Chapter 6

Polarized Singlet DGLAP Evolution Equations

In the last chapters we confined ourselves to the unpolarized DIS and the corresponding spin averaged parton distribution functions and the proton structure functions with the help of the DGLAP evolution equations for the unpolarized parton distributions. But the structure of the proton is more profound to be understood only from the unpolarized DIS as revealed first by the measurement of the polarized structure functions by the European Muon Collaboration in 1988. Their result[47, 48], which is often referred as the proton spin problem indicated that the contribution of the quarks to the nucleon spin is very small. The rest of the spin then must be carried by the gluon and/or by the angular momentum of quarks and gluons. The spin distributions among the different contributors is expressed by a sum rule [105, 106]

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L_q + L_g\,, \tag{6.1}$$

where $\Delta\Sigma$ is the contribution from the quarks, ΔG from the gluons and L_q , L_g are the orbital angular momentum of quarks and gluons to the total spin of the nucleon. In the static quark model with SU(6) symmetry, the spin is entirely carried by the quarks i.e $\Delta\Sigma = 1$. In the näive quark-parton model $\Delta\Sigma$ is predicted to be ~ 0.7. These are in contradiction to the EMC result. Based on the next-to-leading order QCD analysis of the deep inelastic polarized data, there are several model parametrization [107, 108, 109, 110, 111, 112] of the polarized parton distribution functions. All of these models predict that the value of $\Delta\Sigma$ is around 0.3 or less; that is, the quark contribution to the nucleon spin is very small contrary to our expectation. In view of this, the polarized gluon distribution has attracted in recent years special interest in polarized DIS, because a rather large contribution of the first moment of the gluon distribution is essential to compensate for the low value of the quark contribution to the total helicity of the nucleon. Recent data [113] and analyses [108, 109, 110, 111, 112] suggest that the contribution of the polarized gluon is quite large, though the error involved is too large to significantly constrain its contribution. It is therefore imperative to know how the gluon is polarized inside the nucleon. In this chapter we investigate the polarized gluon and the singlet contribution to the nucleon spin from the DGLAP evolution equations. We adopt the method of characteristics employed earlier to solve the coupled polarized DGLAP evolution equations. In §6.1 we give our formalism and in §6.2 we discuss the results.

6.1 Formalism

For Q^2 evolution it is convenient to introduce a flavour singlet combination of polarized quark and anti-quarks similar to Eq.(1.48) by

$$\Delta\Sigma(x, Q^2) = \sum_q (\Delta q(x, Q^2) + \Delta\bar{q}(x, Q^2)), \tag{6.2}$$

where the sum is over all quark and anti-quark flavours. Here $\Delta q(x, Q^2) = q^+(x, Q^2) - q^-(x, Q^2)$ where $q^+(x, Q^2)$, ($q^-(x, Q^2)$) refer to the quark distribution with helicity parallel (anti-parallel) to the parent proton. The polarized gluon distribution is similarly defined as in (Eq.1.46) by

$$\Delta g(x, Q^2) = g^{(+)}(x, Q^2) - g^{(-)}(x, Q^2). \tag{6.3}$$

The Q^2 evolution of the polarized singlet and the gluon distribution mix non-trivially and is given by the DGLAP equations Eq.(1.53) i.e

$$\frac{\partial}{\partial t}\begin{pmatrix} \Delta\Sigma(x,t) \\ \Delta g(x,t) \end{pmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{pmatrix} \Delta P_{qq} & 2n_f \Delta P_{qg} \\ \Delta P_{gq} & \Delta P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Delta\Sigma(x,t) \\ \Delta g(x,t) \end{pmatrix}, \tag{6.4}$$

where $t = ln\frac{Q^2}{\Lambda^2}$ and ΔP_{ij} are polarized splitting functions which are known at LO [16] and NLO [24, 25, 26]. The symbol $\otimes$ stands for the usual Mellin convolution in the first variable defined in Eq.(1.43) The polarized parton distributions at an initial scale enter as boundary values in the solution of the above equations. However with exact splitting functions, analytic solutions in the entire range of x is not possible. The splitting functions can be simplified by taking their moments and expanding around the rightmost singularity at $N = 0$ where N is the moment variable. In this procedure the splitting functions[111] get simplified at LO to

$$\left.\begin{array}{ll} \Delta P^{(0)}_{qq}(x) = \frac{4}{3}\left\{1 + \frac{1}{2}\delta(1-x)\right\}, & \Delta P^{(0)}_{qg}(x) = \frac{1}{2}2n_f\left\{-1 + 2\delta(1-x)\right\}, \\ \Delta P^{(0)}_{gq}(x) = \frac{4}{3}\left\{2 - \delta(1-x)\right\}, & \Delta P^{(0)}_{gg}(x) = 3\left\{4 - \frac{13}{6}\delta(1-x)\right\} - \frac{n_f}{3}\delta(1-x) \end{array}\right\}. \tag{6.5}$$

With these simplified splitting functions, Eqs.(6.4) can be solved analytically with some approximations about small x behaviour to be discussed later. First we write the two Eqs.(6.4) in LO separately as

$$\frac{\partial}{\partial t}\Delta\Sigma(x,t) = \frac{\alpha_s(t)}{2\pi}\left[\Delta P^{(0)}_{qq} \otimes \Delta\Sigma(x,t) + 2n_f \Delta P^{(0)}_{qg}(x) \otimes \Delta g(x,t)\right] \tag{6.6}$$

and

$$\frac{\partial}{\partial t}\Delta g(x,t) = \frac{\alpha_s(t)}{2\pi}\left[\Delta P^{(0)}_{gq} \otimes \Delta\Sigma(x,t) + \Delta P^{(0)}_{gg}(x) \otimes \Delta g(x,t)\right]. \tag{6.7}$$

Using the convolution integral (Eq.1.43), we write Eq.(6.6) and Eq.(6.7) as

$$\frac{\partial}{\partial t}\Delta\Sigma(x,t) = \frac{\alpha_s(t)}{2\pi}\left[\int_x^1 dz\,\Delta P_{qq}^{(0)}(z)\,\Delta\Sigma\left(\frac{x}{z},t\right) + 2\,n_f \int_x^1 dz\,\Delta P_{qg}^{(0)}(z)\,\Delta g\left(\frac{x}{z},t\right)\right], \quad (6.8)$$

$$\frac{\partial}{\partial t}\Delta g(x,t) = \frac{\alpha_s(t)}{2\pi}\left[\int_x^1 dz\,\Delta P_{gq}^{(0)}(z)\,\Delta\Sigma\left(\frac{x}{z},t\right) + \int_x^1 dz\,\Delta P_{gg}^{(0)}(z)\,\Delta g\left(\frac{x}{z},t\right)\right]. \quad (6.9)$$

Now we expand $\Delta\Sigma\left(\frac{x}{z},t\right)$ and $\Delta g\left(\frac{x}{z},t\right)$ in Eq.(6.8) and Eq.(6.9) in Taylor series using Eq.(4.6) and Eq.(4.7) as

$$\begin{aligned}\Delta\Sigma\left(\frac{x}{z},t\right) &= \Delta\Sigma\left(x + x\sum_{k=1}^{\infty} u^k, t\right) \\ &= \Delta\Sigma(x,t) + \left(x\sum_{k=1}^{\infty} u^k\right)\frac{\partial\Delta\Sigma(x,t)}{\partial x} + \frac{1}{2}\left(x\sum_{k=1}^{\infty} u^k\right)^2 \frac{\partial^2\Delta\Sigma(x,t)}{\partial x^2} + \ldots \quad (6.10)\end{aligned}$$

and

$$\begin{aligned}\Delta g\left(\frac{x}{z},t\right) &= \Delta g\left(x + x\sum_{k=1}^{\infty} u^k, t\right) \\ &= \Delta g(x,t) + \left(x\sum_{k=1}^{\infty} u^k\right)\frac{\partial\Delta g(x,t)}{\partial x} + \frac{1}{2}\left(x\sum_{k=1}^{\infty} u^k\right)^2 \frac{\partial^2\Delta g(x,t)}{\partial x^2} + \ldots \quad (6.11)\end{aligned}$$

At low x the above series Eq.(6.10) and Eq.(6.11) are convergent[65]. Hence, neglecting higher order terms, the series can be approximated as

$$\Delta\Sigma\left(\frac{x}{z},t\right) \approx \Delta\Sigma(x,t) + \left(x\sum_{k=1}^{\infty} u^k\right)\frac{\partial\Delta\Sigma(x,t)}{\partial x} \quad (6.12)$$

and

$$\Delta g\left(\frac{x}{z},t\right) \approx \Delta g(x,t) + \left(x\sum_{k=1}^{\infty} u^k\right)\frac{\partial\Delta g(x,t)}{\partial x}. \quad (6.13)$$

Using Eq.(6.12) and Eq.(6.13) and the splitting functions Eqs.(6.5),we can write Eq.(6.8) and Eq.(6.9) as

$$\frac{\beta_0 t}{2}\frac{\partial \Delta\Sigma(x,t)}{\partial t} = I_q^{(1)}(x)\Delta\Sigma(x,t) + I_q^{(2)}\frac{\partial \Delta\Sigma(x,t)}{\partial x} + I_g^{(1)}(x)\Delta g(x,t) + I_g^{(2)}\frac{\partial \Delta g(x,t)}{\partial x} \quad (6.14)$$

and

$$\frac{\beta_0 t}{2}\frac{\partial \Delta g(x,t)}{\partial t} = I_q^{(3)}(x)\Delta\Sigma(x,t) + I_q^{(4)}\frac{\partial \Delta\Sigma(x,t)}{\partial x} + I_g^{(3)}(x)\Delta g(x,t) + I_g^{(4)}\frac{\partial \Delta g(x,t)}{\partial x} \quad (6.15)$$

where we have used $\alpha_s(t) = \dfrac{4\pi}{\beta_0 t}$ in LO. The quantities $I_i^j(i = q, g; j = 1,2,3,4)$ in Eq.(6.14) and Eq.(6.15) are given by

$$I_q^{(1)}(x) = \frac{4}{3}\int_x^1 \frac{dz}{z}\left(1 + \frac{1}{2}\delta(1-z)\right), \quad (6.16)$$

$$I_q^{(2)}(x) = \frac{4}{3}\int_x^1 \frac{dz}{z}\left(1 + \frac{1}{2}\delta(1-z)\right)\left(x\sum_{k=1}^{\infty}u^k\right), \quad (6.17)$$

$$I_g^{(1)}(x) = 2n_f\frac{1}{2}\int_x^1 \frac{dz}{z}(-1 + 2\delta(1-z)), \quad (6.18)$$

$$I_g^{(2)}(x) = 2n_f\frac{1}{2}\int_x^1 \frac{dz}{z}(-1 + 2\delta(1-z))\left(x\sum_{k=1}^{\infty}u^k\right), \quad (6.19)$$

$$I_q^{(3)}(x) = \frac{4}{3}\int_x^1 \frac{dz}{z}(2 - \delta(1-z)), \quad (6.20)$$

$$I_q^{(4)}(x) = \frac{4}{3}\int_x^1 \frac{dz}{z}(2 - \delta(1-z))\left(x\sum_{k=1}^{\infty}u^k\right), \quad (6.21)$$

$$I_g^{(3)}(x) = \int_x^1 \frac{dz}{z}\left[3\left(4 - \frac{13}{6}\delta(1-z)\right) - \frac{n_f}{3}\delta(1-z)\right] \quad (6.22)$$

and

$$I_g^{(4)}(x) = \int_x^1 \frac{dz}{z}\left[3\left(4 - \frac{13}{6}\delta(1-z)\right) - \frac{n_f}{3}\delta(1-z)\right]\left(x\sum_{k=1}^{\infty}u^k\right). \quad (6.23)$$

Carrying out the integrations in Eqs.(6.16-6.23), we recast Eq.(6.14) and Eq.(6.15)) as two coupled partial differential equations in two variables x and t as :

$$a'_{11}\frac{\partial\Delta\Sigma(x,t)}{\partial t}+a'_{12}\frac{\partial\Delta g(x,t)}{\partial t}+b'_{11}\frac{\partial\Delta\Sigma(x,t)}{\partial x}+b'_{12}\frac{\partial\Delta g(x,t)}{\partial x} = R'_{11}\Delta\Sigma(x,t)+R'_{12}\Delta g(x,t) \tag{6.24}$$

and

$$a'_{21}\frac{\partial\Delta\Sigma(x,t)}{\partial t}+a'_{22}\frac{\partial\Delta g(x,t)}{\partial t}+b'_{21}\frac{\partial\Delta\Sigma(x,t)}{\partial x}+b'_{22}\frac{\partial\Delta g(x,t)}{\partial x} = R'_{21}\Delta\Sigma(x,t)+R'_{22}\Delta g(x,t), \tag{6.25}$$

where

$$\left.\begin{array}{ll} a'_{11}=t, & a'_{12}=0 \\ a'_{21}=0, & a'_{22}=t \end{array}\right\}, \tag{6.26}$$

$$\left.\begin{array}{ll} b'_{11}=-\frac{4}{3}\left(1-x-x\,ln\frac{1}{x}\right), & b'_{12}=-2\,n_f\,\frac{1}{2}\left(-1+x+x\,ln\frac{1}{x}\right) \\ b'_{21}=-\frac{4}{3}\left(1-2x-x\,ln\frac{1}{x}\right), & b'_{22}=-12\left(1-x-x\,ln\frac{1}{x}\right) \end{array}\right\} \tag{6.27}$$

and

$$\left.\begin{array}{ll} R'_{11}=\frac{4}{3}\left(\frac{1}{2}+ln\frac{1}{x}\right), & R'_{12}=2\,n_f\,\frac{1}{2}\left(2-ln\frac{1}{x}\right) \\ R'_{21}=\frac{4}{3}\left(2\,ln\frac{1}{x}-1\right), & R'_{22}=\left[12\,ln\frac{1}{x}-\left(\frac{13}{2}+\frac{n_f}{3}\right)\right] \end{array}\right\}. \tag{6.28}$$

Eq.(6.24) and Eq.(6.25) are exactly same in form to Eq.(5.23) and Eq.(5.24). Here we have used primed symbols to distinguish from simialr quantities of chapter 5. Following a similar procedure as in chapter5, we can solve these equations by the method of characteristics. To do that, we introduce a vector $\vec{\Delta u}(x,t)$ similar to Eq.(5.28) by

$$\vec{\Delta u}(x,t) = \begin{pmatrix} \Delta\Sigma(x,t) \\ \Delta g(x,t) \end{pmatrix} \tag{6.29}$$

and express the two equations Eq.(6.24) and Eq.(6.25) in matrix form

$$a' \vec{\Delta u}_t(x,t) + b' \vec{\Delta u}_x(x,t) = R' \vec{\Delta u}(x,t), \tag{6.30}$$

where the matrices a' , b' and R' are

$$a' = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}, \tag{6.31}$$

$$b' = \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix} \tag{6.32}$$

and

$$R' = \begin{pmatrix} R'_{11} & R'_{12} \\ R'_{21} & R'_{22} \end{pmatrix}, \tag{6.33}$$

the elements being given by Eqs.(6.26-6.28) respectively. The matrix a' being non-singular, we multiply Eq.(6.30) from left by a'^{-1} and put in a similar fashion as that of Eq.(5.33) i.e.

$$\vec{\Delta u}_t(x,t) + A' \vec{\Delta u}_x(x,t) = B' \vec{\Delta u}(x,t), \tag{6.34}$$

where the new matrices A' and B' are:

$$A' = a'^{-1} b' \tag{6.35}$$

and

$$B' = a'^{-1} R'. \tag{6.36}$$

Eq.(6.30) represents a system of two coupled first order partial differential equation. By following the same procedure that leads to the derivation of equation Eq.(5.44), we can reduce this system to canonical form. Let $\lambda'^{(i)} (i = 1, 2)$ be the eigenvalues of the matrix A' (Eq.6.35) and P' be the matrix formed by the corresponding eigenvectors. Let $\vec{\Delta v}(x,t)$ be a new function defined in terms of $\vec{\Delta u}(x,t)$ by the equation

$$\Delta\vec{u}(x,t) = P'.\Delta\vec{v}(x,t)\,. \tag{6.37}$$

Then the canonical form of the system of equations Eq.(6.34) is

$$\Delta v_t^{(i)} + \lambda'^{(i)}\,\Delta v_x^{(i)} = \Delta e^{(i)}\,, \qquad (i = 1,2) \tag{6.38}$$

where $\Delta e^{(i)}$ are the components of a two component column matrix similar to Eq.(5.43). i.e.

$$\vec{\Delta e} = P'^{-1}(B'.P' - P'_t - A'.P'_x).\vec{\Delta v}\,. \tag{6.39}$$

As discussed earlier in chapter 5, these equations (Eq.6.38) are reduced to ordinary differential equations

$$\frac{d\Delta v^{(i)}}{dt} = \Delta e^{(i)}(x,t,\Delta v^{(i)}(x,t)) \qquad (i = 1,2) \tag{6.40}$$

along the characteristic curves defined by the equations

$$\lambda'^{(i)}(x,t) = \frac{dx^{(i)}(t)}{dt}\,. \tag{6.41}$$

In order to integrate the ordinary differential equations (Eq.6.40) to get analytical forms for $\Delta v^{(i)}$, one has to express the right hand side in terms of t. This is done from the solution of the characteristic equations Eq.(6.41), which expresses $x^{(i)}$ as a function of t. However, the integration of Eq.(6.41) to get such an expression depends on the nature of the eigenvalues $\lambda'^{(i)}$ of the matrix A'. For simple forms of the eigenvalues, it is possible to integrate Eq.(6.40) analytically. We discuss this below.

The eigenvalues of the matrix A' are given by

$$\lambda'^{(1,2)} = \frac{1}{9t}(b'_{11} + b'_{22} \pm s), \tag{6.42}$$

where

$$s = \sqrt{b'^2_{11} + 4\,b'_{12}\,b'_{21} - 2b'_{11}\,b'_{22} + b'^2_{22}}\,. \tag{6.43}$$

Taking the elements of the matrices a' and b' as given by Eq.(6.26) and Eq.(6.27), the eigenvalues are found to be

$$\lambda'^{(1,2)}(x,t) = \frac{2}{27\,t}\left[\left(-10 + 10\,x + 10\,x\,ln\frac{1}{x}\right) \pm \sqrt{55 - 101\,x + 46\,x^2 + (-110\,x + 101\,x^2)\,ln\frac{1}{x} + 55\,x^2\,(ln\frac{1}{x}^2)^2}\right], \tag{6.44}$$

where the + (−) sign corresponds to $\lambda'^{(1)}(\lambda'^{(2)})$. And the two components of the vector $\vec{\Delta e}$ are

$$\Delta e^{(1,2)} = \frac{1}{t}\left[\left(\mp\frac{(-b'_{11}+b'_{22}+s)\left(-\frac{b'_{21}R'_{11}}{s} - \frac{(-b'_{11}+b'_{22}\mp s)R'_{21}}{2s}\right)}{9b'_{21}} \mp \frac{2b'_{21}R'_{12}}{9s} \mp \frac{(-b'_{11}+b'_{22}\mp s)R'_{22}}{9s}\right)\Delta\nu^{(1)} + \left(\mp\frac{(-b'_{11}+b'_{22}-s)\left(-\frac{b'_{21}R'_{11}}{s} - \frac{(-b'_{11}+b'_{22}\mp s)R'_{21}}{2s}\right)}{9b'_{21}} \mp \frac{2b'_{21}R'_{12}}{9s} \mp \frac{(-b'_{11}+b'_{22}\mp s)R'_{22}}{9s}\right)\Delta\nu^{(2)}\right], \tag{6.45}$$

where the upper sign (-) corresponds to $\Delta e^{(1)}$ and the lower sign (+) to $\Delta e^{(2)}$. We note that both $\lambda'^{(i)}$ and $\Delta e^{(i)}$ are factorisable in x and t, the quantities b'_{ij} and R'_{ij} being functions of x only. As we have already observed, integration of Eq.(6.40) depends on the analytical solution of the characteristic equation . But with the above eigenvalues (Eq.6.44), analytical solution of the characteristic equations (Eq.6.41) cannot be found and so we cannot find the necessary transformation equation to express x as a function of t for the integration of Eq.(6.40). But under certain extreme situations it is possible to get analytical solutions which we discuss below.

6.1.1 Solution when $x \to 0$

The eigen values of the matrix A' can be simplified by making certain approximations about the elements of the matrix b'. Before we do that we write the eigenvalues in simple form. The eigenvalues being factorisable in x and t can be written as

$$\lambda'^{(i)}(x,t) = \lambda'^{(ia)}(x)\,\lambda'^{(ib)}(t)\,, \qquad (i=1,2), \tag{6.46}$$

where

$$\lambda'^{(ia)}(x) = \frac{2}{27}\left[\left(-10 + 10\,x + 10\,x\,ln\frac{1}{x}\right) \pm \sqrt{55 - 101\,x + 46\,x^2 + (-110\,x + 101\,x^2)\,ln\frac{1}{x} + 55\,x^2\,ln\frac{1}{x}^2}\,\right] \tag{6.47}$$

and

$$\lambda'^{(ib)}(t) = \frac{1}{t}\,. \tag{6.48}$$

In Eq.(6.47), +(-) sign corresponds to i=1 (2) respectively. Now in the limit $x \to 0$, Eq.(6.47) takes the values

$$\lambda'^{(ia)} \to \frac{2}{27}\left(-10 \pm \sqrt{55}\right). \tag{6.49}$$

The characteristic equation Eq.(6.41) now takes the form

$$\frac{dx^{(i)}(t)}{\lambda'^{(ia)}(x)} = \lambda'^{(ib)}(t)dt\,. \tag{6.50}$$

Integrating we get

$$x^{(i)}(t) = \alpha^{(i)}\,lnt + C^i\,, \tag{6.51}$$

where

$$\alpha^{(i)} = \frac{2}{27}(-10 \pm \sqrt{55}), \qquad i = 1, 2 \tag{6.52}$$

and $C^{(i)}$ are two constants of integration. To get the constants of integration, let $(\bar{x}, \bar{t})$ be a fixed point [59] in the (x, t) plane through which the two characteristic curves Eq.(6.51) pass through. That is, $x^{(1)}(\bar{t}) = \bar{x}$ and $x^{(2)}(\bar{t}) = \bar{x}$. Then from the Eqs.(6.51) we get

$$x^{(i)}(t) = \bar{x} + \alpha^{(i)}\, ln\left(\frac{t}{\bar{t}}\right). \tag{6.53}$$

These are the two characteristic curves that pass through a common point $(\bar{x}, \bar{t})$ up to which we can evolve the two functions $\Delta\nu^{(1)}$ and $\Delta\nu^{(2)}$ defined by Eq.(6.37). Furthermore, if the two characteristic curves cut the initial line $t = t_0$ ($t_0 = ln\frac{Q_0^2}{\Lambda^2}$) at $x^{(1)}(t_0) = \tau_1'$ and $x^{(2)}(t_0) = \tau_2'$ respectively, then

$$\tau_1' = \bar{x} + \alpha^{(1)}\, ln\left(\frac{t_0}{\bar{t}}\right) \tag{6.54}$$

and

$$\tau_2' = \bar{x} + \alpha^{(2)}\, ln\left(\frac{t_0}{\bar{t}}\right). \tag{6.55}$$

Hence the characteristic equations corresponding to the two eigenvalues are respectively

$$x^{(1)}(t) = \tau_1' + \alpha^{(1)}\, ln\left(\frac{t}{t_0}\right) \tag{6.56}$$

and

$$x^{(2)}(t) = \tau_2' + \alpha^{(2)}\, ln\left(\frac{t}{t_0}\right). \tag{6.57}$$

In this approximation the two components of the vector $\vec{\Delta e}$ defined in Eq.(6.39) are

$$\begin{aligned} \Delta e^{(1)} \;=\; & \frac{1}{2970}\Big[(-2255 + 284\sqrt{55})\Delta\nu^{(1)} + 3(275 - 52\sqrt{55})\Delta\nu^{(2)} \\ & +(4400 - 404\sqrt{55})\Delta\nu^{(1)} + 4(-880 + 119\sqrt{55})\Delta\nu^{(2)} ln\frac{1}{x^{(1)}(t)}\Big]\frac{1}{t} \end{aligned} \tag{6.58}$$

and

$$\Delta e^{(2)} = \frac{1}{2970}\Big[-(2255 + 284\sqrt{55})\Delta\nu^{(2)} + 3(275 + 52\sqrt{55})\Delta\nu^{(1)} \\ +(4400 + 404\sqrt{55})\Delta\nu^{(2)} - 4(880 + 119\sqrt{55})\Delta\nu^{(1)} ln\frac{1}{x^{(2)}(t)}\Big]\frac{1}{t}. \quad (6.59)$$

Using Eq.(6.58) and Eq.(6.59) we rewrite the two equations Eqs.(6.40) separately as

$$\frac{d\Delta\nu^{(1)}(x,t)}{\Delta\nu^{(1)}(x,t)} = \left[A_1(x,t) + B_1(x,t) ln\frac{1}{x^{(1)}(t)}\right]\frac{1}{t}dt \quad (6.60)$$

and

$$\frac{d\Delta\nu^{(2)}(x,t)}{\Delta\nu^{(2)}(x,t)} = \left[A_2(x,t) + B_2(x,t) ln\frac{1}{x^{(2)}(t)}\right]\frac{1}{t}dt, \quad (6.61)$$

where

$$A_1(x,t) = \frac{1}{2970}\left[(-2255 + 284\sqrt{55}) + 3(275 - 52\sqrt{55})\frac{\Delta\nu^{(2)}}{\Delta\nu^{(1)}}\right], \quad (6.62)$$

$$B_1(x,t) = \frac{1}{2970}\left[(4400 - 404\sqrt{55}) + 4(-880 + 119\sqrt{55})\frac{\Delta\nu^{(2)}}{\Delta\nu^{(1)}}\right], \quad (6.63)$$

$$A_2(x,t) = \frac{1}{2970}\left[-(2255 + 284\sqrt{55}) + 3(275 + 52\sqrt{55})\frac{\Delta\nu^{(1)}}{\Delta\nu^{(2)}}\right] \quad (6.64)$$

and

$$B_2(x,t) = \frac{1}{2970}\left[(4400 + 404\sqrt{55}) - 4(880 + 119\sqrt{55})\frac{\Delta\nu^{(1)}}{\Delta\nu^{(2)}}\right]. \quad (6.65)$$

Eq.(6.60) and Eq.(6.61) can now be integrated. Integrating along the characteristics from $t = t_0$ and $t = \bar{t}$ we get

$$\Delta\nu^{(1)}(\bar{x},\bar{t}) = \Delta\nu^{(1)}(\tau_1') \exp\left[\int_{t_0}^{\bar{t}} \left\{A_1(x^{(1)}(t),t) + B_1(x^{(1)}(t),t)\, ln\frac{1}{x^{(1)}(t)}\right\}\frac{dt}{t}\right] \quad (6.66)$$

and

$$\Delta\nu^{(2)}(\bar{x},\bar{t}) = \Delta\nu^{(2)}(\tau_2') \exp\left[\int_{t_0}^{\bar{t}} \left\{A_2(x^{(2)}(t),t) + B_2(x^{(2)}(t),t)\, ln\frac{1}{x^{(2)}(t)}\right\}\frac{dt}{t}\right]. \quad (6.67)$$

Or changing the variables from $(\bar{x},\bar{t})$ to (x,t)

$$\Delta v^{(1)}(x,t) = \Delta v^{(1)}(\tau_1') \exp\left[\int_{t_0}^{t}\left\{A_1(x^{(1)}(t'),t') + B_1(x^{(1)}(t'),t)\, ln\frac{1}{x^{(1)}(t')}\right\}\frac{dt'}{t'}\right] \quad (6.68)$$

and

$$\Delta v^{(2)}(x,t) = \Delta v^{(2)}(\tau_2') \exp\left[\int_{t_0}^{t}\left\{A_2(x^{(2)}(t'),t') + B_2(x^{(2)}(t'),t')\, ln\frac{1}{x^{(2)}(t')}\right\}\frac{dt'}{t'}\right]. \quad (6.69)$$

Eq.(6.68 and Eq.(6.69) give the two unknown functions $\Delta v^{(1)}(x,t)$ and $\Delta v^{(2)}(x,t)$ which are related to the polarized singlet $\Delta\Sigma(x,t)$ and gluon $\Delta g(x,t)$ by Eq.(6.37). Solving Eq.(6.37) for $\Delta\vec{v}(x,t)$ we get

$$\Delta v^{(1)}(x,t) = -\frac{(-b'_{11}+b'_{22}-s)}{2\,s}\Delta g(x,t) - \frac{b'_{21}}{s}\Delta\Sigma(x,t) \quad (6.70)$$

and

$$\Delta v^{(2)}(x,t) = \frac{(-b'_{11}+b'_{22}+s)}{2\,s}\Delta g(x,t) + \frac{b'_{21}}{s}\Delta\Sigma(x,t)\,, \quad (6.71)$$

where s is given by Eq.(6.43). From Eq.(6.70) and Eq.(6.71) we get

$$\Delta\Sigma(x,t) = -\frac{(-b'_{11}+b'_{22}+s)}{2b'_{21}}\Delta v^{(1)}(x,t) - \frac{(-b'_{11}+b'_{22}-s)}{2b'_{21}}\Delta v^{(2)}(x,t) \quad (6.72)$$

$$\Delta g(x,t) = \Delta v^{(1)}(x,t) + \Delta v^{(2)}(x,t)\,. \quad (6.73)$$

In the special case when $x \to 0$,

$$\Delta\Sigma(x,t) = \frac{1}{4}(-8-\sqrt{55})\Delta v^{(1)}(x,t) + \frac{1}{4}(-8+\sqrt{55})\Delta v^{(2)}(x,t)\,, \quad (6.74)$$

$$\Delta g(x,t) = \Delta v^{(1)}(x,t) + \Delta v^{(2)}(x,t)\,. \quad (6.75)$$

That is, at very low x approximation

$$\Delta\Sigma(x,t) = \frac{1}{4}(-8-\sqrt{55})\Delta\nu^{(1)}(\tau_1')\exp\left[\int_{t_0}^{t}\left\{A_1(x^{(1)}(t'),t') + B_1(x^{(1)}(t'),t')\,ln\frac{1}{x^{(1)}(t')}\right\}\frac{dt'}{t'}\right]$$
$$+\frac{1}{4}(-8+\sqrt{55})\Delta\nu^{(2)}(\tau_2')\exp\left[\int_{t_0}^{t}\left\{A_2(x^{(2)}(t'),t') + B_2(x^{(2)}(t'),t')ln\frac{1}{x^{(2)}(t')}\right\}\frac{dt'}{t'}\right] \tag{6.76}$$

and

$$\Delta g(x,t) = \Delta\nu^{(1)}(\tau_1')\exp\left[\int_{t_0}^{t}\left\{A_1(x^{(1)}(t'),t') + B_1(x^{(1)}(t'),t')\,ln\frac{1}{x^{(1)}(t')}\right\}\frac{dt'}{t'}\right]$$
$$+\Delta\nu^{(2)}(\tau_2')\exp\left[\int_{t_0}^{t}\left\{A_2(x^{(2)}(t'),t') + B_2(x^{(2)}(t'),t')\,ln\frac{1}{x^{(2)}(t')}\right\}\frac{dt'}{t'}\right]. \tag{6.77}$$

In Eq.(6.76) and Eq.(6.77), $\Delta\nu^{(1)}(\tau_1')$ and $\Delta\nu^{(2)}(\tau_2')$ are the inputs on the initial curve $t = t_0$. Since $\Delta\nu^{(1)}$ and $\Delta\nu^{(2)}$ are some combinations of $\Delta\Sigma$ and Δg given by Eq.(6.70) and Eq.(6.71), therefore from the input x-distribution for $\Delta\Sigma$ and Δg at $Q^2 = Q_0^2$ $(t = t_0)$, we can get $\Delta\nu^{(1)}(\tau_1')$ and $\Delta\nu^{(2)}(\tau_2')$ simply by substituting $x \to \tau_1'$ and $x \to \tau_2'$[57]. But in carrying out the integration we face a problem similar to the one we faced in chapter 5. Under the integral sign in Eq.(6.76) and Eq.(6.77) we have the ratio $\frac{\nu_1}{\nu_2}$(see Eqs.(6.62-6.65)) which is yet unknown. This ignorance forbids analytical forms for the quantities defined in these equations. However, at the initial scale $t = t_0$ this ratio is known because we know $\Delta\Sigma$ and Δg. We assume that the x and t dependent parts of $\Delta\nu^{(1)}$ and $\Delta\nu^{(2)}$ are factorisable and the x dependent parts do not deviate significantly from their values at $t = t_0$ i.e.

$$\frac{\Delta\nu^{(2)}(x^{(2)}(t),t)}{\Delta\nu^{(1)}(x^{(1)}(t),t)} \approx \frac{\Delta\nu^{(2)}(\tau_2')}{\Delta\nu^{(1)}(\tau_1')}.\Delta f(t), \tag{6.78}$$

where $\Delta f(t)$ is some unknown test function.

6.1.2 Analytical solution when t is very near the boundary $t = t_0$

It is possible to obtain analytical forms for $\Delta\Sigma$ and Δg in a region very close to the boundary $t = t_0$. Expanding $\Delta f(t)$ in a Taylor series about $t = t_0$ and retaining only the first term we get

$$\begin{aligned} \Delta f(t) &= \Delta f(t_0) + (t - t_0)\Delta f'(t) + ... \\ &\approx \Delta f(t_0) = \Delta f_0 . \end{aligned} \tag{6.79}$$

With this approximation the integration of Eq.(6.68) and Eq.(6.69) give the following analytical forms

$$\Delta\nu^{(1)}(x,t) = \Delta\nu^{(1)}(\tau_1') \exp\left[B_{10}\, ln\frac{t}{t_0}\, ln\frac{1}{x}\right] \left(\frac{t}{t_0}\right)^{(A_{10}+B_{10})} \tau'^{\left(\frac{B_{10}\tau_1'}{\alpha^{(1)}}\right)}_{1}\, x^{-\left(\frac{B_{10}\tau_1'}{\alpha^{(1)}}\right)} \tag{6.80}$$

and

$$\Delta\nu^{(2)}(x,t) = \Delta\nu^{(2)}(\tau_2') \exp\left[B_{20}\, ln\frac{t}{t_0}\, ln\frac{1}{x}\right] \left(\frac{t}{t_0}\right)^{(A_{20}+B_{20})} \tau'^{\left(\frac{B_{20}\tau_2'}{\alpha^{(2)}}\right)}_{2}\, x^{-\left(\frac{B_{20}\tau_2'}{\alpha^{(2)}}\right)}, \tag{6.81}$$

where

$$\begin{aligned} A_{10} &= \frac{1}{2970}\left[(-2255 + 284\sqrt{55}) + 3(275 - 52\sqrt{55})\frac{\nu_2(\tau_2')}{\nu_1(\tau_1')}\Delta f_0\right], \\ B_{10} &= \frac{1}{2970}\left[(4400 - 404\sqrt{55}) + 4(-880 + 119\sqrt{55})\frac{\Delta\nu^{(2)}(\tau_2')}{\Delta\nu^{(1)}(\tau_1')}\Delta f_0\right], \\ A_{20} &= \frac{1}{2970}\left[-(2255 + 284\sqrt{55}) + 3(275 + 52\sqrt{55})\frac{\Delta\nu^{(1)}(\tau_1')}{\Delta\nu^{(2)}(\tau_2')}\Delta f_0\right], \\ B_{20} &= \frac{1}{2970}\left[(4400 + 404\sqrt{55}) - 4(880 + 119\sqrt{55})\frac{\Delta\nu^{(1)}(\tau_1')}{\Delta\nu^{(2)}(\tau_2')}\Delta f_0\right], \end{aligned} \tag{6.82}$$

$$\tau_1' = x + \alpha^{(1)}\, ln(t_0/t), \tag{6.83}$$

and

$$\tau_2' = x + \alpha^{(2)}\, ln(t_0/t)\,. \tag{6.84}$$

Using Eq.(6.80) and Eq.(6.81) in Eq.(6.76) and Eq.(6.77) we get the analytical solutions for the polarized singlet and the gluon distribution which are valid when x is very small and t is very close to the boundary $t = t_0$ of perturbative evolution. Explicitly these are

$$\begin{aligned}\Delta\Sigma(x,t) = &\frac{1}{4}(-8-\sqrt{55})\Delta\nu^{(1)}(\tau_1')\exp\left[B_{10}\, ln\frac{t}{t_0}\, ln\frac{1}{x}\right]\left(\frac{t}{t_0}\right)^{(A_{10}+B_{10})}\tau'^{(\frac{B_{10}\tau_1'}{\alpha^{(1)}})}_1\, x^{-(\frac{B_{10}\tau_1'}{\alpha^{(1)}})}\\ &+\frac{1}{4}(-8+\sqrt{55})\Delta\nu^{(2)}(\tau_2')\exp\left[B_{20}\, ln\frac{t}{t_0}\, ln\frac{1}{x}\right]\left(\frac{t}{t_0}\right)^{(A_{20}+B_{20})}\tau'^{(\frac{B_{20}\tau_2'}{\alpha^{(2)}})}_2\, x^{-(\frac{B_{20}\tau_2'}{\alpha^{(2)}})}\end{aligned} \tag{6.85}$$

and

$$\begin{aligned}\Delta g(x,t) = &\Delta\nu^{(1)}(\tau_1')\exp\left[B_{10}\, ln\frac{t}{t_0}\, ln\frac{1}{x}\right]\left(\frac{t}{t_0}\right)^{(A_{10}+B_{10})}\tau'^{(\frac{B_{10}\tau_1'}{\alpha^{(1)}})}_1\, x^{-(\frac{B_{10}\tau_1'}{\alpha^{(1)}})}\\ &+\Delta\nu^{(2)}(\tau_2')\exp\left[B_{20}\, ln\frac{t}{t_0}\, ln\frac{1}{x}\right]\left(\frac{t}{t_0}\right)^{(A_{20}+B_{20})}\tau'^{(\frac{B_{20}\tau_2'}{\alpha^{(2)}})}_2\, x^{-(\frac{B_{20}\tau_2'}{\alpha^{(2)}})}\,.\end{aligned} \tag{6.86}$$

6.2 Results and discussion

In recent years there have been considerable growth of data [113, 114, 115, 116, 117, 118] on inclusive polarized deep inelastic scattering of leptons off nucleons. These data on the polarized asymmetry provide indirect way of extracting the polarized parton distribution functions that are essential to understand the spin decomposition of the nucleon. The distributions are determined from a global QCD analysis of polarized experimental data using the DGLAP evolution equations. Several such distributions are available in the literature[107, 108, 109, 110, 111, 112]. We compare our results with some of these exact solutions[107, 108, 109] of DGLAP equations that are available in the hep database as Fortran codes[119].

We first compare the polarized gluon distribution ΔG (= $x\Delta g$) given by Eq.(6.86) with the exact distribution AAC00[107] at $Q^2 = 5GeV^2$. The analytical expression Eq.(6.86), expected to be valid at very low x and low t has a free parameter Δf_0 introduced through the Eq.(6.79). Since Δf_0 essentially represents the ratio of some combination of singlet to the gluon polarized distribution at the initial scale, we assume that its value cannot be zero or negative. We vary this parameter starting from a low positive value till we get a reasonable agreement with the exact distributions. In the Fig.6.1(a) we show the graphs only for few representative values of Δf_0 along with the exact AAC00LO[107] distribution. To evolve our polarized gluon we use the AACLO input at the initial scale $Q_0^2 = 1GeV^2$ as given in the ref.[107]. From the figure(Fig.6.1(a)) we see that, while at $x < 0.01$, there is no appreciable difference between our distributions for different values of the parameter Δf_0, above $x > 0.01$, the difference is significant. We can see that for $\Delta f_0 \approx 2.4$, our prediction conforms reasonably well with the exact distribution. However, if we change the input distributions, then the value of the parameter Δf_0 for which we get a reasonable agreement with the exact distribution also changes. In Fig.6.1(b-c) we compare our result with two other exact distributions: LSS01[108] and GRSV01[109] at the same scale $Q^2 = 5GeV^2$. For evolution of Δg we use the inputs also from the above two references. From the figure (Fig.6.1(b-c)) we see that agreement between our prediction and the exact solution is good when $\Delta f_0 \simeq 2.2$ in the case of LSS distribution and $\Delta f_0 \simeq 2.0$ in the case of GRSV input distribution. Thus we find that the value of Δf_0 is sensitive to the input distributions and different input distributions give different Δf_0 for a reasonable agreement with the exact distribution. We use these values of Δf_0 for the later part of our discussion to make the analysis self consistent.

Now let us come to the gluon asymmetry $\frac{\Delta G}{G}$. Presently, the experimental data available on the gluon asymmetry are limited and there are only three data sets from SMC[114], HERMES[116, 117, 120] and COMPASS[121, 118] obtained from the asymmetry measurement of high-p_T hadron production. These are shown in table 6.1. The bracket <> in the table means that the value is obtained at an average fraction of the nucleon momentum

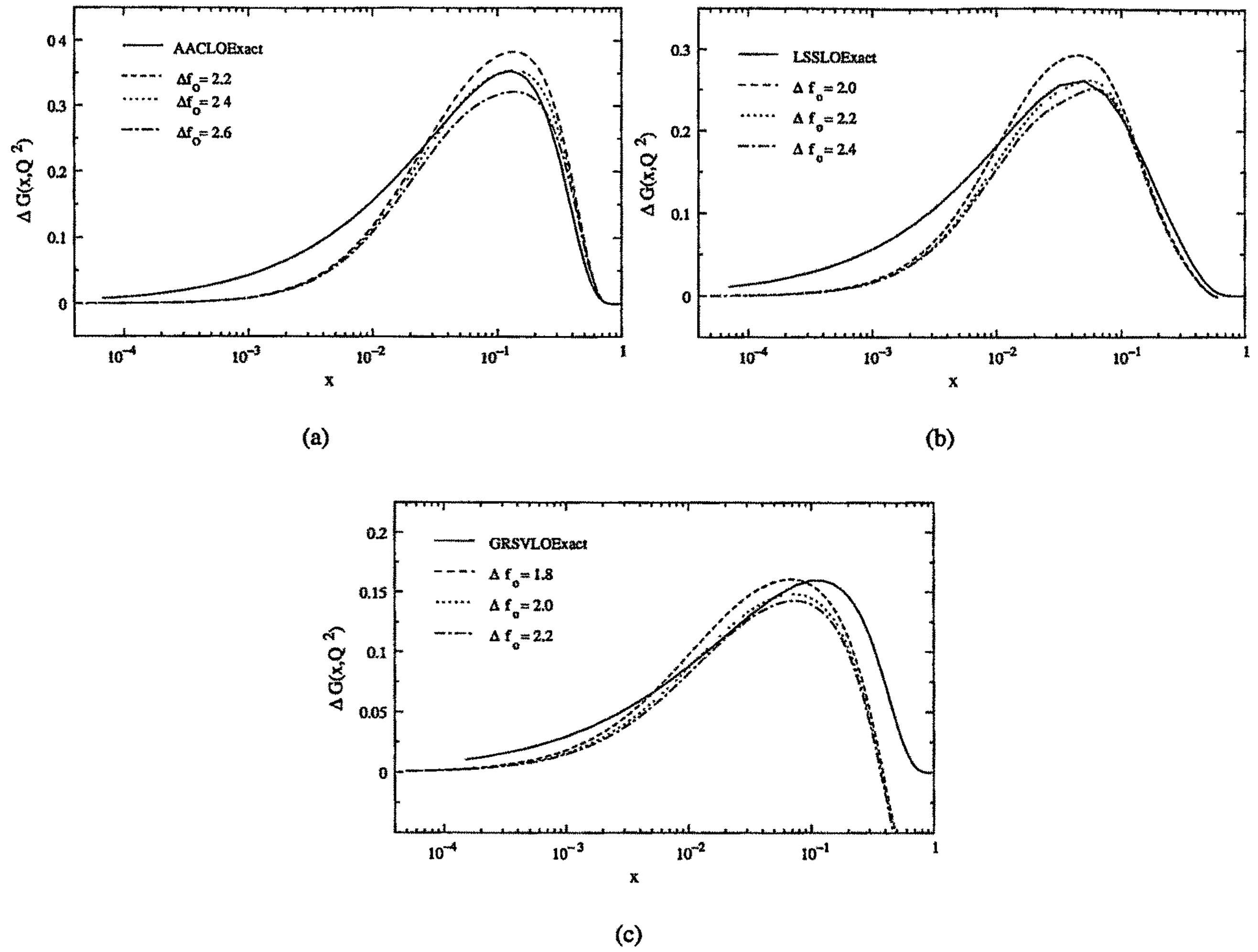

Figure 6.1: Polarized gluon given by the analytic expression Eq.(6.86) compared with the exact (a)AAC00LO[107], (b)LSSLO[108] and (c) GRSVLO[109] distributions at $Q^2 = 5GeV^2$ for different values of the free parameter Δf_0.

Table 6.1: Recent Gluon asymmetry $\frac{\Delta G}{G}$ data from COMPASS, SMC and HERMES

COMPASS	$\frac{\Delta G}{G} = 0.06 \pm 0.31(st) \pm 0.06(syst)$	$< x >= 0.13$
SMC	$\frac{\Delta G}{G} = -0.20 \pm 0.28(st) \pm 0.10(syst)$	$< x >= 0.07$
HERMES	$\frac{\Delta G}{G} = 0.41 \pm 0.018(st) \pm 0.03(syst)$	$0.06 < x < 0.28$

carried by the struck quark. The measurement is carried out at a momentum scale of the order of $Q^2 = 2 \sim 5GeV^2$. We calculate the quantity $\frac{\Delta G}{G}$, with ΔG given by Eq.(6.86) and the unpolarized gluon distribution taken from the solution Eq.(2.36) of the LO gluon evolution equation as obtained in Chapter 2. Unpolarized gluon distribution is evolved with MRSTLO[101] input distribution at $Q^2 = 1GeV^2$. We show our calculated values for $\frac{\Delta G}{G}$ at $Q^2 = 5GeV^2$ along with the available data in Fig.6.2. The vertical error bar in the data represents the statistical error and the horizontal error bar the standard deviation of the x_g distribution at which the measurement is carried out. From the figure we see that our predicted $\frac{\Delta G}{G}$ reproduces the general trend of the values of $\frac{\Delta G}{G}$ indicated by the data. We also see that the gluon asymmetry $\frac{\Delta G}{G}$ invariably remains positive and conforms to the positivity constraint $\frac{\Delta G}{G} \leq 1$. However, this is true only at low momentum scale $Q^2 \leq 5GeV^2$. As the momentum scale is increased, the condition $\frac{\Delta G}{G} < 1$ breaks down at certain values of x which we discuss in the next paragraph.

At this point, few remarks about the positivity condition are in order. The positivity condition has the origin in the probabilistic interpretation of the parton densities. According to it, the magnitude of a polarized cross-section should be smaller than the corresponding unpolarized one, *i.e.* $|\Delta\sigma| \leq \sigma$. In LO, probabilistic interpretation can be applied to parton distributions and hence this condition (i.e. $|\Delta\sigma| \leq \sigma$) implies that the parton densities should satisfy the relation

$$\left|\Delta f\left(x, Q^2\right)\right| \leq f\left(x, Q^2\right).$$

In determining the polarized distributions by different groups, the positivity condition

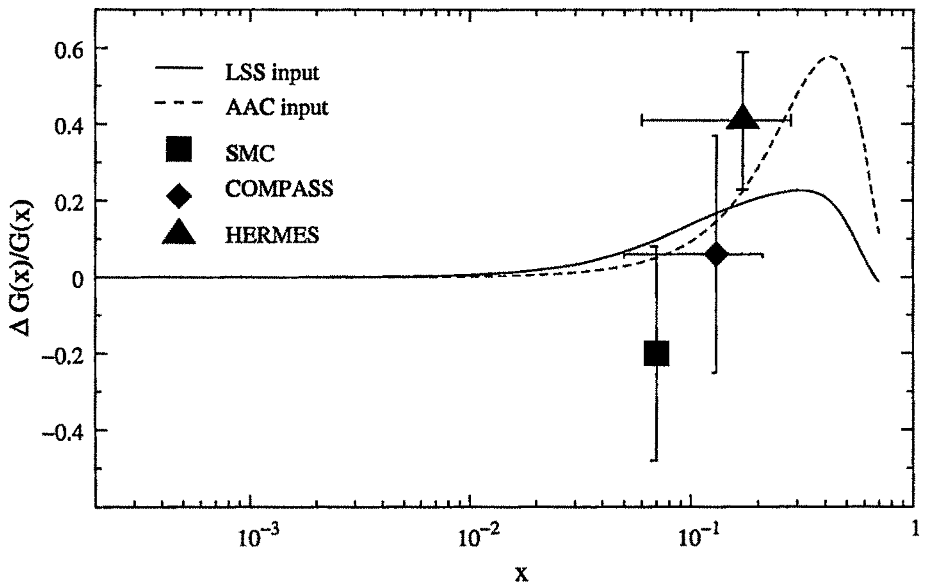

Figure 6.2: Gluon asymmetry with two different inputs compared with data from SMC[115],COMPASS[118] and HERMES[120].Values of Δf_0 are chosen from figure 6.1 as discussed in the text.

is imposed along with other rule such as quark counting rule[122, 123, 124] to find the parameters of the initial distributions. In our study of the polarized DGLAP equations, since we are not making any fit except varying the parameter Δf_0, we see for what kinematic regions, our solutions conform to the positivity condition for ΔG for a particular input distribution. Taking the LSSLO[108] input distribution, we calculate $\frac{\Delta G}{G}$ for a large number of fixed Q^2 and see for what value of x if there is any, the condition $\frac{\Delta G}{G} \leq 1$ breaks down. The value of the free parameter Δf_0 is chosen to be 2.2 (another graph for $\Delta f_0 = 2.0$ also shown) as obtained from the previous analysis. In Fig.6.3 we show the $x - Q^2$ range within which the condition $\frac{\Delta G}{G} \leq 1$ holds good. We see that as Q^2 is increased, the condition $\frac{\Delta G}{G} \leq 1$ breaks down at progressively lower values of x. On the other hand, for low momentum scale $Q^2 < 5GeV^2$ it is always less than one.

Next we consider the flavour singlet polarized quark distribution $\Delta\Sigma(x, t)$ given by Eq.(6.85) and compare it with exact AACLO[107] and LSSLO[108] distributions. Taking the inputs also from AACLO[107] and LSSLO[108] respectively, we evolve $\Delta\Sigma(x, Q^2)$

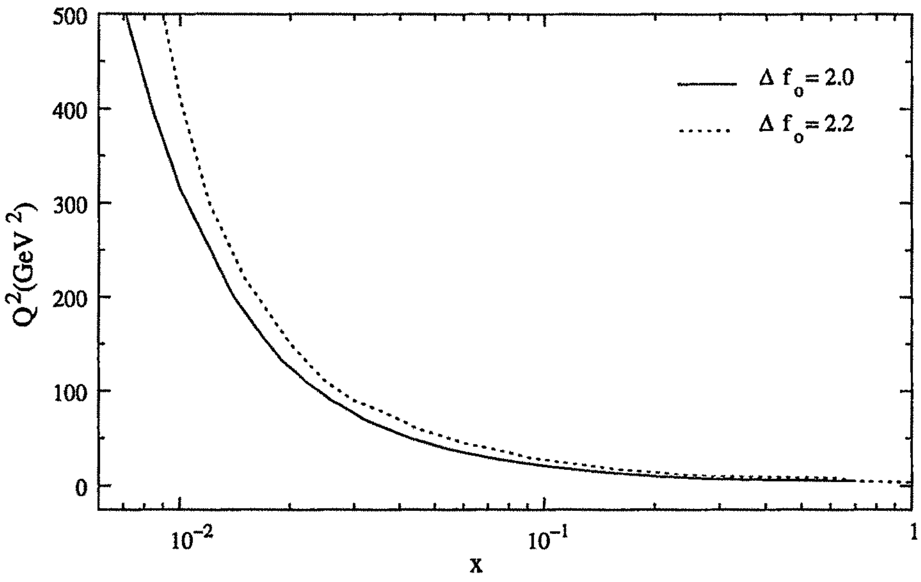

Figure 6.3: $x - Q^2$ range where the positivity condition $\frac{\Delta G}{G} \leq 1$ holds good. In the region below the curve the condition $\frac{\Delta G}{G} \leq 1$ is true. The graphs are drawn with LSS[108] input in Eq.(6.86) for two different values of Δf_0. Unpolarized gluon $G(x,t)$ is taken from Eq.(2.36).

using Eq.(6.85) at two different scales $Q^2 = 2GeV^2$ and $Q^2 = 5GeV^2$ and plot these as function of x along with the exact solutions in Fig.6.4. The value of Δf_0 is taken to be 2.4 for AAC input and 2.2 for LSS input respectively as determined earlier. From the graphs we see that the qualitative features of the LO exact distribution is well reproduced by the analytical form of $\Delta\Sigma(x, Q^2)$ given by Eq.(6.85).

The value of $\Delta\Sigma(x, Q^2)$ and $\Delta g(x, Q^2)$ give us information about the contribution of the quark and the gluon to the total spin of the proton. The contribution of the singlet quark $\Delta\Sigma\left(x, Q^2\right)$ and the gluon distributions $\Delta g(x, Q^2)$ to the total spin of the proton are given by their first moments:

$$\Delta\Sigma(Q^2) = \int_0^1 \Delta\Sigma(x, Q^2)dx$$

$$\Delta g\left(Q^2\right) = \int_0^1 \Delta g\left(x, Q^2\right)dx.$$

Table 6.2: Quark and Gluon helicity distribution

	$Q^2(GeV^2)$	AAC	GRSV	ours
$\Delta\Sigma$	5	0.18	0.259(standard) 0.248(valence)	0.201
Δg	5	1.314	0.684 (standard) 0.963	0.31

To calculate the first moment we have to integrate over the entire range of x from 0 to 1. But in our derivation, we have an upper limit of x up to which we can calculate our functions $\Delta\Sigma(x, Q^2)$ and $\Delta g(x, Q^2)$. This limit is set by the functions τ_1' and τ_2' above which they exceed the physical limit of unity. This upper limit is dependent on Q^2. For $Q^2 = 5GeV^2$ this upper limit is $x_{max} \approx 0.6$. We calculate the first moments for our functions $\Delta\Sigma(x, Q^2)$ and $\Delta g(x, Q^2)$ at $Q^2 = 5GeV^2$ and show in table 6.2 along with the results obtained by GRSV[109] and AAC[107] LO parametrization. We have quoted only the LO result. The usual range of $\Delta\Sigma$ obtained by different groups lies between $0.1 \sim 0.3$ and we see that our calculated $\Delta\Sigma$ is within this range. For the gluon polarization we have obtained a low value compared to the two parametrizations, presumably due to the cut-off imposed on x_{max} in evaluating the integrals.

For completeness, we also note that a parallel analysis is possible if one assumes an *ad-hoc* analytical form of the unknown function $\Delta f(t)$ in Eq.(6.78) instead of the parameter Δf_0 (Eq.6.79) adjusted to the exact result. In Chap 5, a similar analysis was performed in unpolarized structure function assuming $f(t) = (lnt)^k$ where k is fitted from data. In the present chapter, we refrain from doing similar analysis since Q^2 explored in the PDIS is limited near the boundary $t \approx t_o$ and effectively $\Delta f(t) \simeq \Delta f(t_0) = \Delta f_0$.

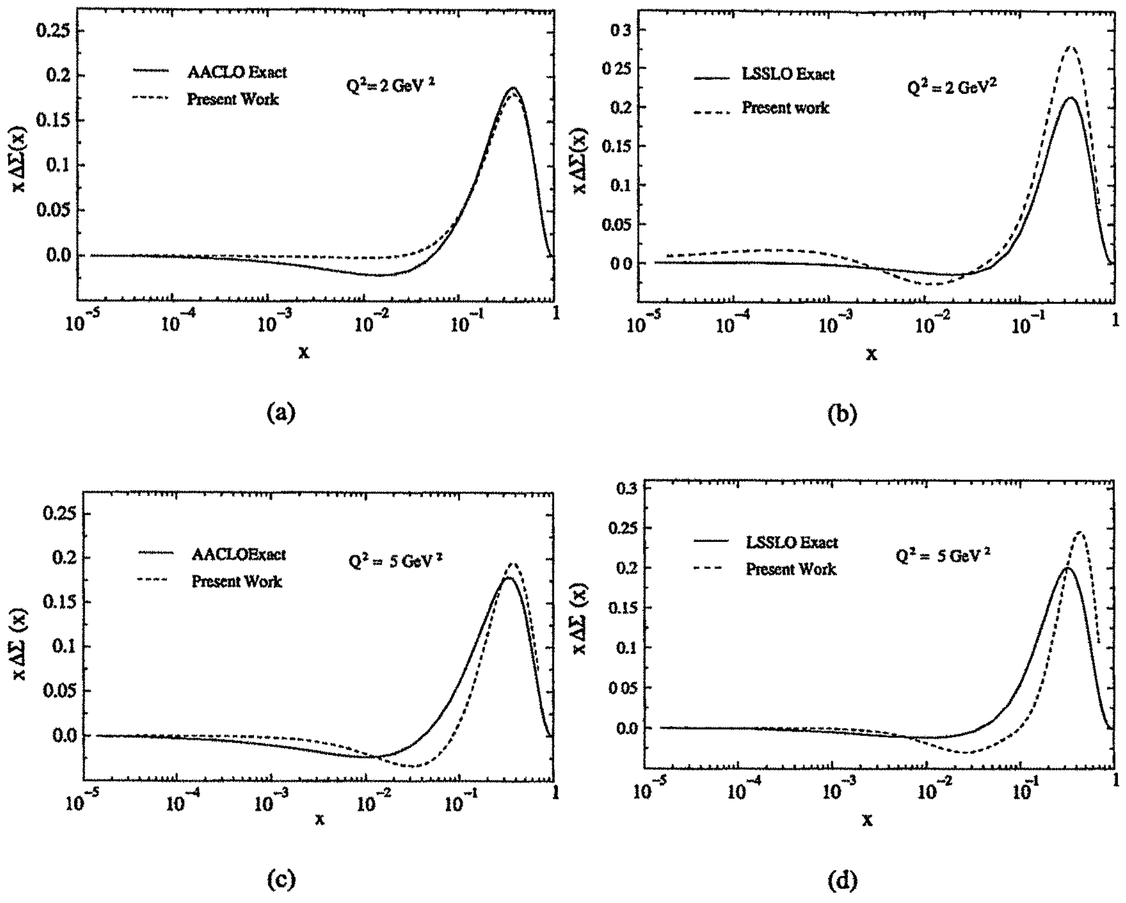

Figure 6.4: Polarized singlet distribution given by Eq.(6.85) as a function of x compared with the exact AAC00LO[107] and LSSLO[108] distributions at two fixed $Q^2 = 2GeV^2$ and $Q^2 = 5GeV^2$. Values of Δf_0 are chosen as discussed in the text.