CHAPTER-3

DEVELOPMENT OF MATHEMATICAL MODEL

3.1 INTRODUCTION:

Mathematical modelling provides a tool by means of which one can study and gain an understanding of a phenomenon, yet to occur. Experience has shown that complete prevention of flood by constructing river dike is practically impossible. Therefore effort should be made to minimise the flood hazard, through proper management practice rather than aiming at its complete prevention. Prior knowledge of flood movement, on downstream of the river dike, would enable one to develop an efficient management plan for mitigation of flood disaster. Therefore, attempt is being made in this chapter to develop a mathematical model for simulating flood propagation on downstream of river dike, either due to its failure or due to release of flood water through a lateral gate.

In the first part of this chapter two-dimensional governing equations adopted for the purpose, have been presented and suitability of these equations, for the situation under study, has been discussed.

The prudent assumptions, made for the purpose of modelling this vexed problem, are then given.
Numerical scheme used for the solution of two-dimensional governing equation has been presented elaborately in this chapter.

A new characteristic based analytical solution of dam-break problem, developed by the author, is then presented. The said solution has been used for computation of initial profile to start the numerical solution of flood wave propagation due to dike failure.

Boundary conditions and stability criteria for the numerical scheme have been presented towards the end of this chapter.

Finally, a concluding remark on the developed mathematical model has been drawn.

3.2 GOVERNING EQUATION:

The problem of simulating flood movement on downstream of a river dike due to its failure can be regarded as a special form of dam-break problem with the following features:-

1) The flow is strictly two-dimensional.

2) The released water moves in a lateral direction from the main stream.

3) Generally flood moves in a flat plain or in a little undular ground, having adverse slope in the lateral direction from main stream.

4) The flow is characterised by small value of water depth and velocities, compared to the flood propagation in a narrow valley.

5) The depth at the upstream end depends on the flow in
the main stream, which can be considered as constant if the flow rate in the main stream remains constant for sufficiently long time after release of water through an opening in the dike.

6) Super-critical and sub-critical flow are present simultaneously at the initial stage.

Considering the above characteristics, the two-dimensional continuity and momentum equations of shallow water wave, in the conservation form, have been used as the governing equations. These equations, in vector notation, can be expressed as

\[ \frac{\delta U}{\delta t} + \frac{\delta E}{\delta x} + \frac{\delta F}{\delta y} + S = 0 \quad \text{(3.1)} \]

where

\[ U = \begin{bmatrix} h \\ hV_x \\ hV_y \end{bmatrix}, E = \begin{bmatrix} hV_x \\ hV_x^2 + gh^2/2 \\ hV_x V_y \end{bmatrix}, F = \begin{bmatrix} hV_y \\ hV_x V_y \\ hV_y^2 + gh^2/2 \end{bmatrix} \]

\[ S = \begin{bmatrix} 0 \\ gh(S_r - S_{ox}) \\ gh(S_r - S_{oy}) \end{bmatrix} \]

Where \( h \) is the flow depth, \( V_x \) and \( V_y \) are the flow velocities in \( x \) and \( y \) direction respectively, \( g \) is the gravitational acceleration, \( S_{ox} \) and \( S_{oy} \) are the bed slopes in \( x \) and \( y \) directions respectively.

However, the Eq. 3.1 does not conserve momentum in true sense due to presence of "S" term\(^{18}\). As the bed slope plays an important role in the whole flow phenomenon, presence of "S" term has been considered to be more important than use of a
true conservation form. Of course since contribution of this term is usually small, the conservation properties are not significantly weakened.

3.3 ASSUMPTIONS MADE FOR MODELLING PURPOSE:

For implementing a numerical scheme, for the solution of governing equations, some prudent assumptions have been made regarding physical condition at the time of flood propagation.

3.3.1. Modelling Consideration at Downstream:

In case of flow over an initially dry bed, complexity of computation always arises at the tip of the wave front, where the depth drops to zero. Commonly, the specification of zero depth of flow in the mathematical model at this boundary causes a singularity in the solution. To overcome this difficulty, generally either computation is terminated a short distance behind the wave front or a small but finite depth is specified at the wave front.

The flow situation considered under the study is also a case of flow over dry bed. But failure of embankment occur invariably during rainy season. Therefore it can reasonably be assumed that a thin layer of water will always be there on the downstream of river dike during its failure. Thus, for the purpose of modelling, the dry downstream condition has been represented by still water of negligibly small depth.
3.3.2 Modelling Consideration at Upstream:

At the upstream end assumption of instantaneous failure makes the modelling much easier. But earthen embankments neither fail completely nor do they fail instantaneously. Gradual embankment failure occur over a period of time. Various considerations made for breach modelling will be discussed in details in the next chapter. However, in absence of reliable data regarding some important breach parameters, assumption of instantaneous failure (sudden opening) may still be considered for embankment breach, as this will represent the hypothetically possible extreme case. The sudden opening will also represent the opening of side gate (in case of gate opening) in true sense. Therefore, scope of handling both instantaneous and gradual opening has been incorporated in the model.

3.4 Numerical Scheme for Solution of Governing Equations:

For the modelling purpose, two-dimensional shallow water equations of gradually varied flow have been used. This set of equations has an inherent directional property of signal propagation. For instance, in case of sub-critical flow, information comes from both upstream and downstream, while information comes only from upstream in case of super-critical flow. This property must be handled properly while estimating flux. Riemann solver, based on characteristic theory, is one such algorithm, commonly used for the purpose. In recent years,
different investigators\textsuperscript{59,97,99}, have successfully used approximate Riemann solver like "Flux difference splitting" (FDS), "Flux vector splitting" (FVS), Oscar scheme etc., for computing numerical flux. Although application of such technique brings accuracy in finer level, it makes the mathematical treatment a little more complex and computation time is also significantly increased in many cases\textsuperscript{37}.

Careful observation has shown that the property of "direction of propagation of information" can be taken into account, while using finite difference scheme, by introducing a simple logical condition to select the appropriate finite difference approximation, depending on whether the flow condition is super-critical or sub-critical. Such technique will make the scheme applicable in the flow region where both sub-critical and super-critical flow may be present simultaneously. This crude method of handling mixed flow region, will of course introduce some computational error. But for practical applications this error can be considered negligible as compared to the error that occurs due to approximation made in the representation of topographical data and statistical method adopted for estimating the value of hydrological parameters.

On the basis of the above analysis, two-step predictor corrector explicit finite difference scheme has been considered suitable for solution of the two-dimensional governing equations. The proposed scheme can be regarded as a modified form of popular Mac Cormac scheme. In the sub-critical region, the backward finite
difference approximation has been used for the predictor step and forward F.D. approximation, using predicted flux matrices, has been adopted for the corrector step. In the super-critical region, as the control is always on the upstream side, use of forward F.D. approximation in the corrector step has been omitted to eliminate influence of downstream flux on the computed value.

3.4.1 Finite Difference Formulation of Governing Equations:

Using backward F. D. approximation, the finite difference form of equation 3.1 can be written to obtain the predicted U vector (denoted by UP) as —

\[ U_{ij}^{P} = U_{ij} - \frac{\Delta t}{\Delta x} [E_{ij} - E_{i-1,j}] - \frac{\Delta t}{\Delta y} [F_{ij} - F_{i,j-1}] - \Delta t(S_{ij}) \] ......(3.2)

Using primitive flow variables \((h, V_x, V_y)\) of predicted matrix, UP, the predicted flux matrix EP and FP and predicted “S” matrix SP are then calculated. Before applying the corrector step at any node, flow condition at that node is checked by using the following conditions.

Flow in the node \((i, j)\) is sub-critical, if —

\[ \sqrt{(Vx_{i,j}^2 + Vy_{i,j}^2)} \leq \sqrt{gh_{i,j}} \] ..........(3.3)

and, flow in the node \((i, j)\) is super-critical if

\[ \sqrt{(Vx_{i,j}^2 + Vy_{i,j}^2)} > \sqrt{gh_{i,j}} \] ..........(3.4)
The corrector step is then applied for computing corrected U matrix (denoted by UC), by using forward finite difference approximation, if Eq. 3.3 is satisfied. This implies corrector step with forward F.D. approximation is applied only when the flow is not in super-critical condition.

Thus,

\[
UC_{ij} = U_{ij} - \frac{\Delta t}{\Delta x} [EP_{i+1,j} - E_{ij}] - \frac{\Delta t}{\Delta y} [FP_{i,j+1} - FP_{ij}] - \Delta t(SP_{ij}) \quad (3.5)
\]

If Eq. 3.4 is satisfied then,

\[
UC_{ij} = UP_{ij} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.6)
\]

Then, the U vector, containing value of primitive flow variables in the next time step, is calculated using the following relation

\[
UN_{ij} = \frac{(UP_{ij} + UC_{ij})}{2} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.7)
\]

Where UN is the U matrix containing values of h, V_x and V_y in the next time step. With the above numerical scheme the values of h, V_x and V_y at different time levels and for different nodes can be computed explicitly.

3.5 INITIAL CONDITION:

In the proposed problem the computation domain lies only on the downstream side of the separating barrier. Therefore, at time \( t = 0 \), the value of primitive variables h, V_x and V_y at all the
nodes of the computation domain are equal. Thus, implementation of the proposed shock capturing scheme is not possible unless an initial profile corresponding to a small value of time $t$, is introduced as initial condition.

For obtaining this initial profile, analytical solution of dam-break problem, developed by Ritter$^{62}$, Dressler$^{19}$ and Su and burness$^{65}$ or Whitham's solution$^{89}$ for tip region, are generally used. Different investigators$^{1,9,22}$ have successfully applied these solutions for computing initial profile in dam-break flow simulation, both in one-dimensional and two-dimensional situation.

But, the proposed problem differs from the popular dam-break problem in the sense that here negative wave does not travel back towards the river side after removal of separating barrier. Therefore the characteristic based Ritter solution cannot be considered as applicable in the said situation. Dressler's perturbation solution also considers Ritter's solution as the zero order solution. Solution given by Su and Burness is only a modified form of dressler's solution. On the other hand so called Whitham's solution, in the true sense, is applicable in the tip region only. Thus none of the above solutions appear to be applicable for computation of initial profile for the problem under study. Therefore a new characteristic based analytical solution, applicable in case of dike breach situation, has been developed.
3.5.1 Analytical Solution of Flood Propagation due to Dike-Failure.

The analytical solution has first been developed for one-dimensional situation and then the developed solution has been applied to two-dimensional situation by assuming radial out flow from the opening.

For development of the one-dimensional solution following assumptions have been made.

1) The flow is one-dimensional.
2) The sectional area of flow is rectangular.
3) St.-Venant equations are valid.
4) Bed slope is small ($S_b \approx 0$).
5) Frictional Resistance does not influence the flow ($S_f=0$).
6) Lagrange's equation of celerity is applicable.

Starting from St.-Venant equations, and applying the above assumptions, the characteristic equations can be derived as

\[
\frac{dx}{dt} = V + C \quad (+\text{ve characteristic})
\]

\[
V + 2C = \text{Constant} \quad (+\text{ve Riemann invariant})
\]

\[
\frac{dx}{dt} = V - C \quad (-\text{ve characteristic})
\]

\[
V - 2C = \text{Constant} \quad (-\text{ve Riemann invariant})
\]

...(3.8)

Fig. 3.1 represents the flood movement due to dike failure and Fig. 3.2 represents characteristic curves for this situation.
Considering the situation just at the time of failure, we have at $x=0$ and $t = 0$
flow velocity $V_o=0$, and Celerity $C_o=\sqrt{gh_o}$
At the point $P$ in the $x$-$t$ plane at a distance $x$ and after a time $t$, let velocity and celerity be $V$ and $C$.
Considering positive characteristic through "P"

\[ V + 2C = V_0 + 2C_0 \]

\[ \Rightarrow V + 2C = 2C_0 \]

\[ \Rightarrow V = 2C_0 - 2C \] \hspace{1cm} ..........(3.9)

Again along the negative characteristic through point 'P'

\[ \frac{dx}{dt} = V - C \] \hspace{1cm} ..........(3.10)

Putting the value of \( V \) in terms of \( C_0 \) and \( C \) from Eq.3.9 in Eq.3.10

\[ \frac{dx}{dt} = 2C_0 - C - 2C \]

\[ \Rightarrow dx = (2C_0 - 3C) dt \]

Integrating

\[ x = 2C_0 t - 3Ct + A \] \hspace{1cm} ..........(3.11)

Putting boundary condition

at \( x = 0, \; h = h_0 \) and \( C = \sqrt{gh_0} \)

\[ \therefore 0 = 2 \sqrt{gh_0} t - 3 \sqrt{gh_0} t + A \]

\[ \Rightarrow A = \sqrt{gh_0} t \]

\[ \therefore \text{Eq 3.11 can be written as} \]

\[ x = 2C_0 t - 3Ct + \sqrt{gh_0} t \]

\[ \Rightarrow \frac{x}{t} = 2 \sqrt{gh_0} - 3 \sqrt{gh} + \sqrt{gh_0} \]

\[ \Rightarrow \frac{x}{t} = 3 \sqrt{gh_0} - 3 \sqrt{gh} \]

\[ \Rightarrow h = \frac{1}{9g} \left[ 3 \sqrt{gh_0} - \frac{x}{t} \right]^2 \] \hspace{1cm} ........(3.12)
Again

\[ V + 2C = V_o + 2C_0 \]
\[ \Rightarrow V + 2C = 2C_0 \]
\[ \Rightarrow 2C = 2C_0 - V \]
\[ \Rightarrow C = C_0 - \frac{1}{2} V \] ..........(3.13)

Considering negative characteristic through 'P'

\[ \frac{dx}{dt} = V - C \] ..........(3.14)

Putting the value of C in terms of \( C_o \) and \( V \) from Eq. 3.13 in Eq.3.14 we have

\[ \frac{dx}{dt} = V - C_o + \frac{1}{2} V \]
\[ \Rightarrow dx = \left( \frac{3}{2} V - C_o \right) dt \]

Integrating,

\[ x = \frac{3}{2} Vt - \sqrt{gh_o} t + B \] ..........(3.15)

Putting boundary condition

at \( x = 0 \), using the equation of broad crested weir, outflow velocity

\[ V = 0.544\sqrt{gh_o} \approx 0.5 \sqrt{gh_o} \]

\[ \therefore 0 = \frac{3}{4} \sqrt{gh_o} t - \sqrt{gh_o} t + B \]
\[ \Rightarrow B = \frac{1}{4} \sqrt{gh_o} t \]

From Eq. 3.15

\[ x = \frac{3}{2} V t - \sqrt{gh_o} t + \frac{1}{4} \sqrt{gh_o} t \]
Thus, the Eq. 3.12 and Eq. 3.16 give the values of flow depth 'h' and flow velocity 'V' at different times and at different distances.

3.5.2 Special Consideration For The Tip Region:

The analytical solutions, given by Eq. 3.12 and Eq. 3.16, are not applicable at the wave front as resistance cannot be negligible at the wave tip. Therefore, for computing velocity at the wave tip, the so called Whitham equation, as used by Akanbi and Katopodes, has been used (Eq. 3.17).

\[ h = \frac{7}{3} \left[ V_F^2 n^2 (x_{up})^{3/7} \right] \]

Where 'V_F' is the velocity of the wave front, 'n' is the roughness coefficient, 'x_{up}' is the position of wave tip at time 't' and 'h' is the depth at the beginning of the tip region.

The time required by the wave for reaching a predetermined position is calculated by considering the wave propagation speed as average of the velocity at the dike breach position and velocity of the wave front (V_F).
3.5.3. Application of The Proposed Analytical Solution in Two-dimensional Situation:

To apply the proposed one-dimensional solution in two-dimensional situation, assumption of radial outflow has been made. The outflow velocity in any radial direction is calculated on the basis of main stream velocity $V_y$ and lateral outflow velocity $V_x$ at the breach position. If $\theta$ is the angle made by the radial direction with the direction perpendicular to the main stream, then radial outflow velocity $V_r$ is calculated as (Fig. 3.3).

$$V_r = \begin{cases} 
V_x \cos \theta + V_y \sin \theta & \text{in the +ve region} \\
V_x \cos \theta - V_y \sin \theta & \text{in the -ve region} 
\end{cases}$$ ........(3.18)

Distance travelled by the flow in any radial direction in time $t$ is calculated as:

$$x_r = \frac{1}{2} (V_r + V) \times t .... (3.19)$$

![Fig. 3.3: POSITIVE AND NEGATIVE REGION IN THE FLOW FIELD](image)

Where $V$ is the tip velocity computed by Whitham's equation. Velocity and depth at any point in a radial direction is calculated on the basis of computed flow depth at the dike position corresponding to the outflow velocity in that radial direction. The computed depth at dike position ($h_o$) for any radial direction is obtained as
Thus depth and velocity at any point at a radial distance $x_r$ in the direction $\theta$ and at time $t$, can be computed by using the proposed analytical solution as

$$h = \frac{1}{9g} \left[ 3\sqrt{gh_{or}} - \frac{x_r}{t} \right]^2 \quad \ldots \ldots (3.21)$$

$$V_r = \frac{2}{3} \left[ \frac{x_r}{t} + \frac{3}{4} \sqrt{gh_{or}} \right] \quad \ldots \ldots (3.22)$$

$$V_x = V_r \cos \theta \quad \ldots \ldots (3.23)$$

$$V_y = \begin{cases} V_r \sin \theta, & +ve \text{ region} \\ -V_r \sin \theta, & -ve \text{ region} \end{cases} \quad \ldots \ldots (3.24)$$

3.6 BOUNDARY CONDITION :

**Upstream Boundary**: It has been assumed that the flood level in the main stream will remain constant for sufficiently long time, i.e, water level for the computational nodes lying in the breach position can be considered as constant. Outflow through the opening has been computed by using equation of broad crested weir. In case of gradual opening out flow at the breach point increases gradually with increase of breach width and attains an ultimate width after complete development of breach. For the other boundary nodes lying on the separating barrier, the velocity, normal to the barrier at each node, has been considered as zero. Thus at upstream nodes
\[ V_{x_{1,j}} = 0, \quad j = 1 \text{ to } n \] and
\[ V_{x_{1, Bj}} = 0.544 \sqrt{gh(1, Bj)} \]

Where Bj represents the nodes lying in the breached portion.

**Downstream Boundary :**

As the shock capturing scheme has been used, assuming a small depth of still water on the downstream side, the moving wave front does not introduce any complexity in implementation of the proposed numerical scheme. While modelling flood propagation in field situation free boundary has been considered for all the nodes lying on the downstream boundary. In case of numerical simulation of flow occurring in the physical model, reflection boundary has been used for all extreme boundary nodes at downstream to represent the physical boundary of the experimental set-up. Thus, if the physical boundary lies in the (n-1)th grid line in x direction, then

\[ h_n = h_{n-2} \]
\[ V_{x_n} = - V_{x_{n-2}} \]
\[ V_{y_n} = V_{y_{n-2}} \]

**Fig. 3.4 :** DOWNSTREAM BOUNDARY

**3.7 Stability Criteria :**

The Courant number, derived on the basis of linearised form of one-dimensional shallow water gradually varied flow equation,
has been used for deriving a Courant-like number for maintaining stability of the proposed two-dimensional scheme. The expression of Courant-like number for each direction (x and y) can be written as

\[
\begin{align*}
CN_x &= \frac{(V_x \pm C \cos \theta)}{\Delta x/\Delta t} \quad \text{...............(3.25)} \\
CN_y &= \frac{(V_y \pm C \sin \theta)}{\Delta y/\Delta t} \quad \text{...............(3.26)}
\end{align*}
\]

Where \(CN_x\) and \(CN_y\) are the Courant-like number in x and y directions. \(C\) is the celerity of wave propagation and \(\theta\) is the angle made by the resultant velocity at any node with x direction.

Thus \(\theta = \tan^{-1} \frac{V_y}{V_x}\) \text{ ........... (3.27)}

For the scheme to be stable the stability criteria, \(CN_x < 1\) and \(CN_y < 1\), must be satisfied at all grid points. Otherwise \(\Delta t\) must be reduced to make the scheme stable.

3.8 CONCLUSION:

The proposed two-dimensional model has specially been developed for simulating propagation of flood wave resulting from an opening in the river dike. However the developed model can be applied to other situation of dam-break flood also with a little modification in the boundary condition. Effort has also been made towards development of an appropriate analytical solution for this
kind of dam-break problem. This analytical solution has been used for providing the necessary initial condition to start the numerical solution. Use of shock capturing scheme has made the model simple and straightforward. The proposed scheme, being explicit, requires less computer space. Of course computation time required is little more as At cannot be increased to a very high value. However the scheme remains stable for large value of At if Ax and Ay are also comparatively large. Again from physical consideration, putting some restriction on the computed value of h, Vx and Vy, the stability criteria can be relaxed to some extent.