PREFACE

Fuzzy subsets: Among the various paradigmatic changes in science and mathematics, one such change concerns the concept of uncertainty. Identification of this important role of uncertainty by some researchers began the stage of transition from the traditional view to the modern view of uncertainty and such transition is characterized by the emergence of several new theories of uncertainty from probability theory.

An important point in the evolution of the modern concept of uncertainty was the publication of a seminal paper by Lotfi A. Zadeh [42]. In his paper Zadeh introduced a theory whose objects fuzzy sets are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of degree. This concept is being used and is also found to be more appropriate in solving problems of all disciplines.

The concept of fuzzy sets was initiated by Zadeh in 1965 to represent / manipulate data and information possessing non-statistical uncertainties. It was specifically designed to mathematically represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems.

The first publication in fuzzy subset theory by Zadeh (1965) and then by Goguen (1967, 1969) show the intention of the authors to generalize the classical set. In classical set theory, a subset A of a set X can be defined
by its characteristic function \( \chi_A : X \to \{ 0, 1 \} \) is defined by \( \chi_A(x) = 0 \), if \( x \notin A \) and \( \chi_A(x) = 1 \), if \( x \in A \).

The mapping may be represented as a set of ordered pairs \( \{(x, \chi_A(x))\} \) with exactly one ordered pair present for each element of \( X \). The first element of the ordered pair is an element of the set \( X \) and the second is its value in \( \{ 0, 1 \} \). The value ‘0’ is used to represent non-membership and the value ‘1’ is used to represent membership of the element \( A \). The truth or falsity of the statement “\( x \in A \)” is determined by the ordered pair. The statement is true, if the second element of the ordered pair is ‘1’, and the statement is false, if it is ‘0’. Similarly, a fuzzy subset \( A \) of a set \( X \) can be defined as a set of ordered pairs \( \{(x, \chi_A(x)) : x \in X\} \), each with the first element from \( X \) and the second element from the interval \( [0, 1] \) with exactly one ordered pair present for each element of \( X \). This defines a mapping \( \mu_A \) between elements of the set \( X \) and values in the interval \( [0, 1] \). That is, \( \mu_A : X \to [0, 1] \).

The value ‘0’ is used to represent complete non-membership, the value ‘1’ is used to represent complete membership and values in between are used to represent intermediate degrees of membership.

The set \( X \) is referred to as the Universe of discourse for the fuzzy subset \( A \). Frequently, the mapping \( \mu_A \) is described as a function, the membership function of \( A \), the degree to which the statement “\( x \in A \)” is true, is determined by finding the ordered pair \( (x, \mu_A(x)) \). The degree of truth of the statement is the second element of the ordered pair.
**Intuitionistic fuzzy subsets:** Prof. K.T. Atanassov, a Bulgarian Engineer, introduced a new component which determines the degree of non-membership also in defining intuitionistic fuzzy subset (IFS) theory. In 1983, he came across A. Kauffmann’s book “Introduction to the theory of fuzzy subsets” Academic Press, New York, 1975, then he tried to introduce intuitionistic fuzzy subsets to study the properties of the new objects so defined. He defined ordinary operations as $\cap$, $\cup$, $+$ and $-$ over the new sets, then defined operators similar to the operators ‘necessity’ and ‘possibility’.

George Gargov named new sets as the ‘Intuitionistic fuzzy subsets’, as their fuzzification denies the law of the excluded middle, $A \cup A^c = X$. This has encouraged Prof. K.T. Atanassov to continue his work on intuitionistic fuzzy subsets.

An intuitionistic fuzzy subset $A$ of a set $X$ can be defined as a set of ordered pairs \{ $\langle x, \mu_A(x), \nu_A(x) \rangle \}$ with the first element from $X$ and the second element from the interval $[0, 1]$ and the third element from the interval $[0, 1]$ with exactly one ordered pair present for each element of $X$ such that for every $x$ in $X$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. This defines the mapping, $\mu_A$ between elements of the set $X$ and values in the interval $[0, 1]$, is called the degree of membership and $\nu_A$ between elements of the set $X$ and values in the interval $[0, 1]$, is called the degree of non-membership of the element $x$ in $X$. That is $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ respectively and for every $x \in X$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. 
Vasantha kandasamy.W.B., well explained the fuzzy algebraic structure with applications in [38]. In [5], [6], Atanassov.K.T. has defined and discussed the importance of intuitionistic fuzzy sets and their applications. Prince Williams. D.R., introduced S-fuzzy left h ideal of hemirings [30], give the scope of hemirings in the theory of formal languages and automata. Wieslaw dudek explained the concept of intuitionistic fuzzy h-ideals of hemirings[39]. Palaniappan.N & K.Arjunan in [26] , [27] , [28] and Palaniappan. N & R.Muthuraj in [25] have discussed the operations such as homomorphism, anti-homomorphism in fuzzy, anti-fuzzy subgroups as well as fuzzy and anti-fuzzy ideals and intuitionistic fuzzy subgroups which gave an essential foundation to study these operations and structures throughout our research work.

Outline of the thesis:

In chapter 1, we have introduced all the basic definitions and important results to develop the thesis.

In chapter 2, we introduce the concept of anti S-fuzzy subhemirings of a hemiring and establish some results on these.

- Union of any two anti S-fuzzy subhemiring of a hemiring R is an anti S-fuzzy subhemiring of R.
- If A and B are any two anti S-fuzzy subhemirings of the hemirings $R_1$ and $R_2$ respectively, then anti-product $A \times B$ is an anti S-fuzzy subhemiring of $R_1 \times R_2$. 
• Let $A$ be a fuzzy subset of a hemiring $R$ and $V$ be the anti-strongest fuzzy relation of $R$. Then $A$ is an anti $S$-fuzzy subhemiring of $R$ if and only if $V$ is an anti $S$-fuzzy subhemiring of $R \times R$.

• Let $A$ be an anti $S$-fuzzy subhemiring of a hemiring $(R, +, \cdot)$. If $\mu_A(x+y)=1$, then either $\mu_A(x)=1$ or $\mu_A(y)=1$, for all $x$ and $y \in R$.

• Let $A$ be an anti $S$-fuzzy subhemiring of a hemiring $H$ and $f$ is an isomorphism (anti-isomorphism) from a hemiring $R$ onto $H$. Then $A \circ f$ is an anti $S$-fuzzy subhemiring of $R$.

• Let $A$ be an anti $S$-fuzzy subhemiring of a hemiring $(R, +, \cdot)$, then the pseudo anti $S$-fuzzy coset $(aA)^p$ is an anti $S$-fuzzy subhemiring of a hemiring $R$, for every $a \in R$.

• Let $(R, +, \cdot)$ and $(R^l, +, \cdot)$ be any two hemirings. The homomorphic (anti-homomorphic) image (preimage) of an anti $S$-fuzzy subhemiring of $R$ is an anti $S$-fuzzy subhemiring of $R^l$.

• Let $A$ be an anti $S$-fuzzy subhemiring of a hemiring $R$, $A^+$ be a fuzzy set in $R$ defined by $A^+(x) = A(x)+1 - A(0)$, for all $x \in R$. Then $A^+$ is an anti $S$-fuzzy subhemiring of a hemiring $R$.

• Let $A$ be an anti $S$-fuzzy subhemiring of a hemiring $R$, $A^+$ be a fuzzy set in $R$ defined by $A^+(x) = A(x) + 1 - A(0)$, for all $x \in R$. Then $(A^+)^+ = A^+$.

• Let $A$ be an anti $S$-fuzzy subhemiring of a hemiring $R$. Then $A^0$ is an anti $S$-fuzzy subhemiring of the hemiring $R$. 

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• Let A be an anti S-fuzzy subhemiring of a hemiring R. Then for \( \alpha \) in [0,1] such that \( \mu_A(0) \leq \alpha \), \( A_\alpha \) is a lower level subhemiring of R.

• Let A be an anti S-fuzzy subhemiring of a hemiring R. Then two lower level subhemiring \( A_{\alpha_1}, A_{\alpha_2} \) and \( \alpha_1, \alpha_2 \) are in [0,1] such that \( \mu_A(0) \leq \alpha_1, \mu_A(0) \leq \alpha_2 \) with \( \alpha_1 < \alpha_2 \) of A are equal if and only if there is no \( x \in R \) such that \( \alpha_2 > \mu_A(x) > \alpha_1 \).

• Let A be an anti S-fuzzy subhemiring of a hemiring R. If any two lower level subhemirings of A belongs to R, then their intersection is also lower level subhemiring of A \( \in R \).

• Let A be an anti S-fuzzy subhemiring of a hemiring R. If any two lower level subhemirings of A belongs to R, then their union is also a lower level subhemiring of A \( \in R \).

• The homomorphic(anti-homomorphic) image(preimage) of a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring R is a lower level subhemiring of an anti S-fuzzy subhemiring of a hemiring \( R^1 \).

• Let \( (R, +, .) \) be a hemiring. If A and B are two anti S-fuzzy normal subhemirings of R, then their union \( A \cup B \) is an anti S-fuzzy normal subhemiring of R.

• Let A and B be anti S-fuzzy subhemirings of the hemirings G and H, respectively. If A and B are anti S-fuzzy normal subhemirings, then \( A \times B \) is an anti S-fuzzy normal subhemiring of \( G \times H \).
Let A and B be anti S-fuzzy normal subhemiring of the hemirings $R_1$ and $R_2$ respectively. Suppose that $0_1$ and $0_2$ are the zero element of $R_1$ and $R_2$ respectively. If $A \times B$ is an anti S-fuzzy normal subhemiring of $R_1 \times R_2$, then at least one of the following two statements must hold.

(i) $B(0_2) \leq A(x)$, for all $x \in R_1$,

(ii) $A(0_1) \leq B(y)$, for all $y \in R_2$.

Let A be a fuzzy subset in a hemiring R and V be the anti-strongest fuzzy relation on R. Then A is an anti S-fuzzy normal subhemiring of R if and only if V is an anti S-fuzzy normal subhemiring of $R \times R$.

Let $(R, +, \cdot)$ and $(R^l, +, \cdot)$ be any two hemirings. The homomorphic (anti-homomorphic) image(preimage) of an anti S-fuzzy normal subhemiring of R is an anti S-fuzzy normal subhemiring of $R^l$.

Let A be an anti S-fuzzy subhemiring of a hemiring H and $f$ is an isomorphism(anti-isomorphism) from a hemiring R onto H. If A is an anti S-fuzzy normal subhemiring of the hemiring H, then $A \circ f$ is an anti S-fuzzy normal subhemiring of the hemiring R.

Let A be an anti S-fuzzy normal subhemiring of a hemiring R. If any two lower level subhemirings of A belongs to R, then their intersection is also lower level subhemiring of A $\in R$.

Let A be an anti S-fuzzy normal subhemiring of a hemiring R. If any two lower level subhemirings of A belongs to R, then their union is also a lower level subhemiring of A $\in R$. 

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• The homomorphic (anti-homomorphic) image(preimage) of a lower level subhemiring of an anti S-fuzzy normal subhemiring of a hemiring R is a lower level subhemiring of an anti S-fuzzy normal subhemiring of a hemiring $R^1$.

**In chapter 3**, we introduce the concept of anti (T, S)-fuzzy ideals of a hemiring, translations of anti S-fuzzy subhemiring and establish some results on these.

• Union of any two anti (T, S)-fuzzy ideal of a hemiring R is an anti (T, S)-fuzzy ideal of R.

• If A and B are any two anti (T, S)-fuzzy ideals of the hemirings $R_1$ and $R_2$ respectively, then anti-product $A \times B$ is an anti (T, S)-fuzzy ideal of $R_1 \times R_2$.

• Let A and B be two fuzzy subsets of the hemirings $R_1$ and $R_2$ respectively and $A \times B$ is an anti (T, S)-fuzzy ideal of $R_1 \times R_2$. Then the following are true:
  
  (i) if $A(x) \geq B(0)$, then A is an anti (T, S)-fuzzy ideal of $R_1$.
  
  (ii) if $B(x) \geq A(0)$, then B is an anti (T, S)-fuzzy ideal of $R_2$.
  
  (iii) either A is an anti (T, S)-fuzzy ideal of $R_1$ or B is an anti (T, S)-fuzzy ideal of $R_2$.

• Let A be a fuzzy subset of a hemiring R and V be the anti-strongest fuzzy relation of R. Then A is an anti (T, S)-fuzzy ideal of R if and only if V is an anti (T, S)-fuzzy ideal of $R \times R$. 

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Let $A$ be an anti $(T, S)$-fuzzy ideal of a hemiring $(R, +, \cdot)$. If $\mu_A(x+y) = 1$, then either $\mu_A(x) = 1$ or $\mu_A(y) = 1$, for all $x$ and $y \in R$.

Let $A$ be an anti $(T, S)$-fuzzy ideal of a hemiring $H$ and $f$ is an isomorphism (anti-isomorphism) from a hemiring $R$ onto $H$. Then $A \circ f$ is an anti $(T, S)$-fuzzy ideal of $R$.

Let $A$ be an anti $(T, S)$-fuzzy ideal of a hemiring $(R, +, \cdot)$, then the pseudo anti $(T, S)$-fuzzy coset $(aA)^p$ is an anti $(T, S)$-fuzzy ideal of a hemiring $R$, for every $a \in R$.

Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two hemirings. The homomorphic (anti-homomorphic) image (pre-image) of an anti $(T, S)$-fuzzy ideal of $R$ is an anti $(T, S)$-fuzzy ideal of $R^1$.

Let $A$ be an anti $(T, S)$-fuzzy ideal of a hemiring $R$, $A^+$ be a fuzzy set in $R$ defined by $A^+(x) = A(x) + 1 - A(0)$, for all $x \in R$. Then $A^+$ is an anti $(T, S)$-fuzzy ideal of a hemiring $R$.

Let $A$ be an anti $(T, S)$-fuzzy ideal of a hemiring $R$. Then $A^0$ is an anti $(T, S)$-fuzzy ideal of the hemiring $R$.

Let $A$ be an anti $(T, S)$-fuzzy ideal of a hemiring $R$. Then for $\alpha$ in $[0,1]$ such that $\mu_A(0) \leq \alpha$, $A_\alpha$ is a lower level ideal of $R$.

Let $A$ be an anti $(T, S)$-fuzzy ideal of a hemiring $R$. If any two lower level ideals of $A$ belongs to $R$, then their intersection is also lower level ideal of $A \in R$. 


• The homomorphic (anti-homomorphic) image (pre-image) of a lower level ideal of an anti \((T, S)\)-fuzzy ideal of a hemiring \(R\) is a lower level ideal of an anti \((T, S)\)-fuzzy ideal of a hemiring \(R^1\).

• Let \((R, +, .)\) be a hemiring. If \(A\) and \(B\) are two anti \((T, S)\)-fuzzy normal ideals of \(R\). Then \(A \cup B\) is an anti \((T, S)\)-fuzzy normal ideal of \(R\).

• Let \(A\) and \(B\) be anti \((T, S)\)-fuzzy ideals of the hemirings \(G\) and \(H\), respectively. If \(A\) and \(B\) are anti \((T, S)\)-fuzzy normal ideals, then \(A \times B\) is an anti \((T, S)\)-fuzzy normal ideal of \(G \times H\).

• Let \(A\) and \(B\) be two fuzzy subsets of the hemirings \(R_1\) and \(R_2\) respectively and \(A \times B\) is an anti \((T, S)\)-fuzzy normal ideal of \(R_1 \times R_2\).

Then the following are true:

(i) if \(A(x) \geq B(0)\), then \(A\) is an anti \((T, S)\)-fuzzy normal ideal of \(R_1\).

(ii) if \(B(x) \geq A(0)\), then \(B\) is an anti \((T, S)\)-fuzzy normal ideal of \(R_2\).

(iii) either \(A\) is an anti \((T, S)\)-fuzzy normal ideal of \(R_1\) or \(B\) is an anti \((T, S)\)-fuzzy normal ideal of \(R_2\).

• Let \(A\) be a fuzzy subset in a hemiring \(R\) and \(V\) be the anti-strongest fuzzy relation on \(R\). Then \(A\) is an anti \((T, S)\)-fuzzy normal ideal of \(R\) if and only if \(V\) is an anti \((T, S)\)-fuzzy normal ideal of \(R \times R\).

• Let \(A\) be an anti \((T, S)\)-fuzzy ideal of a hemiring \(H\) and \(f\) is an isomorphism from a hemiring \(R\) onto \(H\). If \(A\) is an anti \((T, S)\)-fuzzy
normal ideal of the hemiring \( H \), then \( A \circ f \) is an anti \((T, S)\)-fuzzy normal ideal of the hemiring \( R \).

- The anti-homomorphic image of a lower level ideal of an anti \((T, S)\)-fuzzy normal ideal of a hemiring \( R \) is a lower level ideal of an anti \((T, S)\)-fuzzy normal ideal of a hemiring \( R^l \).

- If \( M \) and \( N \) are two translations of anti \( S \)-fuzzy subhemiring \( A \) of a hemiring \((R, +, \cdot)\), then their intersection \( M \cap N \) is translation of anti \( S \)-fuzzy subhemiring \( A \).

- If \( M \) and \( N \) are two translations of anti \( S \)-fuzzy subhemiring \( A \) of a hemiring \((R, +, \cdot)\), then their union \( M \cup N \) is also a translation of anti \( S \)-fuzzy subhemiring \( A \).

- If \( T^A_\alpha \) is a translation of anti \( S \)-fuzzy subhemiring \( A \) of a hemiring \( R \), then \( T^A_\alpha \) is anti \( S \)-fuzzy subhemiring of \( R \).

- Let \((R, +, \cdot)\) and \((R^l, +, \cdot)\) be any two hemirings. If \( f: R \rightarrow R^l \) is a homomorphism, then the translation of anti \( S \)-fuzzy subhemiring \( A \) of \( R \) under the homomorphic image is anti \( S \)-fuzzy subhemiring of \( f(R) = R^l \).

- If \( M \) and \( N \) are two translations of an anti \( S \)-fuzzy normal subhemiring \( A \) of a hemiring \((R, +, \cdot)\), then their intersection \( M \cap N \) is also a translation of \( A \).

- Let \((R, +, \cdot)\) and \((R^l, +, \cdot)\) be any two hemirings. If \( f: R \rightarrow R^l \) is a homomorphism, then the translation of an anti \( S \)-fuzzy normal
subhemiring $A$ of $R$ under the homomorphic image is an anti $S$-fuzzy normal subhemiring of $f(R) = R^1$.

**In chapter 4,** we introduce the concept of $(T, S)$-intuitionistic fuzzy subhemirings of a hemiring and establish some results on these.

- Intersection of any two $(T, S)$-intuitionistic fuzzy subhemirings of a hemiring $R$ is a $(T, S)$-intuitionistic fuzzy subhemiring of $R$.
- If $A$ and $B$ are any two $(T, S)$-intuitionistic fuzzy subhemirings of the hemirings $R_1$ and $R_2$ respectively, then $A \times B$ is an $(T, S)$-intuitionistic fuzzy subhemiring of $R_1 \times R_2$.
- If $A$ is a $(T, S)$-intuitionistic fuzzy subhemiring of a hemiring $(R, +, \cdot)$, then $\mu_A(x) \leq \mu_A(0)$ and $\nu_A(x) \geq \nu_A(0)$, for $x$ in $R$, the zero element $0 \in R$.
- Let $A$ and $B$ be two intuitionistic fuzzy subsets of the hemirings $R_1$ and $R_2$ respectively and $A \times B$ is an $(T, S)$-intuitionistic fuzzy subhemiring of $R_1 \times R_2$. Then the following are true:
  (i) if $\mu_A(x) \leq \mu_B(0)$ and $\nu_A(x) \geq \nu_B(0)$, then $A$ is an $(T, S)$-intuitionistic fuzzy subhemiring of $R_1$.
  (ii) if $\mu_B(x) \leq \mu_A(0)$ and $\nu_B(x) \geq \nu_A(0)$, then $B$ is an $(T, S)$-intuitionistic fuzzy subhemiring of $R_2$.
  (iii) either $A$ is an $(T, S)$-intuitionistic fuzzy subhemiring of $R_1$ or $B$ is an $(T, S)$-intuitionistic fuzzy subhemiring of $R_2$.
- Let $A$ be an intuitionistic fuzzy subset of a hemiring $R$ and $V$ be the strongest intuitionistic fuzzy relation of $R$. Then $A$ is an
(T, S)-intuitionistic fuzzy subhemiring of R if and only if V is an
(T, S)-intuitionistic fuzzy subhemiring of R×R.

- If A is an (T, S)-intuitionistic fuzzy subhemiring of a hemiring
  (R, +, .), then H = \{ x / x∈R : \mu_A(x) = 1, \nu_A(x) = 0 \} is either empty or
  is a subhemiring of R.

- If A is an (T, S)-intuitionistic fuzzy subhemiring of a hemiring
  (R, +, .), then \Box A is an (T, S)-intuitionistic fuzzy subhemiring of R.

- If A is an (T, S)-intuitionistic fuzzy subhemiring of a hemiring
  (R, +, .), then \Diamond A is an (T, S)-intuitionistic fuzzy subhemiring of R.

- Let A be an (T, S)-intuitionistic fuzzy subhemiring of a hemiring H
  and f is an isomorphism(anti-isomorphism) from a hemiring R onto
  H. Then A\circ f is an (T, S)-intuitionistic fuzzy subhemiring of R.

- Let (R, +, .) and (R^l, +, .) be any two hemirings. The homomorphic
  (anti-homomorphic) image (preimage) of an (T, S)-intuitionistic
  fuzzy subhemiring of R is an (T, S)-intuitionistic fuzzy subhemiring
  of R^l.

- Let A be an (T, S)-intuitionistic fuzzy subhemiring of a hemiring
  (R, +, .). Then for \alpha and \beta in [0,1] such that \mu_A(0) ≥ \alpha and \nu_A(0) ≤ \beta,
  A_{(\alpha, \beta)} is a subhemiring of R.

- Let (R, +, .) be a hemiring and A be an intuitionistic fuzzy subset of
  R such that A_{(\alpha, \beta)} be a subhemiring of R. If \alpha and \beta in [0,1], then A
  is an (T, S)-intuitionistic fuzzy subhemiring of R.
• Let $A$ be an $(T, S)$-intuitionistic fuzzy subhemiring of a hemiring $(R, +, \cdot)$. If any two level subhemiring of $A$ belongs to $R$, then their intersection is also level subhemiring of $A \in R$.

• Let $A$ be an $(T, S)$-intuitionistic fuzzy subhemiring of a hemiring $(R, +, \cdot)$. If $\alpha_i$ and $\beta_j$ in $[0,1]$ such that $\mu_A(0) \geq \alpha_i$ and $\nu_A(0) \leq \beta_j$ and $A_{(\alpha_i, \beta_j)}$, $i$ and $j$ in $I$, is a collection of level subhemirings of $A$, then their union is also a level subhemiring of $A$.

• Let $A$ be an $(T, S)$-intuitionistic fuzzy subhemiring of a hemiring $(R, +, \cdot)$. If $A$ is an $(T, S)$-intuitionistic fuzzy characteristic subhemiring of $R$, then each level subhemiring of $A$ is a characteristic subhemiring of $R$.

• Let $f$ be any mapping from a hemiring $R_1$ to $R_2$ and let $A$ be an $(T, S)$-intuitionistic fuzzy subhemiring of $R_1$. Then for $\alpha$ and $\beta$ in $[0,1]$, we have $f(A_{(\alpha, \beta)}) = \bigcap_{\alpha \geq \varepsilon_1 > 0, \beta \geq \varepsilon_2 > 0} f(A_{(\alpha - \varepsilon_1, \beta + \varepsilon_2)})$.

• Let $A$ and $B$ be $(T, S)$-intuitionistic fuzzy subhemirings of the hemirings $G$ and $H$ respectively and $\alpha$ and $\beta$ in $[0,1]$. Then $(A \times B)_{(\alpha, \beta)} = A_{(\alpha, \beta)} \times B_{(\alpha, \beta)}$.

• Let $A$ be an $(T, S)$-intuitionistic fuzzy subhemiring of a hemiring $(R, +, \cdot)$, then for $\alpha$ and $\beta$ in $[0,1]$, $\mu$-level $\alpha$-cut $U(\mu_A, \alpha)$ is a subhemiring of $R$. 

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• Let $A$ be an $(T, S)$-intuitionistic fuzzy subhemiring of a hemiring $(R, +, \cdot)$, then for $\alpha$ and $\beta$ in $[0,1]$, $\nu$-level $\beta$-cut $L(\nu_A, \beta)$ is a subhemiring of $R$.

• Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings, then the homomorphic (anti-homomorphic) image (preimage) of a level subhemiring of an $(T, S)$-intuitionistic fuzzy subhemiring of $R$ is a level subhemiring of an $(T, S)$-intuitionistic fuzzy subhemiring of $R'$.

**In chapter 5,** we introduce the concept of $(T, S)$-intuitionistic fuzzy normal subhemirings and $(T, S)$-intuitionistic fuzzy ideals of a hemiring and establish some results on these.

• Let $R$ be a hemiring. If $A$ and $B$ are two $(T, S)$-intuitionistic fuzzy normal subhemirings of $R$, then their intersection $A \cap B$ is an $(T, S)$-intuitionistic fuzzy normal subhemiring of $R$.

• Let $A$ and $B$ be $(T, S)$-intuitionistic fuzzy subhemiring of the hemirings $G$ and $H$, respectively. If $A$ and $B$ are $(T, S)$-intuitionistic fuzzy normal subhemirings, then $A \times B$ is an $(T, S)$-intuitionistic fuzzy normal subhemiring of $G \times H$.

• Let $A$ be an intuitionistic fuzzy subset in a hemiring $R$ and $V$ be the strongest intuitionistic fuzzy relation on $R$, then $A$ is an $(T, S)$-intuitionistic fuzzy normal subhemiring of $R$ if and only if $V$ is an $(T, S)$-intuitionistic fuzzy normal subhemiring of $R \times R$.

• Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings, then the homomorphic image (preimage) of an $(T, S)$-intuitionistic fuzzy
normal subhemiring of R is an (T, S)-intuitionistic fuzzy normal subhemiring of \( R^1 \).

- The intersection of a family of (T, S)-intuitionistic fuzzy ideals of hemiring R is an (T, S)-intuitionistic fuzzy ideal of R.

- If A and B are any two (T, S)-intuitionistic fuzzy ideal of the hemirings \( R_1 \) and \( R_2 \) respectively, then \( A \times B \) is an (T, S)-intuitionistic fuzzy ideal of \( R_1 \times R_2 \).

- Let A and B be (T, S)-intuitionistic fuzzy ideals of the hemirings \( R_1 \) and \( R_2 \) respectively. Suppose that 0_1 and 0_2 are the zero element of \( R_1 \) and \( R_2 \) respectively. If \( A \times B \) is an (T, S)-intuitionistic fuzzy ideal of \( R_1 \times R_2 \), then at least one of the following two statements must hold.
  
  i) \( \mu_B(0_2) \geq \mu_A(x) \) and \( \nu_B(0_2) \leq \nu_A(x) \), for all \( x \in R_1 \),
  
  ii) \( \mu_A(0_1) \geq \mu_B(y) \) and \( \nu_A(0_1) \leq \nu_B(y) \), for all \( y \in R_2 \).

- Let A and B be two intuitionistic fuzzy subsets of the hemirings \( R_1 \) and \( R_2 \) respectively and \( A \times B \) is an (T, S)-intuitionistic fuzzy ideal of \( R_1 \times R_2 \). Then the following are true:
  
  i) if \( \mu_A(x) \leq \mu_B(0_2) \) and \( \nu_A(x) \geq \nu_B(0_2) \), then A is an (T, S)-intuitionistic fuzzy ideal of \( R_1 \).
  
  ii) if \( \mu_B(x) \leq \mu_A(0_1) \) and \( \nu_B(x) \geq \nu_A(0_1) \), then B is an (T, S)-intuitionistic fuzzy ideal of \( R_2 \).
  
  iii) either A is an (T, S)-intuitionistic fuzzy ideal of \( R_1 \) or B is an (T, S)-intuitionistic fuzzy ideal of \( R_2 \).
CHAPTER - 1

PRELIMINARIES

A: Crisp set theory.

1.1 Introduction: We list here the basic definitions of crisp sets and results for the sake of completeness and for easy references.

1.1.1 Definition: A non-empty set \( R \) together with two binary operations denoted by + and . are called addition and multiplication which satisfy the following axioms are called a hemiring.

(i) \(( R, + )\) is a semigroup and commutative with zero,

(ii) \(( R, . )\) is a semigroup,

(iii) \((a+b).c = a.c + b.c\) and \(a.(b+c) = a.b + a.c\), for all \(a, b, c \in R\).

1.1.1 Example: \(( Z, +, . )\) is a hemiring under the usual addition and multiplication, where \(Z\) is the set of all integers.

1.1.2 Definition: A non-empty subset \( S \) of a hemiring \(( R, +, . )\) is called a subhemiring if \(S\) itself is a hemiring under the same operation as in \(R\).

1.1.2 Example: \(( 2Z, +, . )\) is a subhemiring of \(( Z, +, . )\), where \(Z\) is the set of all integers.

1.1.3 Definition: Let \(( R, +, . )\) and \(( R^l, +, . )\) be any two hemirings. Then the function \(f : R \rightarrow R^l\) is called a hemiring homomorphism if it satisfies the following axioms:

(i) \(f(x+y) = f(x) + f(y)\),

(ii) \(f(xy) = f(x) f(y)\), for all \(x, y \in R\).
**1.1.3 Example:** Let \( R = \{ \frac{m + n\sqrt{2}}{m}, n \in \mathbb{Z} \} \). \( R \) is a hemiring under usual addition and multiplication. Define \( f : R \to R \) by \( f(m + n\sqrt{2}) = m - n\sqrt{2} \) is hemiring homomorphism, where \( \mathbb{Z} \) is the set of all integers.

**1.1.4 Definition:** Let \(( R, +, \cdot )\) and \(( R', +, \cdot )\) be any two hemirings. Then the function \( f : R \to R' \) is called a **hemiring anti-homomorphism** if it satisfies the following axioms:

\[
\begin{align*}
(i) & \quad f(x + y) = f(y) + f(x), \\
(ii) & \quad f(xy) = f(y) f(x), \text{ for all } x \text{ and } y \in R.
\end{align*}
\]

**1.1.5 Definition:** Let \(( R, +, \cdot )\) and \(( R', +, \cdot )\) be any two hemirings. Then the function \( f : R \to R' \) be a hemiring homomorphism. If \( f \) is onto, then \( f \) is called a **hemiring epimorphism**.

**1.1.6 Definition:** Let \(( R, +, \cdot )\) and \(( R', +, \cdot )\) be any two hemirings. Then the function \( f : R \to R' \) be a hemiring homomorphism. If \( f \) is one-to-one, then \( f \) is called a **hemiring monomorphism**.

**1.1.7 Definition:** Let \(( R, +, \cdot )\) and \(( R', +, \cdot )\) be any two hemirings. Then the function \( f : R \to R' \) be a hemiring homomorphism. If \( f \) is one-to-one and onto, then \( f \) is called a **hemiring isomorphism**.

**1.1.8 Definition:** Let \(( R, +, \cdot )\) and \(( R', +, \cdot )\) be any two hemirings. Then the function \( f : R \to R' \) be a hemiring anti-homomorphism. If \( f \) is onto, then \( f \) is called a **hemiring anti-epimorphism**.

**1.1.9 Definition:** Let \(( R, +, \cdot )\) and \(( R', +, \cdot )\) be any two hemirings. Then the function \( f : R \to R' \) be a hemiring anti-homomorphism. If \( f \) is one-to-one, then \( f \) is called a **hemiring anti-monomorphism**.
1.1.10 Definition: Let $(R, +, .)$ and $(R', +, .)$ be any two hemirings. Then the function $f : R \rightarrow R'$ be a hemiring anti-homomorphism. If $f$ is one-to-one and onto, then $f$ is called a **hemiring anti-isomorphism**.

1.1.11 Definition: A T-norm is a binary operations $T: [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following requirements;

(i) $T(0, x) = 0$, $T(1, x) = x$ (boundary condition)

(ii) $T(x, y) = T(y, x)$ (commutativity)

(iii) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity)

(iv) if $x \leq y$ and $w \leq z$, then $T(x, w) \leq T(y, z)$ (monotonicity).

B. Fuzzy set theory.

1.1.12 Definition: Let $X$ be a non-empty set. A **fuzzy subset** $A$ of $X$ is a function $A : X \rightarrow [0, 1]$.

1.1.13 Definition: The **union** of two fuzzy subsets $A$ and $B$ of a set $X$, denoted by $(A \cup B)(x) = \max\{A(x), B(x)\}$, for all $x \in X$.

1.1.14 Example: Let $A = \{\langle a, 0.4 \rangle, \langle b, 0.7 \rangle, \langle c, 0.3 \rangle\}$ and $B = \{\langle a, 0.5 \rangle, \langle b, 0.3 \rangle, \langle c, 0.43 \rangle\}$ be a fuzzy subsets of $X = \{a, b, c\}$. The union of two fuzzy subsets $A$ and $B$ is $A \cup B = \{\langle a, 0.5 \rangle, \langle b, 0.7 \rangle, \langle c, 0.43 \rangle\}$.

1.1.14 Definition: The **intersection** of two fuzzy subsets $A$ and $B$ of a set $X$, denoted by $(A \cap B)(x) = \min\{A(x), B(x)\}$, for all $x \in X$.

1.1.15 Example: Let $A = \{\langle a, 0.54 \rangle, \langle b, 0.57 \rangle, \langle c, 0.93 \rangle\}$ and $B = \{\langle a, 0.75 \rangle, \langle b, 0.63 \rangle, \langle c, 0.43 \rangle\}$ be a fuzzy subsets of $X = \{a, b, c\}$. The intersection of two fuzzy subsets $A$ and $B$ is $A \cap B = \{\langle a, 0.54 \rangle, \langle b, 0.57 \rangle, \langle c, 0.43 \rangle\}$. 
1.1.15 **Definition:** If $A$ is a fuzzy subset of $X$, then the **complement** of $A$, denoted $A^c$, is the fuzzy set of $X$, given by $A^c(x) = 1 - A(x)$, for all $x \in X$.

1.1.6 **Example:** Let $A = \{ \langle a, 0.4 \rangle, \langle b, 0.7 \rangle, \langle c, 0.8 \rangle \}$ is a fuzzy subset of $X = \{ a, b, c \}$. The complement of $A$ is $A^c = \{ \langle a, 0.6 \rangle, \langle b, 0.3 \rangle, \langle c, 0.2 \rangle \}$.

1.1.16 **Definition:** An intuitionistic fuzzy subset (IFS) $A$ in $X$ is defined as an object of the form $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle / x \in X \}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ respectively and for every $x \in X$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

1.1.17 **Example:** Let $X = \{ a, b, c \}$ be a set. Then $A = \{ \langle a, 0.52, 0.34 \rangle, \langle b, 0.14, 0.71 \rangle, \langle c, 0.25, 0.34 \rangle \}$ is an intuitionistic fuzzy subset of $X$.

1.1.17 **Definition:** If $A$ is an intuitionistic fuzzy subset of $X$, then the complement of $A$, denoted $A^c$, is the intuitionistic fuzzy set of $X$, given by $A^c(x) = \{ < x, \nu_A(x), \mu_A(x) > / x \in X \}$, for all $x \in X$.

1.1.18 **Example:** Let $A = \{ \langle a, 0.7, 0.1 \rangle, \langle b, 0.6, 0.2 \rangle, \langle c, 0.2, 0.3 \rangle \}$ is a fuzzy subset of $X = \{ a, b, c \}$. The complement of $A$ is $A^c = \{ \langle a, 0.1, 0.7 \rangle, \langle b, 0.2, 0.6 \rangle, \langle c, 0.3, 0.2 \rangle \}$

1.1.18 **Definition:** Let $A$ and $B$ be any two intuitionistic fuzzy subsets of a set $X$. We define the following operations:

(i) $A \cap B = \{ \langle x, \min \{ \mu_A(x), \mu_B(x) \} \}, \max \{ \nu_A(x), \nu_B(x) \} \rangle$, for all $x \in X$.

(ii) $A \cup B = \{ \langle x, \max \{ \mu_A(x), \mu_B(x) \} \}, \min \{ \nu_A(x), \nu_B(x) \} \rangle$, for all $x \in X$.

(iii) $\square A = \{ \langle x, \mu_A(x), 1 - \mu_A(x) \rangle / x \in X \}$, for all $x \in X$.

(iv) $\Diamond A = \{ \langle x, 1 - \nu_A(x), \nu_A(x) \rangle / x \in X \}$, for all $x \in X$. 
1.1.9 Example: Let $A = \{ \langle a, 0.5, 0.3 \rangle, \langle b, 0.7, 0.1 \rangle, \langle c, 0.3, 0.2 \rangle \}$ and $B = \{ \langle a, 0.5, 0.1 \rangle, \langle b, 0.6, 0.3 \rangle, \langle c, 0.4, 0.1 \rangle \}$ be two intuitionistic fuzzy subsets of $X = \{a, b, c\}$. Then we have

(i) $A \cap B = \{ \langle a, 0.5, 0.3 \rangle, \langle b, 0.6, 0.3 \rangle, \langle c, 0.3, 0.2 \rangle \}$,

(ii) $A \cup B = \{ \langle a, 0.5, 0.1 \rangle, \langle b, 0.7, 0.1 \rangle, \langle c, 0.4, 0.1 \rangle \}$,

(iii) $\mathcal{F}A = \{ \langle a, 0.5, 0.5 \rangle, \langle b, 0.7, 0.3 \rangle, \langle c, 0.3, 0.7 \rangle \}$,

(iv) $\mathcal{G}A = \{ \langle a, 0.7, 0.3 \rangle, \langle b, 0.9, 0.1 \rangle, \langle c, 0.8, 0.2 \rangle \}$.

1.1.19 Definition: Let $(R, +, \cdot)$ be a hemiring. A fuzzy subset $A$ of $R$ is said to be a $T$-fuzzy subhemiring ($T$FSHR) (fuzzy subhemiring with respect to $T$-norm) of $R$ if it satisfies the following conditions:

(i) $\mu_A(x + y) \geq T(\mu_A(x), \mu_A(y))$,

(ii) $\mu_A(xy) \geq T(\mu_A(x), \mu_A(y))$, for all $x$ and $y \in R$.

1.1.20 Definition: Let $(R, +, \cdot)$ be a hemiring. A $T$-fuzzy subhemiring $A$ of $R$ is said to be a $T$-fuzzy normal subhemiring ($T$FNSHR) of $R$ if $\mu_A(xy) = \mu_A(yx)$, for all $x$ and $y \in R$.

1.1.21 Definition: Let $(R, +, \cdot)$ be a hemiring. A fuzzy subset $A$ of $R$ is said to be a $T$-fuzzy ideal ($T$I)(fuzzy ideal with respect to $T$-norm) of $R$ if it satisfies the following conditions:

(i) $\mu_A(x + y) \geq T(\mu_A(x), \mu_A(y))$,

(ii) $\mu_A(xy) \geq S(\mu_A(x), \mu_A(y))$, for all $x$ and $y \in R$. 

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