Chapter 1

Introduction

The discoveries of the many supposedly elementary particles in the 1950s and onward led to many speculations about the structure of hadrons, more importantly about the structure of nucleons. In 1964, M.Gell-Mann [1] and G.Zweig [2] independently suggested that the hadrons are composites of three more fundamental spin half particles called quarks which carry fractional electric charges and a new degree of freedom called flavour. This model, called the constituent quark model could explain most of the bulk properties of the hadrons like their charge, mass and spin. Since all elementary particles discovered till then carry an integral electronic charge and isolated quarks with fractional electric charges were not observed, even Gell-Mann himself did not believe the quarks to be physical particles but considered them to be convenient mathematical tools with which to model the properties of the hadrons. Then came a startling discovery in 1968 from Stanford Linear Accelerator Center (SLAC) [3, 4] in the deep inelastic scattering experiments where a beam of high energy ($20\text{GeV}^2$) electrons were scattered off a proton target. The experiment was reminiscent of the famous Rutherford alpha ray scattering experiment done half a century before and clearly indicated that there are scattering point-like spin $\frac{1}{2}$ centres inside the proton. This experiment was a strong evidence but not a conclusive one for the quark model: that quarks are real, not just mnemonic devices. These experiments were
repeated in early seventies using neutrino beams at CERN. The deep inelastic scattering (DIS) experiments between electron and proton, which take place through the exchange of a virtual boson had the remarkable property that the scattering cross-sections do not depend on the momentum transfer square ($q^2$) and the energy transfer ($\nu$) from the electron (lepton) to the proton but depend only on their ratio, a variable to be called the Bjorken variable. This property known as scaling was predicted by J. D. Bjorken [5] on the basis of current algebra and was a natural consequence if the proton had point like constituents. Richard P. Feynman explained these scattering experiments and the scaling phenomenon by introducing the parton model[6] of the proton, where the proton is viewed as a collection of almost free point-like constituents called partons off which the electrons scatter incoherently. These Feynman partons were to be identified later with quarks and gluons.

In the quark parton model (QPM) of Feynman, the nucleon is made up of three valence quarks which give the nucleon its flavour properties and a sea of quark-antiquark pairs which have no overall flavour, in contrast to only three constituent quarks in the original quark model of Gell-Mann and Zweig. The quarks in the Feynman's model carry a mass of a few Mev against a mass of about 300 MeV carried by the quarks of Gell-Mann and Zweig. To avoid a conflict with the Pauli's exclusion principle, an additional quantum number called colour was introduced. From a comparison with experiment, it was decided that each quark flavour must possess three colour degrees of freedom [7, 8]. The non-existence of free quark was tried to be explained away by the notion of quark confinement, a subject still not understood properly. The observation of scaling by the SLAC-MIT group, the fact that only half the nucleon's momentum is carried by quarks and the later observations made at FNAL and SLAC(1972) of logarithmic scaling violations were instrumental in developing a dynamical theory of the strong interaction, called Quantum Chromo- Dynamics (QCD) in analogy with the Quantum Electro-Dynamics(QED).
1.1 QCD as a theory of strong interaction

The Bjorken scaling observed in the deep inelastic scattering suggests that the quarks inside the proton are almost free and point like when observed at a high spatial resolution. Hence, if one accepts the parton idea, then the dynamics governing the parton system should have the remarkable property that the interaction gets weaker at short distances i.e. at large momentum transfer. Soon after the foundation of the parton model, the search was on for a theory of the interactions of the partons which can accomodate the above property of asymptotic freedom. The discovery in 1973 by Gross, Wilczek, Politzer [9, 10] and 't Hooft [11] that the non-Abelian gauge field theories have the required property of asymptotic freedom paved the way for the development of the theory of strong interaction now called Quantum Chromodynamics (QCD). In the mean time, Fritzsch and Gell-Mann proposed [12] that the extra symmetry associated with the non-Abelian gauge theory be identified with the colour symmetry possesed by the quark system. Thus QCD was developed as a non-Abelian field theory of strong interaction with six quark flavours based on the colour symmetry. QCD is motivated by the observation that colour is what distinguishes the quarks from the leptons. Since quarks experience the strong interactions but leptons do not, it is natural to regard colour as a strong interaction charge and to take the symmetry as a local gauge symmetry. The unitary symmetry group SU(3) was the natural choice for the gauge group of QCD, because, having a complex fundamental representation, it can distinguish between quarks and anti-quarks. The quarks transform under the fundamental representation 3 of the group and the anti-quarks transform under the conjugate representation 3. Just as the photon mediates the electromagnetic interaction between charged particles in QED, the colour interaction between quarks is mediated by the non-Abelian gauge field called the gluons which are responsible for binding the quarks together. But there is a crucial difference between the photon and the gluon that stems from the non-Abelian nature of QCD. While the photons are themselves electrically neutral, gluons carry colour charges and hence they couple and can interact with
each other even in the absence of quarks. This non-Abelian nature of the gluon-gluon coupling in QCD leads to anti-screening: the colour charge becomes weaker closer we go to the quarks. The gluon-gluon interactions which have no analogue in QED is responsible for giving strong interaction, the properties of asymptotic freedom at short distances and a possibility of quark confinement at long distances and makes QCD an useful tool to make reliable predictions using perturbative techniques for processes involving high momentum transfer. Moreover, the local gauge symmetry is assumed to be exact and there are no hadrons carrying a non-trivial colour quantum number.

The QCD Lagrangian is constructed along similar lines of QED. It has the form

$$\mathcal{L} = \sum_q \bar{q}_a(i\gamma^\mu D_\mu - m_q)_{ab} q_b - \frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu},$$

(1.1)

where the covariant derivative is given by

$$(D_\mu)_{ab} = \delta_{ab} \partial_\mu + ig_s t^A_{ab} G_\mu^A,$$

(1.2)

and the field strength tensor is given by

$$F^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - g_s f^{ABC} G^B_\mu G^C_\nu.$$

(1.3)

In Eq.(1.3), $G_\mu^A$s are the vector fields that represent the gauge particles of QCD i.e. the gluons, $f^{ABC}$ are the structure constants of SU(3) and $g_s$ is the coupling strength of the quark and gluon field. In the above equations $a, b = 1, 2, 3$ are the colour indices, $\mu, \nu = 0, 1, 2, 3$ are the Lorentz indices and $A, B, C = 1, 2, \ldots 8$ are the indices for the generators of $SU(3)$. The $3 \times 3$ matrices $t^A$ are the generators of the fundamental representation of the SU(3) group and they satisfy the commutation relations:

$$[t^A, t^B] = if^{ABC} t^C.$$

(1.4)
The three elementary vertices of QCD given by the Lagrangian (Eq.(1.1)) are shown in

![Elementary vertices of QCD as given by Eq.(1.1)](image)

Figure 1.1: Elementary vertices of QCD as given by Eq.(1.1)

Fig.1.1. The amplitudes for $qg$ and $ggg$ in Fig.(1.1)(a) and (b) are proportional to the
coupling $g_s$, whereas the four gluon vertex Fig.(1.1)(c) is proportional to $g_s^2$. The colour
factors can be determined from the properties of the generators by summing over possible
colour combinations for final state partons from these graphs. For three colours ($N_c = 3$)
these are:

\[
T_F = \frac{1}{2}, \\
C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}, \\
C_A = N_c = 3.
\] (1.5)

The colour factors give the probabilities of branching: $C_F$ for $q \rightarrow qg$, $C_A$ for $g \rightarrow gg$ and
$T_F$ for $g \rightarrow q\bar{q}$.

### 1.1.1 Running coupling and QCD beta function

The renormalized coupling constant $\alpha_s$ of the quark and gluon fields is given by

\[
\alpha_s = \frac{g_s^2}{4\pi}
\] (1.6)
The non-Abelian nature of QCD gives rise to gluon self coupling which in turn predicts the strong coupling constant \( \alpha_s(\mu^2) \) to decrease logarithmically as energy scale increases. The coupling constant determines the value of the \( q\bar{q}g \) vertex diagram. When one computes the series of loop Feynman diagrams (Fig. 1.2) for virtual corrections to the gluon propagator, the amplitudes become ultraviolet divergent. But fortunately these divergences can be absorbed by renormalization group method. As a result, the coupling constant runs \( \alpha_s(\mu^2) \) where \( \mu^2 \) is an arbitrary renormalization scale. The dependence of \( \alpha_s(\mu^2) \) on \( \mu^2 \) is given by the renormalization group equation (RGE)

\[
\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = \beta(\alpha_s).
\]

The function \( \beta(\alpha_s) \) known as the QCD beta function has a perturbative expansion in \( \alpha_s \)

\[
\beta(\alpha_s) = -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 - O(\alpha_s^4).
\]

The coefficients \( \beta_0, \beta_1, \ldots \) of the beta function depend on the renormalization scheme used, however the first two terms are scheme independent:

\[
\beta_0 = \frac{11C_A - 2n_f}{12\pi}, \quad \beta_1 = \frac{17C_A^2 - 5C_An_f - 3C_Fn_f}{24\pi^2},
\]

where \( n_f \) is the number of active flavours with mass sufficiently small compared to the energy scale of the process and \( C_A \) and \( C_F \) are the QCD colour factors given by Eq.
1.1. QCD as a theory of strong interaction

The quark loop correction of Fig. 1.2 gives positive contribution to the beta function whereas the gluon loop gives a negative correction so that the total beta function is negative as long as the condition $11C_A - 2n_f > 0$ i.e. $n_f < 33/2$ is satisfied. With the six known quark flavours QCD satisfies this condition. Thus the negative value of the beta function ensures that the coupling constant falls logarithmically as the momentum scale increases. This is the property of asymptotic freedom which makes QCD a useful tool for calculation in the perturbative region involving high momentum transfer.

The renormalization group equation can be solved to relate $\alpha_s$ at one scale $\mu^2$ to that at another scale $Q^2$. In leading logarithmic approximation $\alpha_s(Q^2)$ is given by

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \alpha_s(\mu^2) \log \frac{Q^2}{\mu^2}},$$

(1.10)

which shows that $\alpha_s(Q^2)$ depends on the choice of the renormalization scale $\mu^2$. We can choose $\alpha_s(Q^2)$ to be a fundamental parameter of QCD at a large mass scale for perturbative calculation to be valid. However, we can choose another parameter $\Lambda$ to replace $\alpha_s(\mu^2)$. This is defined by

$$\log \frac{Q^2}{\Lambda^2} = - \int_{\alpha_s(Q^2)}^\infty \frac{d\alpha_s}{\beta(\alpha_s)}.$$

(1.11)

This gives the leading order solution of Eq.(1.7) as

$$\alpha_s(Q^2) = \frac{1}{\beta_0 \log \frac{Q^2}{\Lambda^2}}.$$

(1.12)

The solution of Eq.(1.7) in the next-to-leading order is

$$\alpha_s(Q^2) = \frac{1}{\beta_0 \log \frac{Q^2}{\Lambda^2}} \left[ 1 - \frac{\beta_1 \log(\log \frac{Q^2}{\Lambda^2})}{\beta_0 \log \frac{Q^2}{\Lambda^2}} \right].$$

(1.13)

The parameter $\Lambda$ introduced by Eq.(1.11) is known as QCD cut-off parameter. Its value depends on the renormalization scale and on the number of active flavours. It contains
1.2 Deep inelastic scattering and the parton model

Since the initial observation of Bjorken scaling, Deep Inelastic Scattering (DIS) has been playing a pivotal role as a testing ground for the strong interaction dynamics and in the determination of momentum distributions of the partons of the nucleon. Precise knowledge of the partonic structure of the nucleon is very important as they enter into any hard processes involving the nucleon. In this section we describe the kinematics of inclusive deep inelastic scattering of a lepton from a nucleon [13, 14] in the light of naïve quark parton model.

The generic diagram for the scattering of a high energy lepton $e^\pm$ from a proton is shown in Fig.1.3. We denote the four momentum of the incoming electron by $k$, that of the outgoing electron by $k'$, that of the proton by $P$ and that of the virtual photon by $q$. The electron does not feel the strong interaction and its interaction with the proton is dominated by the electromagnetic interaction which is mediated by the virtual photon.
The photon is space like \( q^2 < 0 \). Thus the kinematics can be constructed from two four vectors: \( q^\mu \) and \( P^\mu \) and the inclusive deep inelastic scattering (DIS) by neutral current (NC) exchange can be described by two kinematic variables: the negative of the square of the four momentum transfer

\[
Q^2 = -q^2
\]  

and the Bjorken variable \( x \) defined as:

\[
x = \frac{Q^2}{2P.q}.
\]

In the absence of QED radiation, the other basic deep inelastic variables are:

\[
s \approx 4.E_e.E_p,
\]

\[
y = \frac{q.P}{M_p},
\]

\[
y = \frac{q.P}{k.P}
\]
and

\[ W^2 = \frac{Q^2(1-x)}{x} + M_p^2 \approx \frac{Q^2}{x} \quad \text{for} \quad x \ll 1, \quad (1.19) \]

where \( s \) is the square of the \( e-p \) c.m. energy, \( v \) is the energy transferred, \( y \) is the fractional energy transfer from the incident electron to the proton as measured in the proton rest frame, \( W \) is the \( \gamma^* p \) c.m. energy and \( M_p \) is the mass of the proton. Deep inelastic refers to \( Q^2 \gg M_p^2 \) and \( W^2 = (q + P)^2 \gg M_p^2 \) which ensure that perturbative QCD is applicable. We will see that in the deep inelastic region, the Bjorken variable \( x \) can be identified with the fractional momentum carried by the struck quark.

To the lowest order in electromagnetic coupling, the deep inelastic scattering represents the interaction of a leptonic current \( j^\mu \) with a hadronic current \( J^\mu \) through the exchange of the virtual photon. The differential cross-section for the process can be evaluated contracting the leptonic tensor \( L^{\mu\nu} \) prescribed by QED with the hadronic tensor \( W^{\mu\nu} \) describing the hadronic vertex. The cross-section is

\[ d\sigma \sim L^{\mu\nu} W_{\mu\nu}. \quad (1.20) \]

The hadronic tensor \( W_{\mu\nu} \) contains all the informations about the structure of the proton which are non-perturbative in nature and cannot be calculated in QCD. It is customary to express the hadronic tensor in terms of dimensionless scalar functions \( F_1, F_2 \) and \( F_3 \) called structure functions of the proton. Then the differential cross-section for the inclusive process \( ep \rightarrow eX \), neglecting the proton mass and considering only the exchange of \( \gamma^* \) is given by

\[ \frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{xQ^4} \left[ y^2 xF_1(x, Q^2) + (1-y)F_2(x, Q^2) \right], \quad (1.21) \]

where \( \alpha \) is the fine structure constant and \( F_i(x, Q^2) \) are the structure functions which are arbitrary functions of the kinematic variables \( x \) and \( Q^2 \) describing the transition \( \gamma^* p \rightarrow X \) and carry the information of the structure of the proton as seen by the virtual photon. In the Bjorken limit which is defined as \( Q^2, v \rightarrow \infty \) with \( x \) fixed, the structure functions obey
an approximate scaling law i.e. $F_i(x, Q^2) \rightarrow F_i(x)$ [5, 15]. The deep inelastic scattering in the parton model is best viewed in the 'infinite momentum frame' where the proton is moving with a very high velocity so that transverse momenta of the partons can be neglected. Then, at large $Q^2$ in the quark-parton model, the DIS can be viewed as the sum of incoherent elastic scattering of the photon from point-like quark constituents which behave as free particles during the interaction (Fig.1.4(a)). If $f_i(\zeta)$ is the probability of finding a quark $i$ carrying a momentum fraction $\zeta$ of the parent proton, then the scattering cross-section (Eq.1.21) can be written as

$$
\frac{d^2\sigma}{dx dQ^2} = \sum_i \int d\zeta f_i(\zeta) \frac{d^2\sigma_i}{dx dQ^2},
$$

the sum running over all quark flavours. In Eq.(1.22), $\frac{d^2\sigma_i}{dx dQ^2}$ is the differential cross-section of scattering of the electron and quark $i$. The elastic electron-quark scattering cross-section in terms of the Mandelstam variables

$$
\hat{s} = (k + \zeta P) = \frac{\zeta Q^2}{xy},
$$

$$
\hat{t} = (k - k')^2 = -Q^2
$$

$$
\hat{u} = (\zeta P - k')^2 = \hat{s}(y - 1)
$$

is given by

$$
\frac{d\sigma_i}{dQ^2} = \frac{4\pi\alpha^2}{\hat{t}} e_i^2 \frac{1}{2} \left( \frac{\hat{s}^2 + \hat{u}^2}{\hat{s}^2} \right),
$$

where $e_i$ is the fractional charge of the quark $i$. Since the struck quark acquires four momentum $(\zeta P + q)$, momentum conservation implies $(\zeta P + q)^2 = m^2$ where $m$ is the mass of the struck quark. Neglecting $m$ and in the deep inelastic region $Q^2 \gg M_p^2$, we get that $\zeta = \frac{Q^2}{2Qq}(=x)$. That is, the Bjorken variable $x$ is identified with the fraction $\zeta$ of the proton four momentum carried by the struck quark in the infinite momentum frame where masses may be disregarded. Using the Mandelstam variables in terms of the deep
inelastic scattering variables, i.e. $\hat{t} = -Q^2$ and $\hat{u}/\hat{s} = (y - 1)$, the double differential cross-section for quark scattering (Eq.(1.24)) is:

$$\frac{d^2\sigma_{el}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} e_i^2 \frac{1}{2} \left[ 1 + (1 - y)^2 \right] \delta(x - \xi),$$  \hspace{1cm} (1.25)$$

where the $\delta$-function arises because of energy momentum conservation. Inserting Eq.(1.25) into Eq.(1.22) and comparing with Eq.(1.21) we get

$$F_2(x) = 2xF_1(x) = \sum_i \int_0^1 d\xi f_i(\xi) e_i^2 \delta(x - \xi) = \sum_i xe_i^2 f_i(x).$$  \hspace{1cm} (1.26)$$

The equality $F_2(x) = 2xF_1(x)$ is the famous Callan-Gross relation in the naive quark parton model, which is a direct consequence of the spin $\frac{1}{2}$ property of the quarks. We also note that in the naive quark parton model, the structure functions scale and depend only on the scaling variable $x$.

1.2.1 QCD improved parton model and universal parton distributions

In the naïve quark parton model the structure functions scale. Contrary to such expectations of QPM the deep inelastic data show an approximate logarithmic dependence of the structure function on $Q^2$. Such scaling violations can only be accounted by a QCD improved parton model. In QCD, quarks can no longer be considered as free. Quarks radiate gluons and interact with each other through gluon exchange. Radiated gluons can in turn split into quark-antiquark pairs or gluons and so forth. With increasing $Q^2$, the virtual photon probes deeper and deeper inside the proton and resolves more and more sub-structure of it.

The relevant Feynman diagrams for correction at the first order of perturbative expansion in $\alpha_s$ are shown in Fig.1.5. Calculation of these graphs give divergent integrals.
1.2. Deep inelastic scattering and the parton model

Figure 1.5: QCD subprocess diagrams (a,b,c) of DIS at lowest order in $\alpha_s$, which contribute to $F_2$ that depend on quark and gluon densities of the proton.

Most of the singularities can however be removed. Ultraviolet singularities are removed by coupling constant renormalization. Infrared singularities cancel when we sum the virtual contributions given by diagrams in Fig.1.4(a),1.4(b) and real contributions given by diagrams in Fig.1.5(a),1.5(b). Final state mass or collinear singularities cancel in an inclusive process. This leaves us with the initial state mass or collinear singularity which arises when the gluon in $\gamma^* q \rightarrow qg$ is emitted parallel to the incoming quark (i.e. $k_T = 0$).

This divergence comes from the integration over the gluon transverse momentum spectrum $\int_{\mu^2}^{Q^2} \frac{dk_T^2}{k_T^2}$ and is proportional to

$$\frac{\alpha_s}{2\pi} \log \frac{Q^2}{\mu^2},$$

(1.27)

where $\mu^2$ is an arbitrary scale introduced to prevent the integral from blowing up, that is, introduced specifically to avoid the collinear singularity which occurs when $k_T \rightarrow 0$.

Taking these corrections, the parton model formula for structure function $F_2$ (Eq.1.26) becomes

$$F_2(x, Q^2) = x \sum_i \int_{x}^{1} \frac{dz}{z} f_i(z) e_i^2 \left[ \delta \left( 1 - \frac{x}{z} \right) + \frac{\alpha_s}{2\pi} \left\{ P_{ii} \left( \frac{x}{z} \right) \log \frac{Q^2}{\mu^2} + R \left( \frac{x}{z} \right) \right\} \right],$$

(1.28)

where $P_{ii}$ and $R$ are known calculable terms. The collinear singularities appearing in
the term $\log \frac{Q^2}{\mu^2}$ can be removed by factorizing them out of the partonic subprocess and absorbing into the bare parton distributions causing them to run. This renormalization is called "mass factorization". After mass factorization the renormalized parton density can be written as

$$f_q(x, Q^2) = f_q(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\zeta}{\zeta} f_q(\zeta) P_{qq} \left( \frac{x}{\zeta} \right) \log \frac{Q^2}{\mu^2}$$ \hspace{1cm} (1.29)

and the structure function as:

$$F_2(x, Q^2) = x \sum_q e_q^2 \int_x^1 \frac{d\zeta}{\zeta} f_q(\zeta, Q^2) \left[ \delta \left( 1 - \frac{x}{\zeta} \right) + \frac{\alpha_s}{2\pi} R \left( \frac{x}{\zeta} \right) \right],$$ \hspace{1cm} (1.30)

The mass factorization separates the short distance partonic effects from the long distance hadronic effects. The factorization scale at which this is done is arbitrary and is usually assigned a value of the order of the process energy scale. The renormalized parton distribution functions become scale dependent but are universal i.e. process independent. These depend only on the nonperturbative transition they describe but not on the hard scattering process. Once determined in one process, these can be used in any hard process involving the hadron. On the other hand, the coefficient functions contain all the information about the hard scattering process and is independent of the details of the non-perturbative transition.

### 1.2.2 Scaling violation and DGLAP evolution equations

Due to the renormalization at an arbitrary scale $\mu^2$ the absolute values of the parton distributions are not calculable in perturbative QCD. QCD rather determines their scaling violations i.e. evolution with $Q^2$. Given their values at an initial scale $Q_0^2$ where perturbative treatment is still justified, parton distributions at any higher scale can be determined by solving a system of integro-differential equations known as DGLAP [16, 17, 18, 19]
1.2. Deep inelastic scattering and the parton model

equations. The $Q^2$ evolution of the quark densities for the unpolarised case are given by:

$$
\frac{dq_i(x, Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int y \frac{dy}{x} \left[ \sum \frac{x}{y} q_j(y, Q^2) P_{qi}(x) + g(y, Q^2) P_{qg}(x) \right], \quad (1.31)
$$

where $q_i(x, Q^2)$ is the quark distribution of flavour $i$ and the sum runs over all quark flavours. For gluon distribution we have a similar equation

$$
\frac{dg(x, Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int y \frac{dy}{x} \left[ \sum \frac{x}{y} q_j(y, Q^2) P_{gq}(x) + g(y, Q^2) P_{gg}(x) \right], \quad (1.32)
$$

In Eqs.(1.31) and (1.32) the functions $P_{ij}$ are the Altarelli-Parisi splitting functions which have perturbative expansion in the strong coupling constant $\alpha_s$:

$$
P_{ij}(z) = P_{ij}^{(1)} + \frac{\alpha_s(Q^2)}{2\pi} P_{ij}^{(2)} + \ldots. \quad (1.33)
$$

At leading order the splitting function $P_{ij}(z)$ has an attractive interpretation as the probability of a parton of type $j$ emitting a parton of type $i$ with momentum fraction $z$ of the parent proton when the scale changes from $Q^2$ to $Q^2 + d\ln Q^2$. The LO [16] and NLO [20, 21, 22] contributions to the splitting functions have been calculated. At LO these are

$$
P_{qq}^{(1)}(z) = C_F \left[ \frac{(1+z)^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right],
$$

$$
P_{qg}^{(1)}(z) = T_R \left[ z^2 + (1-z)^2 \right], \quad (1.34)
$$

$$
P_{gg}^{(1)}(z) = C_F \left[ \frac{1 + (1-z)^2}{z} \right]
$$

and

$$
P_{gg}^{(1)}(z) = 2N_c \left[ \frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \delta(1-z) \frac{(11N_c - 4n_f T_R)}{6},
$$
The (+) prescription appearing in the above equations is defined as:

\[ \int_0^1 dz \frac{f(z)}{(1 - z)^+} = \int_0^1 \frac{f(z) - f(1)}{(1 - z)}. \]  

(1.35)

The interpretation of the splitting functions at LO as probabilities leads to the following sum rules:

\[ \int_0^1 dz \, P_{qq}^{(1)} = 0, \]  

(1.36)

\[ \int_0^1 dz \, (P_{qq}^{(1)}(z) + P_{gq}^{(1)}(z)) = 0, \]  

(1.37)

\[ \int_0^1 dz \, (2n_f P_{gg}^{(1)}(z) + P_{qg}^{(1)}(z)) = 0. \]  

(1.38)

The DGLAP equations are a system of 2n_f + 1 equations for the distribution functions of all flavours of quark, antiquark and gluons. However, all quark combinations do not evolve independently. Gluons transform as singlet under the SU(n_f) flavour group. In the quark sector it is convenient to define the non-singlet and the singlet combinations as:

\[ q^{NS}(x, Q^2) = \sum_{i \neq j=1}^{n_f} \left( q_i(x, Q^2) - q_j(x, Q^2) \right) \]  

(1.39)

and

\[ \Sigma(x, Q^2) = \sum_{i=1}^{n_f} \left( q_i(x, Q^2) + \bar{q}_i(x, Q^2) \right). \]  

(1.40)

Since the gluon emission is flavour independent, in the non-singlet distribution the gluon contribution cancels out and the evolution of the non-singlet is independent of the gluon. It is

\[ \frac{d}{dt} q^{NS} = \frac{\alpha_s(t)}{2\pi} \left[ P_{qq} \otimes q^{NS} \right], \]  

(1.41)
where \( t = \ln \frac{Q^2}{x} \). In the singlet case the quark and the gluon mix and their evolution equations are coupled

\[
\frac{d}{dt} \left( \begin{array}{c} \Sigma \\ g \end{array} \right) = \frac{\alpha_s(t)}{2\pi} \left( \begin{array}{cc} P_{qq} & 2n_f P_{gq} \\ P_{qg} & P_{gg} \end{array} \right) \otimes \left( \begin{array}{c} \Sigma \\ g \end{array} \right).
\]

The symbol \( \otimes \) in Eqs.(1.41) and (1.42) stands for the convolution integral with respect to the first variable \( x \) defined as

\[
(f \otimes g)(x) = \int_{x}^{1} \frac{dy}{y} f(y)g \left( \frac{x}{y} \right).
\]

The DGLAP equations have simple interpretation. The partonic subprocess is shown in Fig.1.5. The quark carrying a fraction \( x \) of the proton’s momentum is struck by the virtual photon (Fig.1.5(a)). But that quark might have originated from another quark carrying a larger momentum fraction \( y > x \), which has radiated a gluon before it is struck. This is the familiar QCD Compton effect shown in Fig.(1.5(b)). Or the struck quark of momentum fraction \( x \) may be originated from a gluon carrying a larger momentum fraction \( y \) by the process \( g \rightarrow q\bar{q} \). This is the boson-gluon fusion (BGF) (Fig.1.5(c)). Both these processes are accounted in Eq.(1.31). Effectively the DGLAP equations at LO resum all the leading \((\alpha_s \ln \frac{Q^2}{\mu^2})^n\) singular terms. Dokshitzer[19] showed that in an axial gauge where only two physical polarisation states of the gluon exist, the leading \( \ln Q^2 \) contributions correspond to the sum of the ladder diagrams with \( n \) rungs in which the transverse momenta of the emitted partons are strongly ordered along the chain (Fig.1.6) i.e.

\[
Q^2 \gg k_{nT}^2 \gg \ldots k_{2T}^2 \gg k_{1T}^2.
\]

The NLO contribution sums up \((\alpha_s)^n(\ln \frac{Q^2}{\mu^2})^{n-1}\) terms where we lose one power of \( \ln Q^2 \). This happens when a pair of adjacent momenta are comparable \( k_{iT} \approx k_{iT+1} \). Usually the series is terminated at a given power of \((\alpha_s)^n\) which introduces a scheme dependence.
of order $(\alpha_s)^{m+1}$. We use the $\overline{MS}$ scheme. The evolution of the parton distribution by DGLAP equations are causal in the sense that given their values at an initial scale $Q_0^2$, their values at higher scales can be determined. The parton distributions of the proton are determined from a global analysis of all deep inelastic and hard related processes. In the usual procedure the $x$ dependence of the parton distributions are parametrized at some low scales $Q_0^2$, but sufficiently large enough for perturbative QCD to be applicable. Then the parton distributions are evolved using the NLO DGLAP evolution equations to determine these at all values of $x$ and $Q^2$ where deep inelastic and related data exist and then an optimal fit to global data are performed to find the parameters. Usually the DGLAP equations are solved either by numerical integrations or by analytical methods in the Mellin moment space where the convolution integrals turn into ordinary products. Then the solutions are obtained in the $x$ space by inverse Mellin transformations. In our analysis
we solve the equations in the \((x, t)\) space by applying the method of characteristics. We give a brief introduction about this method in §1.5.

1.3 Polarised parton distributions

In polarised DIS, a beam of longitudinally polarised leptons collide with a nucleonic target polarised either longitudinally or transversely to an arbitrary direction. Polarised DIS provides a more complete insight into the structure of the nucleon than unpolarised DIS (for a review see [23]). The hadronic tensor \(W_{\mu\nu}\) in Eq.(1.20) that appears in the cross-section of any DIS can be separated into a symmetric part and an antisymmetric part. The anti-symmetric part contains all the information about the spin structure of the proton and can be expressed in terms of two invariant structure functions \(g_1(x, Q^2)\) and \(g_2(x, Q^2)\). The structure function \(g_1(x, Q^2)\) is related with the longitudinal polarisation of the proton spin with respect to its momentum. In the Bjorken limit where \(Q^2\) and \(v\) are large and the Bjorken variable \(x = \frac{Q^2}{2p\cdot q}\) is fixed, the two structure functions scale. In the quark parton model (QPM), the structure function \(g_1(x)\) effectively measures the quark helicity density and is given by:

\[
g_1^p(x) = \frac{1}{2} \sum_{i=1}^{n_f} e_i^2 (\Delta q_i(x) + \Delta \bar{q}_i(x)), \tag{1.44}
\]

where

\[
\Delta q_i(x) = q_i^+(x) - q_i^-(x), \quad \Delta \bar{q}_i(x) = \bar{q}_i^+ - \bar{q}_i^-(x) \tag{1.45}
\]

Here \(q_i^+(x)\) and \(q_i^-(x)\) are the distributions of quarks with helicities parallel and antiparallel to the proton spin and \(e_i\) is the fractional charge of the quark. Similar to the polarised quark distribution, the polarised gluon is defined as

\[
\Delta g(x) = g^+(x) - g^-(x) \tag{1.46}
\]
where $g^+(g^-)$ is the gluon density with spin parallel (antiparallel) to the proton spin. As in the case of unpolarised distribution, here also it is convenient to introduce flavour nonsinglet and singlet quark contributions by

$$
\Delta q^{NS}(x) = \sum_{i=1}^{n_f} \left( \frac{e_i^2}{<e^2>} - 1 \right) (\Delta q_i(x) + \Delta \bar{q}_i(x))
$$

and

$$
\Delta \Sigma(x) = \sum_{i=1}^{n_f} (\Delta q_i(x) + \Delta \bar{q}_i(x))
$$

where $<e^2> = \frac{1}{n_f} \sum_{i=1}^{n_f} e_i^2$. In QPM, polarised gluon does not enter the structure function $g_1(x, Q^2)$. In the QPM, the integration of the singlet polarised quark distribution over $x$ gives the fraction of the nucleon spin carried by the quarks

$$
\Delta \Sigma = \int_0^1 dx \sum_{i} (\Delta q_i(x) + \Delta \bar{q}_i(x)).
$$

Interpreting the structure function data with the above identification leads to the proton spin problem discovered by EMC. However, in the QCD improved parton model this identification no longer holds. Beyond the quark parton model, the QCD introduces a momentum scale ($Q^2$) dependence into the structure function. It is given as a convolution of both the quarks and the gluon helicities distribution:

$$
g_1(x, Q^2) = \frac{1}{2} \sum_{q=1}^{n} \langle z \rangle \left[ C_q \otimes (\Delta q + \Delta \bar{q}) + \frac{1}{n_f} C_g \otimes \Delta g \right]
$$

where $C_{q,g}$ are the Wilson coefficients. The convolution $\otimes$ is defined as in Eq.1.43.

$$(C \otimes q)(x, Q^2) = \int_x^1 \frac{dz}{z} C \left( \frac{x}{z}, \alpha_s \right) q(z, Q^2)$$
The polarised parton densities evolve according to the DGLAP equations. The nonsinglet distribution evolves independently as

$$\frac{d}{dt} \Delta q^{NS}(x, t) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} \Delta P_{qq}^{NS}(\frac{x}{y}) \Delta q^{NS}(y, t).$$ (1.52)

In the singlet sector quarks and gluon mix and evolve as

$$\frac{d}{dt} \begin{pmatrix} \Delta \Sigma(x, t) \\ \Delta g(x, t) \end{pmatrix} = \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{dy}{y} \begin{pmatrix} \Delta P_{qq}^{S}(y) & 2 n_f \Delta P_{qg}^{S}(y) \\ \Delta P_{gq}^{S}(y) & \Delta P_{gg}^{S}(y) \end{pmatrix} \begin{pmatrix} \Delta \Sigma(x/y, t) \\ \Delta g(x/y, t) \end{pmatrix},$$

(1.53)

where $t = \ln \frac{Q^2}{\Lambda^2}$ and $\Delta P_{ij}$ are polarised splitting functions which are known at LO [16] and NLO [24, 25, 26]. Starting with some initial distributions, these evolution equations can be solved under some approximations of the splitting functions. In our analysis we will concern ourselves only with the singlet evolution equations and find analytic solutions for $\Delta \Sigma(x, Q^2)$ and $\Delta g(x, Q^2)$.

### 1.4 DIS Experiments

Prior to the advent of HERA in 1992, the DIS experiments were fixed target experiments where a beam of high energy electron, positron or neutrino were scattered off a target of hydrogen or deuterium. From the measured cross-sections, the structure functions and the parton distributions are extracted by a global fit to the data with NLO evolution of the DGLAP equations. The $x$ and $Q^2$ range of these fixed target unpolarised scattering experiments and some of the characteristics are listed in table 1.1. Due to the small energy available for collision, the fixed target experiments are restricted to the high $x$ and low $Q^2$ region of the kinematic domain of DIS and served significantly only for the determination of the valence distributions of the partons. To increase the resolving power of the probe and go deep inside the hadron, it became necessary to increase the energy available in
1.4. DIS Experiments

Figure 1.7: Kinematical region covered by the HERA and the fixed target experiments. [Figure taken from hep-ex/0105055]

The Hadron-Elektron-Ring-Anlage (HERA), situated at DESY in Hamburg is the first electron proton collider where electrons (electrons were replaced by positrons from 1994) and protons are accelerated in two separate storage rings to final energies of $27.5\text{GeV}$ and $820\text{GeV}$ respectively. The available centre of mass energy is about $300\text{GeV}$. The beams are brought together for collision at four intersection regions. The H1 and the ZEUS experiments done at two such intersecting regions are dedicated to the study of unpolarised $\text{e-p}$ collision. Since the initial run of HERA in 1992, the H1 [27, 28, 29, 30, 31, 32, 33] and the ZEUS [34, 35, 36, 37, 38] group have published data on the structure function $F_2(x, Q^2)$, its slopes $\frac{\partial F_2(x, Q^2)}{\partial \ln Q^2}$ and $\frac{\partial F_2(x, Q^2)}{\partial \ln^2 x}$ and the strong
coupling constant $\alpha_s(Q^2)$ over a wide kinematic range of $x$ and $Q^2$ previously unexplored by the fixed target experiments. The data from these experiments are widely used together with the data from the earlier fixed target experiments to determine the parton distribution functions by various groups\[39,40,41\]. The range of $x$ and $Q^2$ in the two experiments H1 and ZEUS at HERA is displayed in Fig.1.7 along with the ranges of earlier fixed target experiments.

Pioneering experiments in polarised DIS were first carried out at SLAC \[42,43,44\] in the late seventies of the last century and then by the SLAC-Yale group \[45,46\], where a beam of longitudinally polarised electrons with energy $10 - 26 GeV$ were scattered off a longitudinally polarised target of hydrogen. These experiments revealed large spin dependence in $e - p$ scattering. Following the experiments at SLAC, the European Muon Collaboration(EMC) at CERN \[47,48\] did the first experiments at $x < 0.1$ with a beam of polarised muon of energy $100 - 200 GeV$ on a beam of longitudinally polarised hydrogen target and obtained the surprising result that in the framework of asymptotic QCD, only a fraction of the nucleon spin could be attributed to the intrinsic spin of the quarks. This historic phenomenon labelled as “spin crisis” led to tremendous theoretical and experimental activities in spin physics. These measurements were confirmed by subsequent measurements at CERN, SLAC and DESY. The Spin Muon Collaboration (SMC) experiments \[49,50,51,52\] done at CERN was a follow-on experiment to the EMC spin experiment and they reached a low value of $x < 0.01$. These experiments measure the spin structure

Table 1.1: Fixed target DIS experiments and their $x$- $Q^2$ range for determination of $F_2(x, Q^2)$ data.

<table>
<thead>
<tr>
<th>Beam</th>
<th>Targets</th>
<th>Experiments</th>
<th>$Q^2(GeV^2)$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>p,d,A</td>
<td>SLAC</td>
<td>$0.6-30$</td>
<td>$0.07-0.8$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>p,d,A</td>
<td>BCDMS</td>
<td>$7.5-230$</td>
<td>$0.07-0.6$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>p,d,A</td>
<td>NMC</td>
<td>$0.5-75$</td>
<td>$0.006-0.6$</td>
</tr>
<tr>
<td>$\nu, \bar{\nu}$</td>
<td>Fe</td>
<td>CCFR,CDHSW</td>
<td>$1.0-500$</td>
<td>$0.015-0.6$</td>
</tr>
</tbody>
</table>

1.4. DIS Experiments
function $g_1^{P\prime}(x, Q^2)$ and its first moment.

Since the publication of the proton spin crisis by the EMC and its subsequent verification by other groups, the gluon was suspected to contribute significantly to the spin of the nucleon. An important goal of present day spin experiment is to measure directly the gluon polarisation. But the polarised gluon distribution is difficult to access experimentally and the limited range of $Q^2$ available makes $\Delta g$ almost unconstrained. Several experiments are currently underway to constrain the gluon polarisation of the proton through some less inclusive processes. The COMPASS (Common Muon Proton Apparatus for Structure and Spectroscopy) experiment at CERN [53] is one such program which uses a high energy (190GeV) polarised muon beam on polarised targets to extract the gluon polarisation asymmetry $A_G(x, t) \equiv \frac{\Delta G(x, t)}{G(x, t)}$ through charm production in photon-gluon fusion process. At DESY HERA, the HERMES (HERa MEasurement of Spin)[54] experiment has been taking data since 1995 using the electron and positron beam from the HERA ring. HERMES is a fixed target experiment that uses the 27.5GeV HERA $e^\pm$ beam on polarised fixed targets ($H, D, He^3$) (for a recent review see [55]). HERMES experiment presently has focussed on semi-inclusive measurement in which the scattered $e^\pm$ and a forward hadron are detected from which the gluon polarisation asymmetry $A_G(x, t)$ and the gluon polarisation $\Delta g(x, t)$ can be extracted [56]. The kinematic range of HERMES measurement is $0.1 < Q^2 < 15GeV^2$ and $x > 0.02$.

1.5 Method of characteristics

The method of characteristics is a powerful tool for solving partial differential equations [57, 58, 59]. In our work we will be concerned only with the solution of first order linear partial differential equations involving two variables. The basic philosophy of the method of characteristics is very simple: The integrations of partial differential equations are reduced to a family of initial value problems for a system of ordinary differential equations.
That is effected by changing the old variables (say $x, t$) to appropriate new variables (say $s, \tau$) where the partial differential equation becomes an ordinary differential equation with respect to any one of the new variables. Thus the problem of solution of partial differential equation is reduced to that of ordinary differential equations. To illustrate the method, we consider the first order partial differential equation for the function $u$ in two variables $x, t$:

$$a(x, t)u_x + b(x, t)u_t + c(x, t, u) = 0,$$  \hspace{1cm} (1.54)

where $u_x, u_t$ are derivatives of $u$ w.r.t. $x$ and $t$ respectively. When $c(x, t, u)$ is linear in $u$ then the equation is called a linear equation. From the theory of partial differential equations we know that most of the important properties of the solution depend on the principal part of the equation i.e. the part containing the highest order terms appearing in the equation. In Eq.(1.54), the principal part is $a(x, t)u_x + b(x, t)u_t$. If $u = u(x, t)$ is a solution of the above equation, then at any point $(x, t, u)$ of the solution surface, the vector $(u_x, u_t, 1)$ is along the normal to the surface $u = u(x, t)$. Again the left hand side of Eq.(1.54) is the dot product of the two vectors $(a, b, c)$ and $(u_x, u_t, 1)$ and the equation says that the dot product of the two vectors vanishes. Hence the vector $(a, b, c)$ is perpendicular to $(u_x, u_t, 1)$ and lies along the tangent plane of the surface $u = u(x, t)$. Thus any solution surface of the differential equation must be tangent to a vector defined by $(a, b, c)$ appearing in the differential equation and it changes along the tangent to a particular member of a family of curves defined by the system of ordinary differential equations:

$$\frac{dx}{dt} = \frac{a}{b}.$$  \hspace{1cm} (1.55)

The one parameter family of curves defined by the above equations is called the characteristic curves of the differential equation and along these curves the partial differential equation (Eq.(1.54)) becomes an ordinary differential equation. Theory of ordinary differential equation suggests that for most well behaved functions, there is one and only
one curve through any point. Thus, prescribing \( u \) at any point on the characteristic curve should enable one to find the function at all points on the characteristic curve. Or if the function \( u \) is given on a curve \( G \) which does not intersect any characteristic twice and which is nowhere tangent to a characteristic, then \( u \) can be found in a region covered by the characteristic curves through \( G \). That is, the problem of the solution of the equation is reduced to the Cauchy initial value problem where for a first order partial differential equation in two independent variables, we ask for a solution \( u \) of the differential equation in a domain containing a curve on which values of the function \( u \) are assigned. This curve is called the initial curve (Fig. 1.8). One method of solution of the differential equation is through the introduction of two new independent variables with one of them being constant on the characteristic curves. Let \( s, \tau \) be two new variables such that \( s \) changes along the characteristic curves of the differential equation and \( \tau \) changes along the initial curve. Then the characteristic curves (Eq.(1.55)) are defined by setting

\[
\frac{dx}{ds} = a(x, t) \quad (1.56)
\]

and

\[
\frac{dt}{ds} = b(x, t). \quad (1.57)
\]

On using these Eqs.(1.56), and (1.57), Eq.(1.54) becomes an ordinary differential equation:

\[
\frac{du}{ds} + c(x, t)u = 0. \quad (1.58)
\]

Solution of Eqs.(1.56) and (1.57) give the characteristic curves of Eq.(1.54). To find out the constants of integration Eqs.(1.56) and (1.57) we set on the initial curve (i.e. \( s = 0 \)) \( x(s = 0) = \tau, t(s = 0) = 0 \), where \( \tau \) is a parameter on the initial curve[57]. Then the solutions of Eqs.(1.56) and (1.57) are:

\[
x = x(s, \tau), \quad (1.59)
\]
1.5. Method of characteristics

Figure 1.8: Schematic diagram of a characteristic curve along which a partial differential equation becomes an ordinary differential equation.

\[ t = t(s, \tau). \] (1.60)

With the help of Eqs.(1.59) and (1.60), \( c(x, t) \) appearing in Eq.(1.58) can be expressed in terms of \( s, \tau \) and the equation can be integrated along the characteristic curves to find the solution \( u = u(s, \tau) \) given the value of the unknown function \( u \) on the initial curve. The solution is then found as a function of \( x \) and \( t \) by inverting the relations Eq.(1.59) and Eq.(1.60) to give \( s \) and \( \tau \) as functions of \( x \) and \( t \). Thus, using a single boundary condition, which is the value of the function on the initial curve, we can solve the first order differential equation for a function of two independent variables. The only problem we might have is in the inversion of Eqs.(1.59) and (1.60). In order that the inversion can be carried out at a given point, the Jacobian must be non-zero, i.e.

\[ J = x_s t_\tau - t_s x_\tau = \begin{vmatrix} x_s & t_s \\ x_\tau & t_\tau \end{vmatrix} \neq 0. \] (1.61)

If the condition (i.e. \( J \neq 0 \)) is not fulfilled, then the initial curve itself becomes a characteristic and there exists no unique solution of the differential equation or there exists an infinite number of solutions of the initial value problem.

We have discussed how a linear partial differential equation for a single function of
two independent variables can be solved by the method of characteristics. This method can be extended to the case where we have a system of first order coupled partial differential equations in \( n \) unknowns of two independent variables. Specifically, we will be interested when we have two unknowns of two independent variables described by the equation:

\[
\bar{\vec{u}}_x + A\bar{\vec{u}}_t = \bar{\vec{c}},
\]

(1.62)

where \( \bar{\vec{u}} \) is a two component column matrix containing the two unknowns, \( A \) is a \( 2 \times 2 \) matrix and \( \bar{\vec{c}} \) is another two component column matrix. For a linear system, \( A \) is independent of the unknowns \( u_1, u_2 \) and \( \bar{\vec{c}} \) involves \( u_1 \) and \( u_2 \) only linearly. The system (Eq.1.62) can be reduced to the canonical form by the introduction of new unknowns. The resulting independent equations then can be solved by the method of characteristics. We describe this method briefly in chapter five where we encounter such a system while solving the coupled DGLAP equations for the quark singlet and the gluon.

1.6 Outline of the thesis

The parton distribution functions of the proton are crucial for understanding its internal structure and are essential ingredients for any hard scattering involving the proton. In QCD their absolute values are not calculable, but given their values at an initial scale we can predict their values at any higher scale using the renormalization based DGLAP evolution equations. In this thesis we use the method of characteristics of partial differential equations to find some analytic solutions of the DGLAP equations at low \( x \) for both unpolarised and polarised case. Our general agenda is like this: By a Taylor series expansion valid to be at low \( x \), we convert the integro-differential DGLAP equations into partial differential equations in two variables: Bjorken \( x \) and \( t (= \ln\frac{Q_x^2}{\Lambda^2}) \). This conversion enables one to calculate also the \( x \) evolution from DGLAP equation at low \( x \) beyond its traditional use in \( t \) evolution only. Employing the method of characteristics, the resulting
partial differential equations are converted into ordinary differential equations in a new space where we can solve them with one boundary condition. This boundary condition can be any non-perturbative input distribution of the gluon and the singlet available in the literature. The solution is then transformed back into the \((x, t)\) space using the inverse transformation equations.

In chapter 2 we consider the LO DGLAP gluon evolution equation and find an approximate analytical solution under certain approximations valid to be at low \(x\) by the application of method of characteristics. In some earlier solutions it was assumed without any physical basis that the \(x\) and \(Q^2\) dependence of the gluon can be factorized. By applying the method of characteristics we have shown that without any such \textit{ad-hoc} assumption a unique solution can be found out. We also discuss the convergence of the Taylor series of the gluon considering a most general form of gluon distribution. At LO the scaling violation of the structure function is directly related with the gluon i.e. \(\frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} \propto xg(x, Q^2)\).

We test the validity of our analytical solution by calculating the slope of the structure function and comparing with the HERA data. The result is also compared with the standard DLA result and the exact solution like GRV parametrization of the gluon.

One of the initial observation of HERA structure function measurement was the rise of \(F_2(x, Q^2)\) towards low \(x\) at fixed \(Q^2\). The rise becomes steeper with increasing \(Q^2\) and it continues down to very low \(Q^2 \sim 1 GeV^2\). Both H1 and ZEUS collaboration have extracted the derivative \(\frac{dF_2(x, Q^2)}{d \ln Q^2}\) at fixed \(Q^2\) and \(\frac{dF_2(x, Q^2)}{d \ln Q^2}\) at fixed \(x\). These measurements show that the structure function can be parametrized as \(F_2 \sim x^{-\lambda(x, Q^2)}\). The exponent \(\lambda(x, Q^2)\) being directly measurable from the structure functions gives us helpful insights about the behaviour of structure function specially at low \(x\). In chapter 3 we calculate this exponent both as a function of \(x\) at fixed \(Q^2\) and as function of \(Q^2\) at fixed \(x\) using our gluon momentum distributions derived in the chapter 2. We carry out phenomenological tests of the results using HERA data.

In chapter 4 we extend our LO analysis to NLO of perturbative expansion. The
DGLAP gluon evolution equation at next-to-leading order is solved analytically by applying the method of characteristics. Compatibility of our result with the DLA asymptotic is discussed and comparison with the exact ones like GRVNLO is made. The quark singlet part which was neglected earlier is also taken into account and its effect on the gluon distribution calculated. The result is used to calculate the slope of the structure function using the scaling violation relation and then compared with the HERA structure function data. We also check the consistency of the method applied at NLO by showing that the LO result obtained in chapter 2 can be directly deduced from the NLO result derived in this chapter.

In the previous chapters, we considered only the gluon evolution equation and derived its analytic solution in LO and NLO by applying the method of characteristics. But in the singlet sector the gluon and the singlet quarks mix non-trivially and evolve according to a pair of coupled integro-differential equations. Evolution of the gluon can not be separated from the singlet quarks and vice-versa. In chapter 5, we apply the method already developed, to solve the pair of coupled evolution equations for the gluon and the quark singlet. By a Taylor series expansion valid to be at low $x$, we transform the integro differential singlet evolution equations into a pair of coupled partial differential equations. The equations are reduced to canonical forms and then simultaneously solved by applying the method of characteristics. The results are presented both in analytic and semi-analytic forms. These are compared with the exact result and the HERA structure function data. Consistency of the result derived in this chapter with the earlier results are also discussed.

In chapter 6, we consider the polarised singlet evolution equation at LO. Applying the same technique, we obtain analytic expressions for the polarised singlet quark and gluon distribution. The results are compared with the exact solutions of polarised singlet equations and with the available data on the quark helicity distribution and gluon polarisation.

In chapter 7, we give a summary of our work and discuss the scope of further work that can be done in this line.