERTIES OF ANTI S-FUZZY SUBHEMIRING OF A HEMIRING

2.2.1 Theorem: Union of any two anti S-fuzzy subhemiring of a hemiring R is an anti S-fuzzy subhemiring of R.

Proof: Let A and B be any two anti S-fuzzy subhemirings of a hemiring R and x and y in R. Let $A = \{ (x, \mu_A(x)) / x \in R \}$ and $B = \{ (x, \mu_B(x)) / x \in R \}$ and also let $C = A \cup B = \{(x, \mu_C(x))/x \in R \}$, where $\max\{ \mu_A(x), \mu_B(x) \} = \mu_C(x)$.

Now, $\mu_C(x + y) = \max \{ \mu_A(x + y), \mu_B(x + y) \}$

$\leq \max \{ S (\mu_A(x), \mu_A(y)), S (\mu_B(x), \mu_B(y)) \}$

$\leq S (S (\mu_A(x), \mu_B(x)), S (\mu_A(y), \mu_B(y)))$

$= S (\mu_C(x), \mu_C(y))$.

Therefore, $\mu_C(x + y) \leq S (\mu_C(x), \mu_C(y))$, for all x and $y \in R$.

And, $\mu_C(xy) = \max \{ \mu_A(xy), \mu_B(xy) \}$

$\leq \max \{ S (\mu_A(x), \mu_A(y)), S (\mu_B(x), \mu_B(y)))$

$\leq S (S (\mu_A(x), \mu_B(x)), S (\mu_A(y), \mu_B(y)))$

$= S (\mu_C(x), \mu_C(y))$.

Therefore, $\mu_C(xy) \leq S (\mu_C(x), \mu_C(y))$, for all x and $y \in R$.

Therefore C is an anti S-fuzzy subhemiring of a hemiring R.

Hence the union of any two anti S-fuzzy subhemirings of a hemiring R is an anti S-fuzzy subhemiring of R. □

2.2.2 Theorem: The union of a family of anti S-fuzzy subhemirings of hemiring R is an anti S-fuzzy subhemiring of R.

Proof: The argument is trivial.

2.2.3 Theorem: If A and B are any two anti S-fuzzy subhemirings of the hemirings R_1 and R_2 respectively, then anti-product $A\times B$ is an anti S-fuzzy subhemiring of $R_1\times R_2$.

Proof: Let A and B be two anti S-fuzzy subhemirings of the hemirings R_1 and R_2 respectively. Let x_1 and x_2 be in R_1, y_1 and y_2 be $\in R_2$.

Then (x_1, y_1) and (x_2, y_2) are in $R_1\times R_2$.

Now, $\mu_{A\times B} [(x_1, y_1) + (x_2, y_2)] = \mu_{A\times B} (x_1 + x_2, y_1+ y_2)$

$= \max \{ \mu_A(x_1+ x_2), \mu_B(y_1+ y_2) \}$

$\leq \max \{ S (\mu_A(x_1), \mu_A(x_2)), S (\mu_B(y_1), \mu_B(y_2)) \}$

$\leq S (S (\mu_A(x_1), \mu_B(y_1)), S (\mu_A(x_2), \mu_B(y_2)))$

$= S (\mu_{AxB} (x_1, y_1), \mu_{AxB} (x_2, y_2))$.

Therefore, $\mu_{A\times B} [(x_1, y_1) + (x_2, y_2)] \leq S (\mu_{A\times B} (x_1, y_1), \mu_{A\times B} (x_2, y_2))$.

Also, $\mu_{A\times B} [(x_1, y_1)(x_2, y_2)] = \mu_{A\times B}(x_1x_2, y_1y_2)$

$= \max \{ \mu_A(x_1x_2), \mu_B(y_1y_2) \}$

$\leq \max \{ S (\mu_A(x_1), \mu_A(x_2)), S (\mu_B(y_1), \mu_B(y_2)) \}$

$\leq S (S (\mu_A(x_1), \mu_B(y_1)), S (\mu_A(x_2), \mu_B(y_2)) \}$

$= S (\mu_{A\times B} (x_1, y_1), \mu_{A\times B} (x_2, y_2))$.

Therefore, $\mu_{A\times B} [(x_1, y_1)(x_2, y_2)] \leq S (\mu_{A\times B} (x_1, y_1), \mu_{A\times B} (x_2, y_2))$.

Hence $A\times B$ is an anti S-fuzzy subhemiring of hemiring of $R_1\times R_2$. □

2.2.4 Theorem: If A is an anti S-fuzzy subhemiring of a hemiring $(R, +, \cdot)$, then $\mu_A(x) \geq \mu_A(0)$, for $x \in R$, the zero $0 \in R$.

Proof: For $x \in R$, and 0 is the zero element of R.

Now, $\mu_A(x) = \mu_A(x+0) \leq S(\mu_A(x), \mu_A(0))$, for all $x \in R$.

So, $\mu_A(x) \geq \mu_A(0)$ is only possible.

2.2.5 Theorem: Let A and B be anti S-fuzzy subhemiring of the hemirings R_1 and R_2 respectively. Suppose that 0_1 and 0_2 are the zero elements of R_1 and R_2 respectively. If $A \times B$ is an anti S-fuzzy subhemiring of $R_1 \times R_2$, then at least one of the following two statements must hold.

(i) $\mu_B(0_2) \leq \mu_A(x)$, for all $x \in R_1$,

(ii) $\mu_A(0_1) \leq \mu_B(y)$, for all $y \in R_2$.

Proof: Let $A \times B$ be an anti S-fuzzy subhemiring of $R_1 \times R_2$.

By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find an element $a \in R_1$ and $b \in R_2$ such that $\mu_A(a) < \mu_B(0_2)$ and $\mu_B(b) < \mu_A(0_1)$.

We have, $\mu_{A\times B}(a, b) = \max\{\mu_A(a), \mu_B(b)\}$

$$< \max\{\mu_B(0_2), \mu_A(0_1)\}$$

$$= \max\{\mu_A(0_1), \mu_B(0_2)\}$$

$$= \mu_{A\times B}(0_1, 0_2).$$

Thus $A \times B$ is not an anti S-fuzzy subhemiring of $R_1 \times R_2$.

Hence either $\mu_B(0_2) \leq \mu_A(x)$, $x \in R_1$ or $\mu_A(0_1) \leq \mu_B(y)$, for all $y \in R_2$. □

2.2.6 Theorem: Let A and B be two fuzzy subsets of the hemirings R_1 and R_2 respectively. If $A \times B$ is an anti S-fuzzy subhemiring of $R_1 \times R_2$. Then the following are true:

i. if $\mu_A(x) \geq \mu_B(0_2)$, then A is an anti S-fuzzy subhemiring of R_1.

ii. if $\mu_B(x) \geq \mu_A(0_1)$, then B is an anti S-fuzzy subhemiring of R_2.

iii. either A is an anti S-fuzzy subhemiring of R_1 or B is an anti S-fuzzy subhemiring of R_2.

Proof: Let A×B be an anti S-fuzzy subhemiring of $R_1 \times R_2$ and x and y $\in R_1$ and $0_2 \in R_2$. Then $(x, 0_2)$ and $(y, 0_2) \in R_1 \times R_2$.

Now, using the property that $\mu_A(x) \geq \mu_B(0_2)$, for all $x \in R_1$.

We get, $\mu_A(x+y) = \max\{ \mu_A(x+y), \mu_B(0_2+0_2) \}$

$= \mu_{A\times B}((x+y), (0_2+0_2))$

$= \mu_{A\times B}[(x, 0_2)+(y, 0_2)]$

$\leq S (\mu_{A\times B}(x, 0_2), \mu_{A\times B}(y, 0_2))$

$= S (\max\{\mu_A(x), \mu_B(0_2) \}, \max\{ \mu_A(y), \mu_B(0_2) \})$

$= S (\mu_A(x), \mu_A (y))$.

Therefore, $\mu_A(x+y) \leq S (\mu_A(x), \mu_A(y))$, for all x and $y \in R_1$.

Also, $\mu_A(xy) = \max\{\mu_A(xy), \mu_B(0_2 0_2) \}$

$= \mu_{A\times B}((xy), (0_2 0_2))$

$= \mu_{A\times B}[(x, 0_2)(y, 0_2)]$

$\leq S (\mu_{A\times B}(x, 0_2), \mu_{A\times B}(y, 0_2))$

$= S (\max\{\mu_A(x), \mu_B(0_2) \}, \max\{\mu_A(y), \mu_B(0_2) \})$

$= S (\mu_A(x), \mu_A(y))$.

Therefore, $\mu_A(xy) \leq S (\mu_A(x), \mu_A (y))$, for all x and $y \in R_1$.

Hence A is an anti S-fuzzy subhemiring of R_1. Thus (i) is proved.

Now, $\mu_B(x) \geq \mu_A(0_1)$, for all $x \in R_2$, let x and y in R_2 and 0_1 in R_1. Then $(0_1, x)$ and $(0_1, y)$ are $\in R_1 \times R_2$.

We get, $\mu_B(x+y) = \max\{ \mu_B(x+y), \mu_A(0_1+0_1)\}$

$= \max\{ \mu_A(0_1+0_1), \mu_B(x+y) \}$

$= \mu_{A\times B}((0_1+0_1), (x+y))$

$= \mu_{A\times B}[(0_1, x) + (0_1, y)]$

$\leq S (\mu_{A\times B}(0_1, x), \mu_{A\times B}(0_1, y))$

$= S (\max\{\mu_A(0_1), \mu_B(x) \}, \max\{\mu_A(0_1), \mu_B(y) \})$

$= S (\mu_B(x), \mu_B(y))$.

Therefore, $\mu_B(x+y) \leq S (\mu_B(x), \mu_B(y))$, for all x and y $\in R_2$.

Also, $\mu_B(xy) = \max\{ \mu_B(xy), \mu_A(0_1 0_1)\}$

$= \max\{\mu_A(0_1 0_1), \mu_B(xy)\}$

$= \mu_{A\times B}((0_1 0_1), (xy))$

$= \mu_{A\times B}[(0_1, x)(0_1, y)]$

$\leq S (\mu_{A\times B}(0_1, x), \mu_{A\times B}(0_1, y))$

$= S (\max\{ \mu_A(0_1), \mu_B(x) \}, \max\{ \mu_A(0_1), \mu_B(y)\})$

$= S (\mu_B(x), \mu_B(y))$.

Therefore, $\mu_B(xy) \leq S (\mu_B(x), \mu_B(y))$, for all x and y $\in R_2$.

Hence B is an anti S-fuzzy subhemiring of a hemiring R_2.

Thus (ii) is proved. (iii) is clear. □

2.2.7 Theorem: Let A be a fuzzy subset of a hemiring R and V be the anti-strongest fuzzy relation of R. Then A is an anti S-fuzzy subhemiring of R if and only if V is an anti S-fuzzy subhemiring of R×R.

Proof: Suppose that A is an anti S-fuzzy subhemiring of a hemiring R.

Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in R\times R$,.

we have, $\mu_V(x+y) = \mu_V[(x_1, x_2) + (y_1, y_2)]$

$= \mu_V(x_1+y_1, x_2+y_2)$

$= \max\{\mu_A(x_1+y_1), \mu_A(x_2+y_2)\}$

$\leq \max\{S(\mu_A(x_1), \mu_A(y_1)), S(\mu_A(x_2), \mu_A(y_2))\}$

$\leq S(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\})$

$= S(\mu_V(x_1, x_2), \mu_V(y_1, y_2))$

$= S(\mu_V(x), \mu_V(y))$.

Therefore, $\mu_V(x+y) \leq S(\mu_V(x), \mu_V(y))$, for all x and $y \in R\times R$.

Also, $\mu_V(xy) = \mu_V[(x_1, x_2)(y_1, y_2)]$

$= \mu_V(x_1y_1, x_2y_2)$

$= \max\{\mu_A(x_1y_1), \mu_A(x_2y_2)\}$

$\leq \max\{S(\mu_A(x_1), \mu_A(y_1)), S(\mu_A(x_2), \mu_A(y_2))\}$

$\leq S(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\})$

$= S(\mu_V(x_1, x_2), \mu_V(y_1, y_2))$

$= S(\mu_V(x), \mu_V(y))$.

Therefore, $\mu_V(xy) \leq S(\mu_V(x), \mu_V(y))$, for all x and $\mathbf{y} \in R\times R$.

This proves that V is an anti S-fuzzy subhemiring of R×R.

Conversely assume that V is an anti S-fuzzy subhemiring of R×R, then for any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in R\times R$, we have

$\max\{\mu_A(x_1+y_1), \mu_A(x_2+y_2)\} = \mu_V(x_1+y_1, x_2+y_2)$

$= \mu_V [(x_1, x_2) + (y_1, y_2)] = \mu_V (x+ y)$

$\leq S (\mu_V (x), \mu_V (y)) = S (\mu_V (x_1, x_2) , \mu_V (y_1, y_2))$

$= S (\max\{ \mu_A(x_1), \mu_A(x_2) \}, \max\{ \mu_A(y_1), \mu_A(y_2) \})$.

If $x_2 = 0$, $y_2 = 0$, we get, $\mu_A(x_1+y_1) \leq S (\mu_A(x_1), \mu_A(y_1))$, for all x_1 and $y_1 \in R$.

and, $\max \{ \mu_A(x_1y_1), \mu_A(x_2y_2) \}= \mu_V(x_1y_1 , x_2y_2)$

$= \mu_V[(x_1, x_2) (y_1, y_2)] = \mu_V (x y)$

$\leq S (\mu_V (x), \mu_V (y)) = S (\mu_V (x_1, x_2), \mu_V (y_1, y_2))$

$= S (\max \{ \mu_A(x_1), \mu_A(x_2) \}, \max \{ \mu_A(y_1), \mu_A(y_2) \})$.

Therefore A is an anti S-fuzzy subhemiring of R. □

2.2.8 Theorem: If A is an anti S-fuzzy subhemiring of a hemiring (R, +, .), then H = { x / x∈R: $\mu_A(x) = 0$} is either empty or is a subhemiring of R.

Proof: The argument is trivial.

2.2.9 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring (R, +, .). If $\mu_A(x+y) = 1$, then either $\mu_A(x) =1$ or $\mu_A(y) = 1$, for all x and y ∈ R.

Proof: The argument is trivial.

In the next theorem, we use composition operation in S- fuzzy subhemiring

2.2.10 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring H and f is an isomorphism from a hemiring R onto H. Then A◦f is an anti S-fuzzy subhemiring of R.

Proof: Let x, y in R and A be an anti S-fuzzy subhemiring of a hemiring H.

Then we have, $(\mu_A \circ f)(x+y) = \mu_A (f(x+y)) = \mu_A(f(x)+ f(y))$

$\leq S (\mu_A(f(x)), \mu_A(f(y))) \leq S ((\mu_A \circ f)(x), (\mu_A \circ f)(y))$, which implies that $(\mu_A \circ f)(x+y) \leq S((\mu_A \circ f)(x), (\mu_A \circ f)(y))$.

And, $(\mu_A \circ f)(xy) = \mu_A(f(xy)) = \mu_A(f(x)f(y))$

$$\leq S(\mu_A(f(x)), \mu_A(f(y))) \leq S((\mu_A \circ f)(x), (\mu_A \circ f)(y)),$$

which implies that $(\mu_A \circ f)(xy) \leq S((\mu_A \circ f)(x), (\mu_A \circ f)(y))$.

Therefore $(A \circ f)$ is an anti S-fuzzy subhemiring of a hemiring R. □

2.2.11 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring H and f is an anti S-isomorphism from a hemiring R onto H. Then $A \circ f$ is an anti S-fuzzy subhemiring of R.

Proof: Let x, y in R and A be an anti S-fuzzy subhemiring of a hemiring H.

Then we have, $(\mu_A \circ f)(x+y) = \mu_A(f(x+y))$

$$= \mu_A(f(y)+f(x)), \text{ as f is an anti-isomorphism}$$

$$\leq S(\mu_A(f(x)), \mu_A(f(y))),$$

$$\leq S((\mu_A \circ f)(x), (\mu_A \circ f)(y)),$$

which implies that $(\mu_A \circ f)(x+y) \leq S((\mu_A \circ f)(x), (\mu_A \circ f)(y))$.

Now, $(\mu_A \circ f)(xy) = \mu_A(f(xy)) = \mu_A(f(y)f(x))$

$$\leq S(\mu_A(f(x)), \mu_A(f(y))),$$

$$\leq S((\mu_A \circ f)(x), (\mu_A \circ f)(y)),$$

which implies that $(\mu_A \circ f)(xy) \leq S((\mu_A \circ f)(x), (\mu_A \circ f)(y))$.

Therefore $A \circ f$ is an anti S-fuzzy subhemiring of a hemiring R. □

2.2.12 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring (R, +, .), then the pseudo anti S-fuzzy coset $(aA)^p$ is an anti S-fuzzy subhemiring of a hemiring R, for every a in R.

Proof: Let A be an anti S-fuzzy subhemiring of a hemiring R.

For every x and y in R, we have, $((a\mu_A)^p)(x+y) = p(a)\mu_A(x+y)$

$$\leq p(a)\, S(\mu_A(x), \mu_A(y)) = S(p(a)\mu_A(x), p(a)\mu_A(y))$$

$$= S(((a\mu_A)^p)(x), ((a\mu_A)^p)(y)).$$

Therefore, $((a\mu_A)^p)(x+y) \leq S(((a\mu_A)^p)(x), ((a\mu_A)^p)(y))$.

Now, $((a\mu_A)^p)(xy) = p(a)\mu_A(xy) \leq p(a)\, S(\mu_A(x), \mu_A(y))$

$$= S(p(a)\mu_A(x), p(a)\mu_A(y))$$

$$= S(((a\mu_A)^p)(x), ((a\mu_A)^p)(y)).$$

Therefore, $((a\mu_A)^p)(xy) \leq S(((a\mu_A)^p)(x), ((a\mu_A)^p)(y))$.

Hence $(aA)^p$ is an anti S-fuzzy subhemiring of a hemiring R. □

2.2.13 Theorem: Let (R, + , .) and (R^l, +, .) be any two hemirings. The homomorphic image of an anti S-fuzzy subhemiring of R is an anti S-fuzzy subhemiring of R^l.

Proof: Let (R, +, .) and (R^l, +, .) be any two hemirings. Let $f : R \rightarrow R^l$ be a homomorphism. Then, f(x+y) = f(x) + f(y) and f(xy) = f(x)f(y), for all x and y in R. Let V = f(A), where A is an anti S-fuzzy subhemiring of R. We have to prove that V is an anti S-fuzzy subhemiring of R^l. Now, for f(x), f(y) in R^l, $\mu_v(f(x) + f(y)) = \mu_v(f(x+y))$, as f is a homomorphism

$$\leq \mu_A(x+y) \leq S(\mu_A(x), \mu_A(y)),$$

which implies that $\mu_v(f(x) + f(y)) \leq S(\mu_v(f(x)), \mu_v(f(y)))$.

Again, $\mu_v(f(x)f(y)) = \mu_v(f(xy))$, as f is a homomorphism

$$\leq \mu_A(xy) \leq S(\mu_A(x), \mu_A(y)),$$

which implies that $\mu_v(f(x)f(y)) \leq S (\mu_v(f(x)), \mu_v(f(y)))$.

Hence V is an anti S-fuzzy subhemiring of R^l. □

2.2.14 Theorem: Let (R, +, .) and (R^l, +, .) be any two hemirings. The homomorphic preimage of an anti S-fuzzy subhemiring of R^l is an anti S-fuzzy subhemiring of R.

Proof: Let (R, +, .) and (R^l, +, .) be any two hemirings. Let $f : R \to R^l$ be a homomorphism. Then, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and $y \in R$. Let $V = f(A)$, where V is an anti S-fuzzy subhemiring of R^l. We have to prove that A is an anti S-fuzzy subhemiring of R. Let x and $y \in R$.

Then, $\mu_A(x+ y) = \mu_v(f(x + y))$, since $\mu_v(f(x)) = \mu_A(x)$

$= \mu_v(f(x) + f(y))$, as f is a homomorphism

$\leq S (\mu_v(f(x)), \mu_v(f(y)))$

$= S (\mu_A(x), \mu_A(y))$, since $\mu_v(f(x)) = \mu_A(x)$

which implies that $\mu_A(x+ y) \leq S (\mu_A(x), \mu_A(y))$.

Again, $\mu_A(xy) = \mu_v(f(xy))$, since $\mu_v(f(x)) = \mu_A(x)$

$= \mu_v(f(x)f(y))$, as f is a homomorphism

$\leq S (\mu_v(f(x)), \mu_v(f(y)))$

$= S (\mu_A(x), \mu_A(y))$, since $\mu_v(f(x)) = \mu_A(x)$

which implies that $\mu_A(xy) \leq S (\mu_A(x), \mu_A(y))$.

Hence A is an anti S-fuzzy subhemiring of R. □

2.2.15 Theorem: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. The anti - homomorphic image of an anti S-fuzzy subhemiring of R is an anti S-fuzzy subhemiring of R^{l}.

Proof: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. Let $f : R \to R^{l}$ be an anti-homomorphism. Then, f(x+y) = f(y) + f(x) and f(xy) = f(y) f(x), for all x and y in R. Let V = f(A), where A is an anti S-fuzzy subhemiring of R. We have to prove that V is an anti S-fuzzy subhemiring of R^{l}.

Now, for f(x), f(y) in R^{l}, $\mu_v(f(x) + f(y)) = \mu_v(f(y + x))$

$\leq \mu_A(y+x) \leq S(\mu_A(y), \mu_A(x))$

$= S (\mu_A(x), \mu_A(y))$,

which implies that $\mu_v(f(x) + f(y)) \leq S (\mu_v(f(x)), \mu_v(f(y)))$.

Again, $\mu_v(f(x)f(y))= \mu_v(f(yx))$, as f is an anti-homomorphism

$\leq \mu_A(yx) \leq S (\mu_A(y), \mu_A(x))$

$= S (\mu_A(x), \mu_A(y))$,

which implies that $\mu_v(f(x)f(y)) \leq S (\mu_v(f(x)), \mu_v(f(y)))$.

Hence V is an anti S-fuzzy subhemiring of R^{l}. □

2.2.16 Theorem: Let (R, + , .) and (R^{l}, +, .) be any two hemirings. The anti-homomorphic preimage of an anti S-fuzzy subhemiring of R^{l} is an anti S-fuzzy subhemiring of R.

Proof: Let (R, +, .) and (R^{l}, +, .) be any two hemirings. Let $f : R \to R^{l}$ be an anti-homomorphism. Then, f(x+y) = f(y) +f(x) and f(xy) = f(y) f(x), for all x and y $\in$ R. Let V = f(A), where V is an anti S-fuzzy subhemiring of R^{l}. We have to prove that A is an anti S-fuzzy subhemiring of R. Let x and y $\in$ R.

Then, $\mu_A(x+y) = \mu_v(f(x+y))$, since $\mu_v(f(x)) = \mu_A(x)$

$= \mu_v(f(y) + f(x))$, as f is an anti-homomorphism

$\leq S(\mu_v(f(y)), \mu_v(f(x)))$

$= S(\mu_v(f(x)), \mu_v(f(y)))$

$= S(\mu_A(x), \mu_A(y))$, since $\mu_v(f(x)) = \mu_A(x)$

which implies that $\mu_A(x+y) \leq S(\mu_A(x), \mu_A(y))$.

Again, $\mu_A(xy) = \mu_v(f(xy))$, since $\mu_v(f(x)) = \mu_A(x)$

$= \mu_v(f(y)f(x))$, as f is an anti-homomorphism

$\leq S(\mu_v(f(y)), \mu_v(f(x)))$

$= S(\mu_v(f(x)), \mu_v(f(y)))$

$= S(\mu_A(x), \mu_A(y))$, since $\mu_v(f(x)) = \mu_A(x)$

which implies that $\mu_A(xy) \leq S(\mu_A(x), \mu_A(y))$.

Hence A is an anti S-fuzzy subhemiring of R. □

2.2.17 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R, A^+ be a fuzzy set in R defined by $A^+(x) = A(x)+1 - A(0)$, for all $x \in R$. Then A^+ is an anti S-fuzzy subhemiring of a hemiring R.

Proof : Let x and y in R. We have,

$A^+(x+y) = A(x+y) +1- A(0) \leq S(A(x), A(y)) + 1- A(0)$

$\leq S((A(x) +1- A(0)), (A(y) +1- A(0)))$

$= S(A^+(x), A^+(y))$.

Therefore , $A^+(x+y) \leq S(A^+(x), A^+(y))$, for all $x, y \in R$.

Similarly, $A^+(xy) = A(xy) +1- A(0) \leq S(A(x), A(y))+1- A(0)$

$\leq S ((A(x) +1- A(0)), (A(y) +1- A(0)))$

$= S (A^+(x), A^+(y)).$

Therefore, $A^+(xy) \leq S (A^+(x), A^+(y))$, for all x, y $\in$ R.

Hence A^+ is an anti S-fuzzy subhemiring of a hemiring R. □

2.2.18 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R, A^+ be a fuzzy set in R defined by $A^+(x) = A(x) +1 - A(0)$, for all x $\in$ R. Then there exists 0 in R such that A(0) = 1 if and only if $A^+(x) = A(x)$.

Proof : The argument is trivial.

2.2.19 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R, A^+ be a fuzzy set in R defined by $A^+(x) = A(x) +1 - A(0)$, for all x $\in$ R. Then there exists x $\in$ R such that $A^+(x) = 1$ if and only if x = 0.

Proof: The argument is trivial.

2.2.20 Theorem : Let A be an anti S-fuzzy subhemiring of a hemiring R, A^+ be a fuzzy set in R defined by $A^+(x) = A(x) +1 - A(0)$, for all x $\in$ R. Then $(A^+)^+ = A^+$.

Proof: Let x and y $\in$ R. We have, $(A^+)^+(x) = A^+(x) +1 - A^+(0) = \{ A(x) +1 - A(0) \}+1 - \{A(0) +1 - A(0) \}= A(x) +1 - A(0) = A^+(x)$. Hence $(A^+)^+ = A^+$.

2.2.21 Theorem : Let A be an anti S-fuzzy subhemiring of a hemiring R. Then A^0 is an anti S-fuzzy subhemiring of the hemiring R .

Proof: For any x $\in$ R, we have

$A^0(x+y) = A(x+y)A(0) \leq [A(0)]\, S (A(x), A(y))$

$\leq S ([A(x)A(0)], [A(y)A(0)]) = S (A^0(x), A^0(y)).$

That is $A^0(x+y) \leq S (A^0(x), A^0(y))$, for all x, y $\in$ R.

Similarly, $A^0(xy) = A(xy)A(0) \leq [A(0)]\ S\ (A(x), A(y))$

$$\leq S\ ([A(x)A(0)], [A(y)A(0)]) = S\ (A^0(x), A^0(y)).$$

That is $A^0(xy) \leq S\ (A^0(x), A^0(y))$, for all $x, y \in R$.

Hence A^0 is an anti S-fuzzy subhemiring of the hemiring R. □

2.3 LOWER LEVEL SUBHEMIRINGS OF ANTI S-FUZZY SUBHEMIRING OF A HEMIRING

2.3.1 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R. Then for α in [0,1] such that $\mu_A(0) \leq \alpha$, A_α is a lower level subhemiring of R.

Proof: For all x and y in A_α, we have, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$.

Now, $\mu_A(x+y) \leq S(\mu_A(x), \mu_A(y)) \leq S(\alpha, \alpha) = \alpha$, which implies that $\mu_A(x+y) \leq \alpha$.

And, $\mu_A(xy) \leq S\ (\mu_A(x), \mu_A(y)) \leq S(\alpha, \alpha) = \alpha$, which implies that $\mu_A(xy) \leq \alpha$.

Therefore, $\mu_A(x+ y) \leq \alpha$ and $\mu_A(xy) \leq \alpha$. Therefore , $x + y$ and $xy \in A_\alpha$.

Hence A_α is a lower level subhemiring of a hemiring R. □

2.3.2 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R. Then two lower level subhemiring $A_{\alpha 1}$, $A_{\alpha 2}$ and α_1, α_2 are in [0,1] such that $\mu_A(0) \leq \alpha_1$, $\mu_A(0) \leq \alpha_2$ with $\alpha_1 < \alpha_2$ of A are equal if and only if there is no $x \in R$ such that $\alpha_2 > \mu_A(x) > \alpha_1$.

Proof: Assume that $A_{\alpha 1} = A_{\alpha 2}$. Suppose there exists x in R such that $\alpha_2 > \mu_A(x) > \alpha_1$. Then $A_{\alpha 1} \subseteq A_{\alpha 2}$ implies x belongs to $A_{\alpha 2}$, but not in $A_{\alpha 1}$. This is contradiction to $A_{\alpha 1} = A_{\alpha 2}$. Therefore there is no $x \in R$ such that $\alpha_2 > \mu_A(x) > \alpha_1$. Conversely if there is no $x \in R$ such that $\alpha_2 > \mu_A(x) > \alpha_1$. Then $A_{\alpha 1} = A_{\alpha 2}$. (by the definition of lower level set). □

2.3.3 Theorem: Let R be a hemiring and A be a fuzzy subset of R such that A_α be a subhemiring of R. If α in [0,1], then A is an anti S-fuzzy subhemiring of R.

Proof: The argument is trivial.

2.3.4 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R. If any two lower level subhemirings of A belongs to R, then their intersection is also lower level subhemiring of $A \in R$.

Proof : Let $\alpha_1, \alpha_2 \in [0,1]$.

Case (i): If $\alpha_1 < \mu_A(x) < \alpha_2$, then $A_{\alpha 1} \subseteq A_{\alpha 2}$.

Therefore, $A_{\alpha 1} \cap A_{\alpha 2} = A_{\alpha 1}$, but $A_{\alpha 1}$ is a lower level subhemiring of A.

Case (ii): If $\alpha_1 > \mu_A(x) > \alpha_2$, then $A_{\alpha 2} \subseteq A_{\alpha 1}$.

Therefore, $A_{\alpha 1} \cap A_{\alpha 2} = A_{\alpha 2}$, but $A_{\alpha 2}$ is a lower level subhemiring of A.

Case (iii): If $\alpha_1 = \alpha_2$, then $A_{\alpha 1} = A_{\alpha 2}$.

In all cases, intersection of any two lower level subhemirings is a lower level subhemiring of A. □

2.3.5 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R. If $\alpha_i \in [0,1]$ and $A_{\alpha i}$, $i \in I$ is a collection of lower level subhemirings of A, then their intersection is also a lower level subhemiring of A.

Proof: The argument is trivial.

2.3.6 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R. If any two lower level subhemirings of A belongs to R, then their union is also a lower level subhemiring of $A \in R$.

Proof: Let $\alpha_1, \alpha_2 \in [0,1]$.

Case (i): If $\alpha_1 < \mu_A(x) < \alpha_2$, then $A_{\alpha 1} \subseteq A_{\alpha 2}$.

Therefore, $A_{\alpha 1} \cup A_{\alpha 2} = A_{\alpha 2}$, but $A_{\alpha 2}$ is a lower level subhemiring of A.

Case (ii): If $\alpha_1 > \mu_A(x) > \alpha_2$, then $A_{\alpha 2} \subseteq A_{\alpha 1}$.

Therefore , $A_{\alpha 1} \cup A_{\alpha 2} = A_{\alpha 1}$, but $A_{\alpha 1}$ is a lower level subhemiring of A.

Case (iii): If $\alpha_1 = \alpha_2$, then $A_{\alpha 1} = A_{\alpha 2}$.

In all cases, union of any two lower level subhemiring is also a lower level subhemiring of A. □

2.3.7 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring R. If $\alpha_i \in [0,1]$ and $A_{\alpha i}$, $i \in I$ is a collection of lower level subhemirings of A, then their union is also a lower level subhemiring of A.

Proof: The argument is trivial.

2.3.8 Theorem: The homomorphic image of a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring R is a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring R^l.

Proof: Let (R, +, .) and (R^l , +, .) be any two hemirings and $f : R \rightarrow R^l$ be a homomorphism. That is, f(x+y)=f(x)+f(y) and f(xy)=f(x)f(y), for all x and y $\in$ R. Let V = f(A), where A is an anti S-fuzzy subhemiring of a hemiring R. Clearly V is an anti S-fuzzy subhemiring of a hemiring R^l. Let x and y $\in$ R, implies f(x) and f(y) $\in R^l$. Let A_α is a lower level subhemiring of A.

That is, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$; $\mu_A(x+ y) \leq \alpha$, $\mu_A(xy) \leq \alpha$.

We have to prove that f (A_α) is a lower level subhemiring of V.

Now, $\mu_V(f(x)) \leq \mu_A(x) \leq \alpha$, which implies that $\mu_V(f(x)) \leq \alpha$;

and $\mu_V(f(y)) \leq \mu_A(y) \leq \alpha$, which implies that $\mu_V(f(y)) \leq \alpha$ and

$\mu_V(f(x) + f(y)) = \mu_V(f(x + y))$, as f is a homomorphism

$\leq \mu_A(x+y) \leq \alpha$, which implies that $\mu_V(f(x) + f(y)) \leq \alpha$.

Also, $\mu_V(f(x) f(y)) = \mu_V(f(xy))$, as f is a homomorphism

$\leq \mu_A(xy) \leq \alpha$, which implies that $\mu_V(f(x)f(y)) \leq \alpha$.

Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$, $\mu_V(f(x) f(y)) \leq \alpha$.

Hence f (A_α) is a lower level subhemiring of an anti S-fuzzy subhemiring V of a hemiring R^l. □

2.3.9 Theorem: The homomorphic pre-image of a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring R^l is a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring R.

Proof: Let (R, +, .) and (R^l , +, .) be any two hemirings and $f : R \rightarrow R^l$ be a homomorphism.That is, f(x+y)=f(x)+f(y) and f(xy) = f(x)f(y) for all x and y in R. Let V = f(A), where V is an anti S-fuzzy subhemiring of a hemiring R^l. Clearly A is an anti S-fuzzy subhemiring of a hemiring R. Let f(x) and f(y) in R^l, implies x and y in R. Let f(A_α) is a lower level subhemiring of V.

That is, $\mu_V(f(x)) \leq \alpha$ and $\mu_V(f(y)) \leq \alpha$;

$\mu_V(f(x) + f(y)) \leq \alpha$, $\mu_V(f(x) f(y)) \leq \alpha$.

We have to prove that A_α is a lower level subhemiring of A.

Now, $\mu_A(x) = \mu_V(f(x)) \leq \alpha$, implies that $\mu_A(x) \leq \alpha$;

$\mu_A(y) = \mu_V(f(y)) \leq \alpha$, implies that $\mu_A(y) \leq \alpha$

and $\mu_A(x + y) = \mu_V(f(x + y)) = \mu_V(f(x) + f(y)) \leq \alpha$,

which implies that $\mu_A(x + y) \leq \alpha$.

Also, $\mu_A(xy) = \mu_V(f(xy)) = \mu_V(f(x)f(y)) \leq \alpha$, which implies that $\mu_A(xy) \leq \alpha$.

Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$, $\mu_V(f(x)f(y)) \leq \alpha$. Hence, A_α is a lower level subhemiring of an anti S-fuzzy subhemiring A of R. □

2.3.10 Theorem: The anti S-homomorphic image of a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring R is a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring R^{I}.

Proof: Let (R, +, .) and (R^{I} , +, .) be any two hemirings and $f : R \to R^{I}$ be an anti-homomorphism. That is, $f(x+y)=f(y)+f(x)$ and $f(xy)=f(y)f(x)$, for all x and y $\in$ R. Let V = f(A), where A is an anti S-fuzzy subhemiring of R. Clearly V is an anti S-fuzzy subhemiring of R^{I}. Let x and y $\in$ R, implies f(x) and f(y) in R^{I}. Let A_α is a lower level subhemiring of A.

That is, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$, $\mu_A(y + x) \leq \alpha$, $\mu_A(yx) \leq \alpha$.

We have to prove that $f(A_\alpha)$ is a lower level subhemiring of V.

Now, $\mu_V(f(x)) \leq \mu_A(x) \leq \alpha$, which implies that $\mu_V(f(x)) \leq \alpha$;

$\mu_V(f(y)) \leq \mu_A(y) \leq \alpha$, which implies that $\mu_V(f(y)) \leq \alpha$.

Now, $\mu_V(f(x) + f(y)) = \mu_V(f(y + x))$, as f is an anti-homomorphism

$\leq \mu_A(y+x) \leq \alpha$, which implies that, $\mu_V(f(x) + f(y)) \leq \alpha$.

Also, $\mu_V(f(x)f(y)) = \mu_V(f(yx))$, as f is an anti-homomorphism

$\leq \mu_A(yx) \leq \alpha$, which implies that $\mu_V(f(x)f(y)) \leq \alpha$.

Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x)f(y)) \leq \alpha$. Hence $f(A_\alpha)$ is a lower level subhemiring of an anti S-fuzzy subhemiring V of R^{I}. □

2.3.11 Theorem: The anti-homomorphic pre-image of a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring R^l is a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring R.

Proof: Let (R, +, .) and (R^l , +, .) be any two hemirings and $f : R \to R^l$ be an anti-homomorphism. That is, f(x+y) = f(y)+ f(x) and f(xy) = f(y)f(x), for all x and y $\in$ R. Let V = f(A), where V is an anti S-fuzzy subhemiring of a hemiring R^l. Clearly A is an anti S-fuzzy subhemiring of a hemiring R. Let f(x) and f(y) in R^l, implies x and y in R. Let $f(A_\alpha)$ is a lower level subhemiring of V. That is, $\mu_V(f(x)) \leq \alpha$ and $\mu_V(f(y)) \leq \alpha$; $\mu_V(f(y)+f(x)) \leq \alpha$, $\mu_V(f(y) f(x)) \leq \alpha$. We have to prove that A_α is a lower level subhemiring of A. Now, $\mu_A(x) = \mu_V(f(x)) \leq \alpha$, which implies that $\mu_A(x) \leq \alpha$;

$\mu_A(y) = \mu_V(f(y)) \leq \alpha$, which implies that $\mu_A(y) \leq \alpha$.

Now, $\mu_A(x + y) = \mu_V(f(x + y)) = \mu_V(f(y) + f(x)) \leq \alpha$,

which implies that $\mu_A(x + y) \leq \alpha$.

Also, $\mu_A(xy) = \mu_V(f(xy))= \mu_V(f(y)f(x)) \leq \alpha$,

which implies that $\mu_A(xy) \leq \alpha$.

Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x) f(y)) \leq \alpha$. Hence A_α is a lower level subhemiring of an anti S-fuzzy subhemiring A of R. □

2.3.12 Theorem: Let (R, +, .) be a hemiring and A be a non empty subset of R. Then A is a subhemiring of R if and only if B = $<\overline{\chi_A}>$ is an anti S-fuzzy subhemiring of R, where χ_A is the characteristic function.

Proof: The argument is trivial.

2.4 ANTI S-FUZZY NORMAL SUBHEMIRINGS OF A HEMIRING

2.4.1 Theorem: Let (R, +, .) be a hemiring. If A and B are two anti S-fuzzy normal subhemirings of R, then their union $A \cup B$ is an anti S-fuzzy normal subhemiring of R.

Proof: Let x and $y \in R$. Let $A = \{ \langle x, \mu_A(x) \rangle / x \in R \}$ and $B = \{ \langle x, \mu_B(x) \rangle / x \in R \}$ be anti S-fuzzy normal subhemirings of a hemiring R. Let $C = A \cup B$ and $C = \{ \langle x, \mu_C(x) \rangle / x \in R \}$, where $\mu_C(x) = \max\{ \mu_A(x), \mu_B(x) \}$. Then, Clearly C is an anti S-fuzzy subhemiring of a hemiring R, since A and B are two anti S-fuzzy subhemirings of the hemiring R.

And, $\mu_C(xy) = \max \{\mu_A(xy), \mu_B(xy)\} = \max \{\mu_A(yx), \mu_B(yx)\} = \mu_C(yx)$, for all x and y in R. Therefore, $\mu_C(xy) = \mu_C(yx)$, for all x and $y \in R$.

Hence $A \cup B$ is an anti S-fuzzy normal subhemiring of the hemiring R. □

2.4.2 Theorem: Let (R, +, .) be a hemiring. The union of a family of anti S-fuzzy normal subhemirings of R is an anti S-fuzzy normal subhemiring of R.

Proof: The argument is trivial.

2.4.3 Theorem: Let A and B be anti S-fuzzy subhemirings of the hemirings G and H, respectively. If A and B are anti S-fuzzy normal subhemirings, then $A \times B$ is an anti S-fuzzy normal subhemiring of $G \times H$.

Proof: Let A and B be anti S-fuzzy normal subhemirings of the hemirings G and H respectively. Clearly $A\times B$ is an anti S-fuzzy subhemiring of $G\times H$. Let x_1 and $x_2 \in G$, y_1 and $y_2 \in H$. Then (x_1, y_1) and $(x_2, y_2) \in G\times H$.

$$
\begin{aligned}
\text{Now, } \mu_{A\times B} [(x_1, y_1)(x_2, y_2)] &= \mu_{A\times B} (x_1x_2, y_1y_2) \\
&= \max \{ \mu_A(x_1x_2), \mu_B(y_1y_2) \} \\
&= \max \{ \mu_A(x_2x_1), \mu_B(y_2y_1) \}, \\
&= \mu_{A\times B} (x_2x_1, y_2y_1) \\
&= \mu_{A\times B} [(x_2, y_2)(x_1, y_1)].
\end{aligned}
$$

Therefore, $\mu_{A\times B} [(x_1, y_1)(x_2, y_2)] = \mu_{A\times B} [(x_2, y_2)(x_1, y_1)]$.

Hence $A\times B$ is an anti S-fuzzy normal subhemiring of $G\times H$. □

2.4.4 Theorem: Let A and B be anti S-fuzzy normal subhemiring of the hemirings R_1 and R_2 respectively. Suppose that 0_1 and 0_2 are the zero element of R_1 and R_2 respectively. If $A\times B$ is an anti S-fuzzy normal subhemiring of $R_1\times R_2$, then at least one of the following two statements must hold.

(i) $\mu_B(0_l) \leq \mu_A(x)$, for all $x \in R_1$,

(ii) $\mu_A(0) \leq \mu_B(y)$, for all $y \in R_2$.

Proof: The argument is trivial.

2.4.5 Theorem: Let A and B be two fuzzy subsets of the hemirings R_1 and R_2 respectively and $A\times B$ is an anti S-fuzzy normal subhemiring of $R_1\times R_2$. Then the following are true:

(i) if $\mu_A(x) \geq \mu_B(0_l)$, then A is an anti S-fuzzy normal subhemiring of R_1.

(ii) if $\mu_B(x) \geq \mu_A(0)$, then B is an anti S-fuzzy normal subhemiring of R_2.

(iii) either A is an anti S-fuzzy normal subhemiring of R_1 or B is an anti S-fuzzy normal subhemiring of R_2.

Proof: The argument is trivial.

2.4.6 Theorem: Let A be a fuzzy subset in a hemiring R and V be the anti-strongest fuzzy relation on R. Then A is an anti S-fuzzy normal subhemiring of R if and only if V is an anti S-fuzzy normal subhemiring of R×R.

Proof: The argument is trivial.

2.4.7 Theorem: Let (R, +, .) and (R^{I}, +, .) be any two hemirings. The homomorphic image of an anti S-fuzzy normal subhemiring of R is an anti S-fuzzy normal subhemiring of R^{I}.

Proof: Let (R, +, .) and (R^{I}, +, .) be any two hemirings and f : R → R^{I} be a homomorphism. Then, f(x+y) = f(x)+f(y) and f(xy) = f(x)f(y), for all

x and y ∈ R. Let V = f(A), where A is an anti S-fuzzy normal subhemiring of a hemiring R. We have to prove that V is an anti S-fuzzy normal subhemiring of a hemiring R^{I}. Now, for f(x), f(y) in R^{I}, clearly V is an anti S-fuzzy subhemiring of a hemiring R^{I}, since A is an anti S-fuzzy subhemiring of a hemiring R. Now, $\mu_v(f(x)f(y)) = \mu_v(f(xy)) \leq \mu_A(xy) = \mu_A(yx) \geq \mu_v(f(yx))$

$= \mu_v(f(y) f(x))$, which implies that $\mu_v(f(x)f(y)) = \mu_v(f(y) f(x))$, for all f(x) and f(y) in R^{I}. Hence V is an anti S-fuzzy normal subhemiring of a hemiring R^{I}. □

2.4.8 Theorem: Let (R, +, .) and (R^{I}, +, .) be any two hemirings. The homomorphic preimage of an anti S-fuzzy normal subhemiring of R^{I} is an anti S-fuzzy normal subhemiring of R.

Proof: Let (R, +, .) and (R^l, +, .) be any two hemirings and f : R $\rightarrow$ R^l be a homomorphism. Then, f(x+y) = f(x)+f(y) and f(xy) = f(x) f(y), for all x and y $\in$ R. Let V = f(A), where V is an anti S-fuzzy normal subhemiring of a hemiring R^l. We have to prove that A is an anti S-fuzzy normal subhemirring of a hemiring R. Let x and y $\in$ R. Then, clearly A is an anti S-fuzzy subhemiring of a hemiring R, since V is an anti S-fuzzy subhemiring of a hemiring R^l. Now, $\mu_A(xy) = \mu_v(f(xy)) = \mu_v(f(x)f(y)) = \mu_v(f(y)f(x))$ $= \mu_v(f(yx)) = \mu_A(yx)$, which implies that $\mu_A(xy) = \mu_A(yx)$, for all x and y in R. Hence A is an anti S-fuzzy normal subhemiring of a hemiring R. □

2.4.9 Theorem: Let (R, +, .) and (R^l, +, .) be any two hemirings. The anti-homomorphic image of an anti S-fuzzy normal subhemiring of R is an anti S-fuzzy normal subhemiring of R^l.

Proof: Let (R, +, .) and (R^l, +, .) be any two hemirings and f : R $\rightarrow$ R^l be an anti-homomorphism. Then, f(x+y) = f(y) + f(x) and f(xy) = f(y) f(x), for all x and y in R. Let V = f(A), where A is an anti S-fuzzy normal subhemiring of a hemiring R. We have to prove that V is an anti S-fuzzy normal subhemiring of a hemiring R^l. Now, for f(x) and f(y) in R^l, clearly V is an anti S-fuzzy subhemiring of a hemiring R^l, since A is an anti S-fuzzy subhemiring of a hemiring R.

Now, $\mu_v(f(x)f(y)) = \mu_v(f(yx))$, as f is an anti-homomorphism

$\leq \mu_A(yx)$

$= \mu_A(xy)$

$\geq \mu_v(f(xy))$

$= \mu_v(f(y) f(x))$, as f is an anti-homomorphism

which implies that $\mu_v(f(x)f(y) = \mu_v(f(y)f(x))$, for all f(x) and f(y) $\in R^l$.

Hence V is an anti S-fuzzy normal subhemiring of a hemiring R^l. □

2.4.10 Theorem: Let (R, +, .) and (R^l, +, .) be any two hemirings. The anti-homomorphic preimage of an anti S-fuzzy normal subhemiring of R^l is an anti S-fuzzy normal subhemiring of R.

Proof: Let (R, +, .) and (R^l, +, .) be any two hemirings and $f : R \to R^l$ be an anti-homomorphism. Then, f (x+y) = f(y) + f(x) and f(xy) = f(y) f(x), for all x and y in R. Let V = f(A), where V is an anti S-fuzzy normal subhemiring of a hemiring R^l. We have to prove that A is an anti S-fuzzy normal subhemiring of a hemiring R. Let x and y in R, then, clearly A is an anti S-fuzzy subhemiring of a hemiring R, since V is an anti S-fuzzy subhemiring of a hemiring R^l.

Now, $\mu_A(xy) = \mu_v(f(xy))$, since $\mu_A(x) = \mu_v(f(x))$

$= \mu_v(f(y)f(x))$, as f is an anti-homomorphism

$= \mu_v(f(x)f(y))$

$= \mu_v(f(yx))$, as f is an anti-homomorphism

$= \mu_A(yx)$, since $\mu_A(x) = \mu_v(f(x))$

which implies that $\mu_A(xy) = \mu_A(yx)$, for all x and y in R.

Hence A is an anti S-fuzzy normal subhemiring of a hemiring R. □

In the next theorem we introduce a new composition operation in S- fuzzy normal subhemiring

2.4.11 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring H and f is an isomorphism from a hemiring R onto H. If A is an anti S-fuzzy normal subhemiring of the hemiring H, then A◦f is an anti S-fuzzy normal subhemiring of the hemiring R.

Proof: Let x and y ∈ R and A be an anti S-fuzzy normal subhemiring of a hemiring H. Then clearly A◦f is an anti S-fuzzy subhemiring of a hemiring R.

Now since f is anti-isomorphism, $(\mu_A \circ f)(xy) = \mu_A(f(xy))$

$= \mu_A(f(x)f(y))$, as f is an isomorphism

$= \mu_A(f(y)f(x))$

$= \mu_A(f(yx))$, as f is an isomorphism

$= (\mu_A \circ f)(yx)$,

which implies that $(\mu_A \circ f)(xy) = (\mu_A \circ f)(yx)$, for all x and y ∈ R.

Hence A◦f is an anti S-fuzzy normal subhemiring of a hemiring R. □

2.4.12 Theorem: Let A be an anti S-fuzzy subhemiring of a hemiring H and f is an anti-isomorphism from a hemiring R onto H. If A is an anti S-fuzzy normal subhemiring of the hemiring H, then A◦f is an anti S-fuzzy normal subhemiring of the hemiring R.

Proof: Let x and y in R and A be an anti S-fuzzy normal subhemiring of a hemiring H. Then clearly A◦f is an anti S-fuzzy subhemiring of the hemiring R. Now since f is anti-isomorphism, $(\mu_A \circ f)(xy) = \mu_A(f(xy))$

$= \mu_A(f(y)f(x))$,

$= \mu_A(f(x)f(y))$

$= \mu_A(f(yx))$,

$= (\mu_A \circ f) (yx)$,

which implies that $(\mu_A \circ f)(xy) = (\mu_A \circ f)(yx)$, for all x and y $\in$ R.

Hence A∘f is an anti S-fuzzy normal subhemiring of the hemiring R. □

2.4.13 Theorem: Let A be an anti S-fuzzy normal subhemiring of a hemiring R. Then for $\alpha \in [0,1]$ such that $\mu_A(0) \leq \alpha$, A_α is a lower level subhemiring of R.

Proof: The argument is trivial.

2.4.14 Theorem: Let A be an anti S-fuzzy normal subhemiring of a hemiring R, then two lower level subhemiring $A_{\alpha 1}$, $A_{\alpha 2}$ and α_1, α_2 are in [0,1] such that $\mu_A(0) \leq \alpha_1$, $\mu_A(0) \leq \alpha_2$ with $\alpha_1 < \alpha_2$ of A are equal if and only if there is no x in R such that $\alpha_2 > \mu_A(x) > \alpha_1$.

Proof: The argument is trivial.

2.4.15 Theorem: Let A be an anti S-fuzzy normal subhemiring of a hemiring R. If any two lower level subhemirings of A belongs to R, then their intersection is also lower level subhemiring of A $\in$ R.

Proof : The argument is trivial.

2.4.16 Theorem: Let A be an anti S-fuzzy normal subhemiring of a hemiring R. If $\alpha_i \in [0,1]$, and $A_{\alpha i}$, $i \in I$ is a collection of lower level subhemirings of A, then their intersection is also a lower level subhemiring of A.

Proof: The argument is trivial.

2.4.17 Theorem: Let A be an anti S-fuzzy normal subhemiring of a hemiring R. If any two lower level subhemirings of A belongs to R, then their union is also a lower level subhemiring of A ∈ R.

Proof: The argument is trivial.

2.4.18 Theorem: Let A be an anti S-fuzzy normal subhemiring of a hemiring R. If $\alpha_i \in [0,1]$ and $A_{\alpha i}$, $i \in I$ is a collection of lower level subhemirings of A, then their union is also a lower level subhemiring of A.

Proof: The argument is trivial.

2.4.19 Theorem: The homomorphic image of a lower level subhemiring of an anti S-fuzzy normal subhemiring of a hemiring R is a lower level subhemiring of an anti S-fuzzy normal subhemiring of R^l.

Proof: The argument is trivial.

2.4.20 Theorem: The homomorphic pre-image of a lower level subhemiring of an anti S-fuzzy normal subhemiring of a hemiring R^l is a lower level subhemiring of an anti S-fuzzy normal subhemiring of R.

Proof: The argument is trivial.

2.4.21 Theorem: The anti-homomorphic image of a lower level subhemiring of an anti S-fuzzy normal subhemiring of a hemiring R is a lower level subhemiring of an anti S-fuzzy normal subhemiring of R^l.

Proof: The argument is trivial.

2.4.22 Theorem: The anti-homomorphic pre-image of a lower level subhemiring of an anti S-fuzzy normal subhemiring of a hemiring R^l is a lower level subhemiring of an anti S-fuzzy normal subhemiring of R.

Proof: The argument is trivial.