# Chapter 7

# Summary and outlook

The partonic structure of the nucleon is an essential ingredient in any hard process involving the proton and hence its precise determination has a fundamental role in understanding the structure of the nucleon and the underlying dynamics of quarks and gluons. Deep inelastic scattering have been playing a major role in exploring the partonic structure of the nucleon and establishing QCD as a theory of strong interaction. The parton distribution functions of the nucleon are universal, i.e. once they are determined in a process, they can be used in any process involving the nucleon. However, in QCD, these functions are not calculable from first principle. Given their values at an initial scale $Q_0^2$, which are to be determined from experiment, the DGLAP equations predict their evolution to any higher scale $Q^2$. Thus, the DGLAP equations serve as important theoretical tools in determining the parton distribution functions of the nucleon. These equations ($2n_f + 1$ numbers, $n_f$= number of flavours) are integro-differential equations that involve convolution integrals involving the parton distribution functions and the Altarelli-Parisi splitting functions. There are different techniques to solve these equations. One can solve them numerically by performing the convolution integrals starting from some input distributions obtained from experiment. Alternatively, analytical solutions can be found in some regions of $x - Q^2$ space. Analytical solutions are usually obtained in the Mellin moment space where the convolution products turn into ordinary products.

In this thesis, we have presented analytical solutions of the gluon evolution equations in LO and NLO and of the singlet coupled evolution equations both for unpolarized and polarized case. Instead of working in the Mellin-moment space, we have solved these equations in $(x, t)$ ($t = ln\frac{Q^2}{\Lambda^2}$) space itself applying the method of characteristics and considering the $x$ space expressions of the splitting kernels. Our method has been like this: By a Taylor series expansion of the gluon $G(x/z, t)$ and the singlet $F_2^S(x/z, t)$ ($\Delta G(x/z, t)$ and $\Delta F^S(x/z, t)$ in polarized case) that appear under the convolution integrals, we converted the integro-differential DGLAP equations into partial differential equations in two variables $x$ and $t$. In doing so, the Taylor series are cut after the second term, which are valid assumptions at low $x$ and the convergence criteria discussed taking most general form of input distributions. The resulting equations thus allowed both $x$ and $t$ evolution contrary to only $t$ evolution in traditional case. However, solution of the first order partial differential equation in two variables needs two boundary conditions whereas we have only one at our hand, the non-perturbative input distribution. The problem is kind of a initial value problem where we ask for a solution of a first order partial differential equation on a curve provided the value of the unknown function is assigned at some initial point of the curve. The method of characteristics of partial differential equation is a powerful tool to handle such situation and we have applied this method to derive analytic solutions of the evolution equations.

In the first chapter, we have presented a brief introduction of the parton model interactions of quarks and gluons, kinematics of DIS of leptons off a proton and the DGLAP evolution equations. We have also briefly mentioned various DIS experiments data from which we have utilized in phenomenological analysis of our results. A brief introduction about the method of characteristics is also given, which we have used all throughout our work.

In chapter 2, we have presented an analytic solution of the LO gluon evolution equation valid to be at low $x$. The quark singlet part is neglected which is quite justified at low $x$. The first order partial differential equation that is obtained from the evolution equation by Taylor series expansion is converted into an ordinary differential equation by introducing two new variables $(s, \tau)$. The solution obtained in $(s, \tau)$ space is transformed back to the $(x, t)$ space by using the inverse of the characteristic equations. The result is compared with earlier results[64, 65, 72, 71] and standard DLLA [60]. We have also used the result to calculate the slope $\frac{\partial F_2}{\partial lnQ^2}$ of the structure function using the scaling violation relation [73]. Comparing the slope with the H1[27] and ZEUS [34] data , specifically we have demonstrated that $\alpha$, the point of expansion of the gluon in the scaling violation relation is an important one and our analytical solution can reproduce the data fairly well if $\alpha$ is chosen around $\approx 0.7$. This value is in conformity with the result obtained from Glauber-Müller [75, 77] approach and also with that of Ryskin *et al*[78], which shows that the longitudinal gluon momentum is approximately three times larger than the Bjorken $x$ of the quark or anti-quark probed in DIS. The result in this chapter has also overcome two drawbacks of earlier solutions in refs.[64, 65, 72, 71]: the result obtained here is unique and there is no *ad-hoc* assumption of the factorizability of $x$ and $Q^2$ dependence of the gluon. Clearly, this is an improvement over earlier results.

The rise of the structure function $F_2(x, Q^2)$ toward low $x$ down to very small values of $Q^2 \sim 1GeV^2$ is an important observation of HERA experiment and the exponent $\lambda(x, Q^2)$ in $F_2 \sim x^{-\lambda(x,Q^2)}$ has played as a crucial discriminator for such behaviour. In chapter 3, we have calculated the exponent using the result derived in chapter 2 and the scaling violation [73]. Comparing with H1 [33] data on the exponent in the kinematic range $3. \times 10^{-5} \leq x \leq 0.2$ and $1.5GeV^2 \leq Q^2 \leq 150GeV^2$, we have found that the calculated value of $\lambda(x, Q^2)$ compares well with data for $x \lesssim 10^{-2}$ and $Q^2 \geq 8.5GeV^2$. However, in this kinematic range, our observation suggests a $x$-dependent form of the exponent as $\lambda(x, Q^2) \sim a(x)ln\frac{Q^2}{\Lambda^2}$ rather than $\lambda(Q^2) \sim aln\frac{Q^2}{\Lambda^2}$ ($a = 0.0481$) obtained by H1 [33], where $a(x)$ is a very slowly falling function of $x$ with increasing $x$. Such $x$ dependence of the derivative $\lambda(x, Q^2)$ is also suggested by Desgrolard *et al.*[84] based on various Regge type models [85, 86].

Chapter 4 deals with the NLO gluon evolution equation. Analytic solution near the boundary of the perturbative evolution ($Q^2 = Q_0^2$ i.e. $s = 0$) is obtained without any approximation of the splitting kernels $p_{gg}^{(1)}(z)$ and $p_{gg}^{(2)}(z)$. But to obtain solution away from the initial curve, we had to retain only the leading terms of $p_{gg}^{(2)}(z)$ as $z \to 0$, while retaining the full form of $p_{gg}^{(1)}(z)$. We have demonstrated that the NLO expression (Eq.4.37) of the gluon reduces to the LO result (Eq.2.36) when we neglect the two loop splitting kernel $p_{gg}^{(2)}(z)$. The quark part in the evolution equation is also taken into account and is found that it has a significant effect only in the range $5GeV^2 \sim 100GeV^2$ for $x \leq 10^{-2}$ but at higher $Q^2$, its effect is negligible. Compatibility with the DLA asymptotic is discussed (see Eq.4.45) indicating the possible reasons for the disagreement between the two at LO. The NLO $Q^2$ slope of $F_2$ is computed and compared with the exact slope derived from the GRV98NLO[40] parametrization and H1 [29, 32] data in the range $7GeV^2 < Q^2 < 150GeV^2$ and found that it compares well with both in the range $10GeV^2 \leq Q^2 \leq 100GeV^2$ for $x \geq 10^{-4}$. The $x$-slope i.e. the derivative $\frac{\partial \ln F_2}{\partial \ln 1/x}$ is found to be almost constant for $x \leq 10^{-2}$ in the $Q^2$ range $5GeV^2 < Q^2 < 150GeV^2$ consistent with H1[33] data and those extracted by Desgrolard *et al*[84].

In chapter 5, we have presented solution of the LO singlet coupled evolution equations. Solutions are presented both in analytic and semi-analytic forms. However, the analytic solutions (Eqs.5.88 and 5.89) are dependent on an *ad-hoc* assumption (Eq.5.83) which may be considered as an inherent limitation of the approach. There is also another limitation in the derivation. Taking the full forms of the eigenvalues $\lambda_1$ and $\lambda_2$ of the matrix $A$ (Eq.5.34)[see Appendix], the characteristic equations of the problem cannot be solved analytically. Hence, we had to make some further approximations shown in Eq.(5.48) and Eq.(5.55). The solutions thus obtained are compared with the exact MRST01LO[101] distributions and range of validity within about 10% of the exact solutions obtained. We find that the free parameter $k$ and $f_0$ introduced through the test function $f(t)$ (Eq.5.83) and by Eq.(5.87) play an important role for the validity of the solution. Solution of non-singlet evolution equation is combined with $F_2^S$ to calculate $F_2^P$ which is then compared both with the exact solution [101] and H1[32] data. There is a reasonable agreement with HERA data in the kinematic region $5 \leq Q^2 \leq 60GeV^2$ and $10^{-4} \leq x \leq 10^{-2}$, if we choose the same value of $k$ (= 1.2) and $f_0$ (= 0.8) as obtained from the comparison of $F_2^S$ with the exact result. We have also shown that the earlier LO solution of the gluon derived in chapter 2 (Eq.2.36) without considering the quark singlet can be recovered from the present result for gluon if we put $F_2^S = 0$. This demonstrates the consistency of the method applied.

The method is then applied in chapter 6 to the polarized coupled singlet evolution equations at LO. We take the simplest forms of the eigenvalues of the matrix $A'$ (Eq.6.35) to get analytical solutions of the characteristic curves, which render integration of the ODE (Eq.6.40) possible. Analytic solutions for $\Delta\Sigma(x, t)$ and $\Delta g(x, t)$ are obtained for $x \to 0$ and $t$ very close to the boundary $t = t_0$. Comparing with the exact results LSS LO[108], AAC LO[107] and GRSV LO [109] at $Q^2 = 2GeV^2$ and $5GeV^2$, we find that the results are in good agreement qualitatively. The gluon polarization asymmetry $\frac{\Delta G(x)}{G(x)}$ at $Q^2 = 5GeV^2$ can explain the general trend of the presently available data [115, 120, 118]. The quark helicity distribution $\Delta\Sigma(Q^2)$ at($Q^2 = 5GeV^2$) is found to be ($\simeq$ 0.201) and it lies within the range 0.1 $\sim$ 0.3 presently obtained by different parametrization groups[107, 108, 109]. However, the gluon polarization $\Delta g(Q^2)$ ( at $Q^2 = 5GeV^2$) ($\Delta g(5) = 0.31$) is below the value obtained by GRSV[109] and AACLO[107], probably due to the upper cut off in our integration.

To summarize, we have demonstrated in this thesis, how the method of characteristics of partial differential equations can be successfully applied to find analytical and semi-analytical solutions of the DGLAP evolution equations. The work presented here has admittedly many limitations which arise from the various approximations to be assumed at low $x$ to obtain the solutions. However, the method has a sound theoretical basis as has been demonstrated all throughout by showing how consistently we can move from LO gluon to NLO gluon and then to the coupled equations both in unpolarized and polarized case, theoretically as well as phenomenologically. As for the future work, there is ample scope to improve upon our result by avoiding many of the small $x$ assumptions we used. For example, consideration of higher order terms in Taylor expansion might give some sizable correction to our result. Moreover, in the case of coupled equations, both polarized and unpolarized, we have applied the method only at LO. Certainly, the method can be extended beyond leading order to NLO and possibly to NNLO as the splitting kernels at these levels are also presently known.