Chapter 2

Approximate solution of DGLAP equation for gluon at low x by the method of characteristics

It is well known for a long time [60] that with reasonable boundary conditions perturbative QCD predicts a universal growth of gluon structure function at large t ($t = ln(Q^2/\Lambda^2)$) and small x faster than any power of $ln(1/x)$ but slower than any inverse power of x. More recently this perturbative prediction was brought to the phenomenological front [61, 62] through double asymptotic scaling. One of the recent important observation from this prediction is the rise of the exponent of the structure function with increasing t ($t = ln(Q^2/\Lambda^2)$), consistent with HERA data at low x. In recent years, an approximate method of solving DGLAP equations[16, 17, 18, 19] at low x has been pursued [63, 64, 65] with considerable phenomenological success. However that approach had some limitations:the solution reported was not unique[66] and there was an *ad-hoc* assumption about the factorizability of x and Q^2 dependence of the gluon momentum distribution $G(x, Q^2) = xg(x, Q^2)$. The solutions were selected as the simplest ones with a single boundary condition on the non-perturbative $x-$ distribution of the structure function at some $Q^2 = Q_0^2$. However, complete solution of DGLAP equations with two differential variables generally needs two boundary conditions[67], one at $x \to 0, t \to \infty$ limit of double asymptotic scaling and the other at any fixed $Q^2 = Q_0^2$. In this chapter, we develop a method of obtaining a unique solution with a single boundary condition without any assumption of factorizabilty. We apply the method of characteristics as discussed in §1.5 of chapter 1. In §2.1 we summarise the theory and in §2.2 we discuss our result.

2.1 Formalism

2.1.1 Gluon momentum density

The DGLAP equations for the gluon structure functions $G(x, Q^2)(= xg(x, Q^2))$ taking only the leading term of the gluonic kernel [68] and neglecting the singlet structure function $F_2^S(x, Q^2)$ is

$$\begin{aligned}\frac{\partial G(x,t)}{\partial t} = \; & \frac{3\,\alpha_s(t)}{\pi}\left[\left\{\left(\frac{11}{12}-\frac{n_f}{18}\right)+ln(1-x)\right\}G(x,t)\right.\\ & \left.+\int_x^1 dz\left\{\left(\frac{zG(x/z,t)-G(x,t)}{1-z}\right)+\left(z(1-z)+\frac{1-z}{z}\right)G(x/z,t)\right\}\right]. \quad (2.1)\end{aligned}$$

In Eq.(2.1)

$$\alpha_s(t) = \frac{4\pi}{\beta_0 t} = \frac{\gamma^2\pi}{3t}, \tag{2.2}$$

where $t = ln\frac{Q^2}{\Lambda^2}$, $\gamma = \sqrt{\frac{12}{\beta_0}}$ and $\beta_0 = 11 - \frac{2n_f}{3}$, n_f being the number of active flavours. Eq.(2.1) can be written as:

$$\frac{\partial G(x,t)}{\partial t} - \left(\frac{\gamma^2}{t}\right)\left[\left\{\left(\frac{11}{12}-\frac{n_f}{18}\right)+ln(1-x)\right\}G(x,t)+I_g(x,t)\right] = 0, \tag{2.3}$$

where

$$I_g(x,t) = \int_x^1 dz \left\{ \left(\frac{zG(x/z,t) - G(x,t)}{1-z} \right) + \left(z(1-z) + \frac{1-z}{z} \right) G(x/z,t) \right\}. \tag{2.4}$$

We now introduce a variable u as

$$u = 1 - z. \tag{2.5}$$

Since $x < z < 1$, therefore $0 < u < 1 - x$ and hence we can approximate $\frac{x}{z}$ as

$$\frac{x}{z} = \frac{x}{1-u} \approx x(1+u). \tag{2.6}$$

Using Eq.(2.6) we expand $G(x/z,t)$ in a Taylor series as

$$G\left(\frac{x}{z},t\right) = G(x,t) + xu\frac{\partial G(x,t)}{\partial x} + \frac{1}{2}x^2u^2\frac{\partial^2 G(x,t)}{\partial x^2} + \ldots \quad , \tag{2.7}$$

which covers the entire range of u, i.e. $0 < u < 1 - x$. In order to check the convergence of the series(2.7) let us a use a general form of the gluon density

$$G(x,t) \sim x^{\lambda(x,t)}, \tag{2.8}$$

where the slope function $\lambda(x,t)$ is in general x and t dependent. Convergence of the series Eq.(2.7) requires that

$$\left| xu\frac{1}{2}\frac{\partial^2 G(x,t)}{\partial x^2} \right| < 1. \tag{2.9}$$

This condition leads to

$$u < u_0, \tag{2.10}$$

where

$$u_0 = \frac{2\left| lnx\frac{\partial\lambda(x,t)}{\partial x} + \frac{\lambda(x,t)}{x} \right|}{x\left| \left(lnx\frac{\partial\lambda(x,t)}{\partial x} + \frac{\lambda(x,t)}{x} \right)^2 + \left(lnx\frac{\partial^2\lambda(x,t)}{\partial x^2} + \frac{2}{x}\frac{\partial\lambda(x,t)}{\partial x} - \frac{\lambda(x,t)}{x^2} \right) \right|} \quad . \tag{2.11}$$

Eq.(2.11) gives the upper limit of u upto which the series(2.7) is convergent. However, as $0 < u < 1 - x$, therefore Eq.(2.9) yields an upper limit of x as well

$$x_{max} < 1 - u_0 \,. \tag{2.12}$$

For x independent λ, Eq.(2.11) simplifies to

$$u_0 = \frac{2}{|\lambda - 1|} \,. \tag{2.13}$$

The above analysis demonstrates that series(2.7) is convergent so long x does not exceed the limit set by Eq.(2.12). In case λ is x dependent, Eq.(2.11) indicates that both u_0 and x_{max} also develop x dependence. In that case only the region of x satisfying

$$x \leq x_{max} \tag{2.14}$$

for the particular value of Q^2 will have validity of the present formalism. On the other hand for simpler case when λ is x independent then Eq.(2.13) implies that convergence is assumed only for $3 < \lambda < -1$, suggesting its limited utility. Assuming the validity of convergence and neglecting the terms $O(u^2)$ and higher, Eq.(2.7) has the form [65]

$$G\left(\frac{x}{z}, t\right) \approx G(x,t) + xu\frac{\partial G(x,t)}{\partial x}. \tag{2.15}$$

Using Eq.(2.6) and Eq.(2.15) in Eq.(2.4) and performing the u integration we obtain

$$I_g(x,t) = R_g(x)G(x,t) + P_g(x)\frac{\partial G(x,t)}{\partial x} \,, \tag{2.16}$$

where we have used the identity

$$\sum_{k=1}^{\infty} \frac{u^k}{k} = ln\frac{1}{1-u} \,. \tag{2.17}$$

The functions $R_g(x)$ and $P_g(x)$ have the explicit forms

$$R_g(x) = -2(1-x) + \frac{1}{2}(1-x^2) - \frac{1}{3}(1-x^3) + ln\frac{1}{x} \tag{2.18}$$

and

$$P_g(x) = \left(-\frac{11}{12} + 2x - \frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{x^4}{4} + ln\frac{1}{x}\right)x. \tag{2.19}$$

Using Eq.(2.16) in Eq.(2.3) the DGLAP equation for the gluon at low x can be written as

$$t\frac{\partial G(x,t)}{\partial t} - \gamma^2 P_g(x)\frac{\partial G(x,t)}{\partial x} - \gamma^2\left\{\left(\frac{11}{12} - \frac{n_f}{18}\right) + ln(1-x) + R_g(x)\right\}G(x,t) = 0. \tag{2.20}$$

We note that Eq.(2.20) has the derivative $\frac{\partial G(x,t)}{\partial x}$ which enables one to calculate the x evolution at low x [64] beyond its traditional use in t evolution only. In order to solve Eq.(2.20) by the method of characteristics [57], we recast it in the form:

$$a(x,t)\frac{\partial G(x,t)}{\partial x} + b(x,t)\frac{\partial G(x,t)}{\partial t} + c(x,t)G(x,t) = 0, \tag{2.21}$$

where

$$a(x,t) = \gamma^2 P_g(x), \tag{2.22}$$

$$b(x,t) = -t \tag{2.23}$$

and

$$c(x,t) = \gamma^2\left\{\left(\frac{11}{12} - \frac{n_f}{18}\right) + ln(1-x) + R_g(x)\right\}. \tag{2.24}$$

As an initial value to the problem we set

$$G(x,t)\mid_{t=t_0} = G(x), \tag{2.25}$$

which would correspond to some specific non-perturbative inputs[39, 40, 69, 70] available in the current literature. Eq.(2.21) is a homogeneous first order linear partial differential equation in the variables x and t. To convert it into an ordinary differential equation we now introduce two new variables s and τ such that

$$\frac{dx}{ds} = a(x,t)\,, \tag{2.26}$$

$$\frac{dt}{ds} = b(x,t)\,. \tag{2.27}$$

Eq.(2.26) and Eq.(2.27) define the characteristic curves of the differential equation(2.21). On using Eqs.(2.26) and (2.27) in Eq.(2.21) we get

$$\frac{dG(s,\tau)}{ds} + c(s,\tau)G(s,\tau) = 0\,, \tag{2.28}$$

which is an ordinary differential equation in the variable s along the characteristic curves defined by the solution of Eqs.(2.26) and (2.27). In Eq.(2.28) $c(s,\tau)$ is the function $c(x,t)$ (Eq.(2.24)) expressed in terms of the new variables s,τ with the help of the transformation equations which are the solutions of Eqs.(2.26) and (2.27). Solving the parametric Eqs.(2.26) and (2.27) and setting along the characteristic curves at $s = 0$,

$$x(s=0) = \tau, \qquad t(s=0) = t_0 \tag{2.29}$$

to find out the constants of integrations, we get the transformation equations as

$$s = -ln\left(\frac{t}{t_0}\right) \tag{2.30}$$

and

$$ln\tau + \frac{11}{12} = \left(ln\frac{1}{x} + \frac{11}{12}\right)\left(\frac{t_0}{t}\right)^{\gamma^2}\,, \tag{2.31}$$

where we have approximated $P_g(x)$ and $R_g(x)$ in the limit $x \to 0$ as

$$P_g(x) \approx x\left(ln\frac{1}{x} - \frac{11}{12}\right), \tag{2.32}$$

$$R_g(x) \approx \left(ln\frac{1}{x} - \frac{11}{6}\right). \tag{2.33}$$

Using Eq.(2.30) and Eq.(2.31) we can express $c(x,t)$ defined in Eq.(2.24) in terms of s and τ as

$$c(s,\tau) = -\gamma^2\left\{\left(ln\tau + \frac{11}{12}\right)exp(-\gamma^2 s) + \frac{n_f}{18}\right\}. \tag{2.34}$$

Putting Eq.(2.34) in Eq.(2.28) and solving the differential equation yields

$$G(s,\tau) = G(\tau)exp\left[-\left(ln\tau + \frac{11}{12}\right)exp(-\gamma^2 s) + \left(\frac{\gamma^2 n_f}{18}\right)s + \left(ln\tau + \frac{11}{12}\right)\right]. \tag{2.35}$$

This solution is in the (s,τ) space. Transforming back into the (x,t) space with the help of the inverse of the transformation Eqs.(2.30) and (2.31) we get

$$G(x,t) = G(\tau)x^{-\left(1-(\frac{t_0}{t})^{\gamma^2}\right)}\left(\frac{t_0}{t}\right)^{\frac{\gamma^2 n_f}{18}} exp\left[-\frac{11}{12}\left\{1 - (t_0/t)^{\gamma^2}\right\}\right], \tag{2.36}$$

where τ appearing in $G(\tau)$ is to be expressed in terms of x and t using Eq.(2.31) as

$$\tau = exp\left[\left(-ln\frac{1}{x} + \frac{11}{12}\right)\left(\frac{t_0}{t}\right)^{\gamma^2} - \frac{11}{12}\right]. \tag{2.37}$$

Here it is to be noted that $G(\tau)$ is the initial condition of the problem and is obtained from any input x distribution simply by substituting τ for x [57]. Eq.(2.36) is our main result.

This result is to be compared with the known DLLA[60] form

$$G(x,t) = G(x,t_0)exp\left[\sqrt{4\gamma^2 ln\left(\frac{t}{t_0}\right)ln\left(\frac{1}{x}\right)}\right] \tag{2.38}$$

provided the gluon is not singular at $t = t_0$.

Our present result is different from the solution obtained earlier using the method reported in refs.[64, 65], which is

$$G(x,t) = G(x,t_0)\left(\frac{t}{t_0}\right) \tag{2.39}$$

or the result derived in [71, 72] with the factorization ansatz, which is

$$G(x,t) = G(x,t_0)\left(\frac{t}{t_0}\right)^{\frac{12}{\beta_0}ln(\frac{1}{x})} . \tag{2.40}$$

In §2.2 we study the quantitative differences of these alternative forms of gluon.

2.1.2 Slope of the structure function at low x from gluon density

There are different formulae [73, 74, 75] that relate the gluon density to the scaling violation of $F_2(x,Q^2)$ at low x. Prytz formula [73] reads

$$\frac{\partial F_2(x,Q^2)}{\partial lnQ^2} \approx \frac{10\alpha_s}{27\pi} G(2x), \tag{2.41}$$

while that of Bora and Choudhury [74] suggests

$$\frac{\partial F_2(x,Q^2)}{\partial lnQ^2} \approx \frac{5\alpha_s}{3\pi} G\left(\frac{4}{3}x\right) . \tag{2.42}$$

Gay Ducati and Goncalves [75] have obtained a general relation between gluon and the scaling vilolation of $F_2(x,Q^2)$ which incorporates both of the above results Eq.(2.41) and Eq.(2.42). They expanded the gluon $G(\frac{1}{1-z})$ in the scaling violation relation at an arbitrary point of expansion $z = \alpha$ and obtained in the small x limit the formula:

$$\frac{\partial F_2(x,Q^2)}{\partial lnQ^2} \approx \frac{10\alpha_s}{27\pi} G\left[\frac{x}{1-\alpha}\left(\frac{3}{2} - \alpha\right)\right] , \tag{2.43}$$

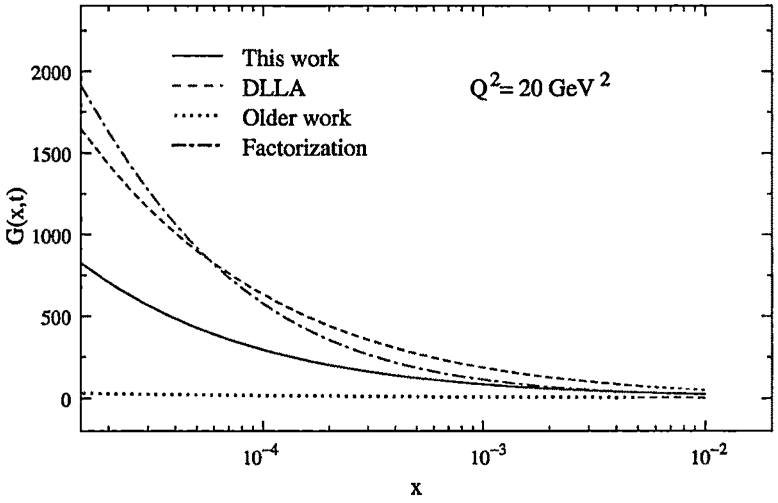

Figure 2.1: Gluon distributions from Eqs.(2.36), (2.38), (2.39) and (2.40) at $Q^2 = 20GeV^2$. Input gluon is taken from ref.[70]. In the label, Eqs.(2.36), (2.38), (2.39) and (2.40) are respectively referred as this work, DLLA, older work and factorization.

where $\alpha < 1$ is an arbitrary point of expansion. Using Eq.(2.43) we will estimate the logarithmic slope of the structure function from the proposed gluon distribution Eq.(2.36) at several points of expansion α and compare with data at $Q^2 = 20GeV^2$.

2.2 Results and discussion

In this chapter we have obtained an analytical solution of the gluon given by Eq.(2.36) valid to be at low x. Already existing forms of gluon available in literature are Eqs.(2.38), (2.39) and (2.40). While Eq.(2.38) is the standard form [60] in QCD, Eq.(2.39) is based on the solution of Lagrange's auxillary system of equations [66] in x and t. Similarly, Eq.(2.40) is derived on the *ad-hoc* assumption of factorizability of gluon distribution in x and t. In the new approximation Eq.(2.36), the inherent difficulty of non-unique nature of solutions of Lagrange's auxillary system of equations [64, 65] while deriving Eq.(2.39) as well as the *ad-hoc* assumption of factorizability in Eq.(2.40) are avoided. In this way the new approximation is a definite improvement over the two forms Eq.(2.39) and Eq.(2.40).

In Fig.2.1 we compare the gluon distribution given by Eq.(2.36) with the already

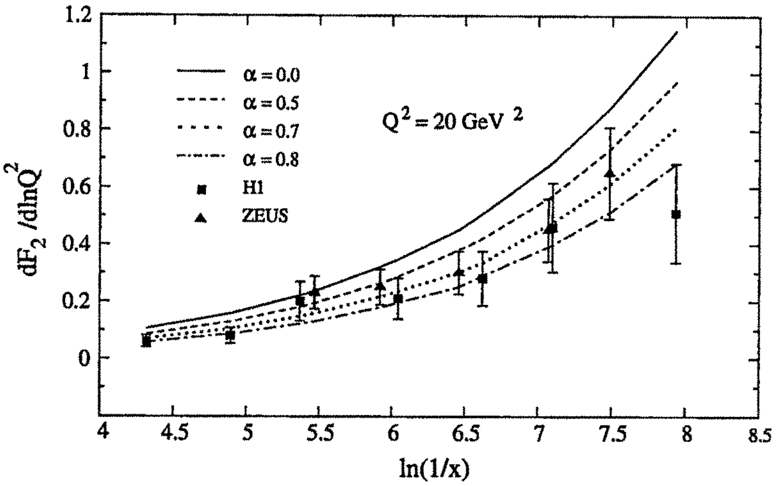

Figure 2.2: Slope of the structure function $dF_2(x, Q^2)/dlnQ^2$ using Eq.(2.36) and the relation Eq.(2.43) obtained at several points of expansion α. Data from H1 [27] and ZEUS [34].

existing forms of gluons given by Eqs.(2.38), (2.39) and (2.40) at $Q^2 = 20GeV^2$. For our evolution we take the MRSA[70] gluon at $Q_0^2 = 4GeV^2$. While evolving the gluon using Eq.(2.36) we need $G(\tau)$ as input rather than $G(x, t_0)$ which appear in Eqs.(2.38), (2.39) and (2.40). We note that $G(\tau)$ contains both x and t variables through Eq.(2.37). At $t = t_0$, τ reduces to x and $G(\tau)$ reduces to $G(x)$ which is the standard form of x-distribution of input. However, there is no additional arbitrariness in the choice of $G(\tau)$ as compared with $G(x, t_0) = G(x)$ of Eqs.(2.38), (2.39) and (2.40), since it is obtained by a formal replacement $x \rightarrow \tau$[57] in the same input $G(x)$ [70]. We also note that taking some explicit parametrization for $G(\tau)$ from some other sources e.g. GRVLO [76] is undoubtedly unjustified. In such a case there would have been no predictive value at all for this approximation.

To test the predicted gluon momentum distribution given by Eq.(2.36) further, we use the method of ref[75] and calculate the slopes $\dfrac{dF_2(x, Q^2)}{dlnQ^2}$ using Eq.(2.43) for several points of expansion $\alpha = 0.0, 0.5, 0.7$ and 0.8. We take our input gluon from MRSA [70] at $Q_0^2 = 4GeV^2$ with $n_f = 4$ and compare our result with H1 [27] and ZEUS [34] data at $Q^2 = 20GeV^2$. This is shown in Fig.2.2. We note from the figure that the data favour

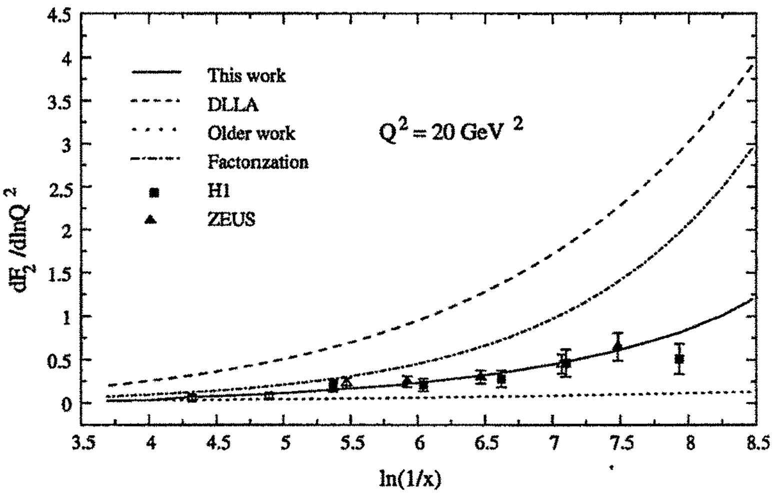

Figure 2.3: Slope of the structure function $dF_2(x, Q^2)/dlnQ^2$ using Eq.(2.36), (2.38), (2.39) and (2.40) at the point of expansion $\alpha = 0.7$. Data from H1[27] and ZEUS [34].

Eq.(2.36) when the point of expansion of the gluon is chosen around $\alpha \approx 0.7$. In Fig.2.3 we have plotted the structure function slopes using the four different gluon density given by Eqs.(2.36), (2.38), (2.39) and (2.40) with the expansion point $\alpha = 0.7$ and compare the predictions with the same set of data as in Fig.2.2 at $Q^2 = 20GeV^2$. At this point the question that naturally arises is, whether other values of α yield better agreement for Eqs.(2.38), (2.39) and (2.40) compared with Eq.(2.36). To explore this we vary the point of expansion α from very low value of 0.2 to 0.99 and calculate the slopes of the structure functions using all the Eqs.(2.36), (2.38), (2.39) and (2.40). We observe that the x dependence of the slopes flatten out with increasing values of α. We display in Fig.2.4(a-d) four such graphs for values of $\alpha = 0.5, 0.7, 0.9$ and 0.98 respectively along with H1[27] and ZEUS data [34] at $Q^2 = 20GeV^2$. We note that the value of α for which the calculated slopes agree with the experimental data is different for different equations. Eq.(2.38) i.e. the DLLA result gives better agreement for $\alpha = 0.98$, whereas Eq.(2.40), i.e. the factorization result is good when the value of α is chosen around 0.9 while the older result Eq.(2.39) fails. The present work i.e. Eq.(2.36) fits best with the data when the value of α is around 0.7.

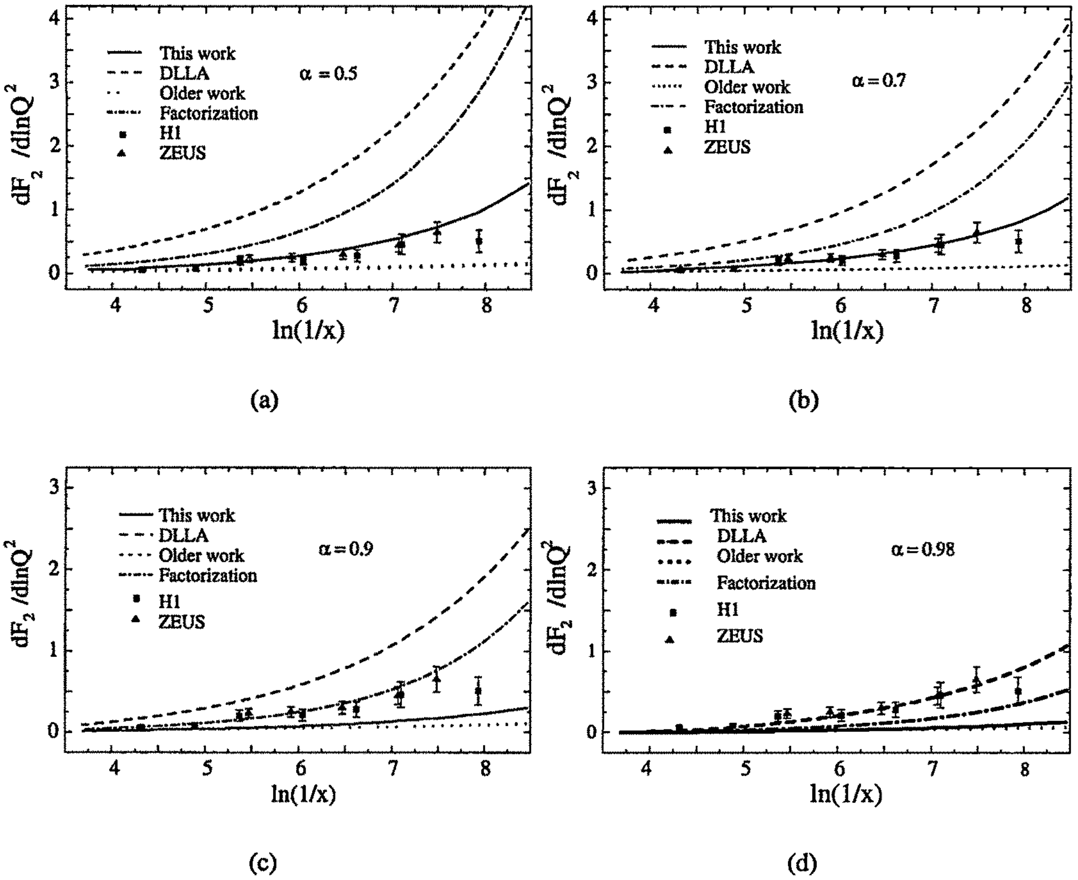

Figure 2.4: (a-d)Slope of the structure function $dF_2(x, Q^2)/dlnQ^2$ using Eq.(2.36), (2.38), (2.39) and (2.40) at the point of expansion α = 0.5, 0.7, 0.9 and 0.98 respectively and comparison with data from H1[27] and ZEUS [34].

Let us therefore consider the relative merits of Eq.(2.36) over the others, in the light of analysis of Gay Ducati and Goncalves [75] where the points of expansion α in the range $0.5 \leq \alpha \leq 0.8$ are favoured. Since the longitudinal momentum of the gluon x_g is given by the quantity within the parenthesis of G occured in Eq.(2.43) i.e. $x_g = \frac{x}{1-\alpha}\left(\frac{3}{2} - \alpha\right)$, value of α in the above range implies that x_g is more than twice the value of the longitudinal momentum of the probed quark (or antiquark) in DIS. It is concluded in ref.[75] in conformity with the result obtained from Glauber-Müeller approach [77] which includes shadowing corrections, that the more suitable points of expansion of the gluon are in the range $0.5 \leq \alpha \leq 0.8$ (i.e.$2x \leq x_g \leq 3.5x$). This conclusion also agrees with that of Ryskin *et al* [78] where the estimated value of the longitudinal gluon momentum x_g is shown to be approximately three times larger than the Bjorken x_B, which corresponds to the expansion at $\alpha = 0.75$. In the light of these analyses of refs. [75, 77, 78], Eq.(2.36) is definitely preferred over Eq.(2.39) and Eq.(2.40). The approximate form Eq.(2.36) is admissible only if it helps in explaining general trend of data or else explains the data precisely in a limited kinematic range. One also needs to know how the approximate formula Eq.(2.36) compares with the exact ones. In Fig.(2.5) we present the gluon density obtained from Eq.(2.36) with the MRSA input at $Q_0^2 = 4GeV^2$ and compare with the exact results GRV98LO [40]. We compare the two at representative values of $Q^2 = 3.5, 5, 10, 20, 40$ and $80GeV^2$ and $10^{-5} \leq x \leq 10^{-1}$. At low Q^2, the difference between the two is much prominent specially for $x < 10^{-4}$. For $Q^2 < 20GeV^2$, the predicted gluon density Eq.(2.36) lies below GRVLO. As Q^2 increases, the difference between the two gradually shrinks and for $Q^2 \geq 20GeV^2$ our predicted gluon rises faster than the GRV gluon. Above $Q^2 = 20GeV^2$, the deviation from the exact result is much prominent in the small x regime $x < 10^{-4}$ and the difference increases with increasing Q^2. From the study of the cross over points we infer that for $3.5 < Q^2 < 80GeV^2$ and $x > 10^{-3}$ our prediction lies within about 10 percent of the exact result.

2.3 Conclusion

In this chapter we have proposed an analytical solution of the gluon distribution (Eq.2.36) at low and moderate x which is theoretically and phenomenologically favoured over the earlier forms Eqs(2.39) and (2.40). We have also seen that the scaling violation relationship Eq.(2.43) which was earlier used to extract the gluon momentum density can also be used to describe the logarithmic slope of the structure function F_2 data using our proposed gluon momentum density if we properly choose the point of expansion of $G(\frac{x}{1-z})$

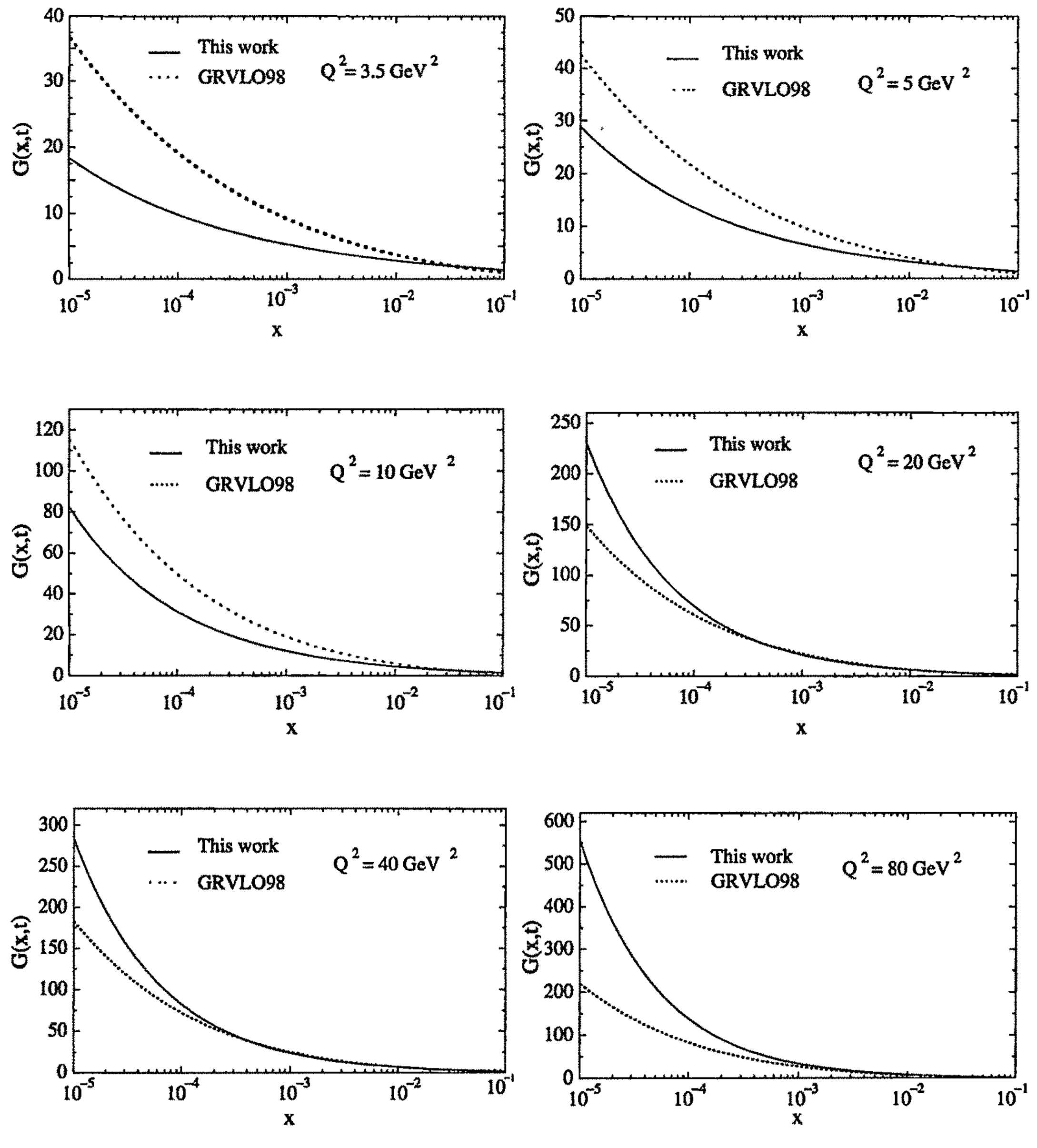

Figure 2.5: Gluon distribution obtained from Eq.(2.36) and compared with GRVLO98 at $Q^2 = 3.5, 5.0, 10, 20, 40$ and $80 GeV^2$.

in conformity with the approach of refs.[77, 78]. The approximate formula has a limited range of validity and breaks down particularly at small x which is to be expected in the DGLAP framework.