Chapter 2

Pseudo Algebraic Spaces

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2.1 Introduction:

In this chapter, introducing the concepts of Pseudo topology (p-topology), Pseudo topological space (p-topological space) and Pseudo algebraic space (p-a space), we have discussed some of their properties like p-normal set and Pseudo continuity (p-continuity). A special kind of mappings called p-a homomorphism with their properties is introduced. We have also introduced the notion of p-Kernel of a p-a homomorphism with example.

2.2 Pseudo Topological Spaces:

Following the definitions of general topology and group, we have defined as follows:

Definition 2.2.1 Let $X$ be a non-empty set and $T$ a class of subsets of $X$ such that

i) $X \in T$

ii) there exists an $A_0 \in T$ such that $A_0 \subseteq A$ for every $A \in T$

iii) any finite intersection of members of $T$ is in $T$.

The class $T$ is called a Pseudo topology on $X$ and the pair $(X, T)$ is called a Pseudo topological space. When there is no scope for confusion, $X$ may be simply called a p-topological space. The members of $T$ are called Pseudo open sets (p-open sets) in $X$. A set $A_0$ with the property (ii) is called a minimal p-open set.

Proposition 2.2.1 In a p-topological space, there is one and only one minimal p-open set.
**Proof:** Let $T$ be a $p$-topology on a non-empty set $X$ with two minimal $p$-open sets $A_o$ and $A'_o$. Since both $A_o$ and $A'_o$ belong to $T$ and both are minimal, $A_o \subseteq A'_o$ and $A'_o \subseteq A_o$ so that $A_o = A'_o$. This completes the proof.

**Remark 2.2.1** In view of the above proposition, a minimal $p$-open set is referred as the minimal $p$-open set.

**Example 2.2.1** A topological space is a $p$-topological space. The null set may be taken as the minimal $p$-open set, the topology as the $p$-topology.

Let $T = \{\emptyset, \{a\}, \{b, c\}, X\}$ be a topology on $X = \{a, b, c\}$. Then $T$ is a $p$-topology on $X$ and $(X, T)$ is a $p$-topological space with $\emptyset$ as the minimal $p$-open set.

**Example 2.2.2** A group is a $p$-topological space. The class of all subgroups may be taken as a $p$-topology with identity subgroup as the minimal $p$-open set. The $p$-topology $T$ consisting of all subgroups of a group $G$ is called the usual $p$-topology on $G$.

Let $G = \{1, -1, i, -i\}$ where $i = \sqrt{-1}$ be a group under multiplication. Let $T = \{\{1\}, \{1, -1\}, G\}$ be a $p$-topology on $G$, because $\{1\}$ is the minimal $p$-open set and any finite intersection of members of $T$ is again in $T$.

Example of a $p$-topology which is not a topology

**Example 2.2.3** Let $X = \{a, b, c, d\}$ and

$$T = \{\{a\}, \{a, b\}, \{a, c\}, X\}$$

Then $T$ is a $p$-topology on $X$ with $\{a\}$ as the minimal $p$-open set. But $T$ is not a topology on $X$. 

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Example 2.2.4  On the set of real numbers \( \mathbb{R} \),

Let \( T_1 = \{ \{1\}, \{1\} \cup \{0, \frac{1}{n}\}, n = 1, 2, 3, \ldots \} \)

\[ T_2 = \{ \{2\}, \{2\} \cup \{0, \frac{1}{n}\}, n = 1, 2, 3, \ldots \} \]

Then \( T_1 \) is a \( p \)-topology on \( \mathbb{R} \) with \( \{1\} \) as the minimal \( p \)-open set
and \( T_2 \) is also a \( p \)-topology on \( \mathbb{R} \) with \( \{2\} \) as the minimal \( p \)-open set.

From this definition, we observe that different \( p \)-topologies can be generated on the same set.

Definition 2.2.2  Two \( p \)-topologies on a set \( X \) are called compatible
if they have the same minimal \( p \)-open set.

Example 2.2.5  Let \( X = \{a, b, c, d\} \) and

\[ T_1 = \{ \{a\}, \{a, b\}, \{a, c\}, X \} \]

\[ T_2 = \{ \{a\}, \{a, b\}, \{a, d\}, X \} \]

Then \( T_1 \) and \( T_2 \) are compatible with common minimal \( p \)-open set \( \{a\} \).

Remark 2.2.2  In example 2.2.4 the two \( p \)-topologies \( T_1 \) and \( T_2 \) are
not compatible because their minimal \( p \)-open sets are different.

Proposition 2.2.2  The intersection of two compatible \( p \)-topologies
on a set is a \( p \)-topology on that set.

Proof:  Let \( T_1 \) and \( T_2 \) be two compatible \( p \)-topologies on a set \( X \) with
common minimal \( p \)-open set \( A_0 \). Clearly \( X \in T_1 \cap T_2 \) and \( A_0 \) is
the minimal element of \( T_1 \cap T_2 \). If \( A \) is any member of \( T_1 \cap T_2 \),
then \( A \in T_1 \) and \( A \in T_2 \), consequently \( A_0 \subseteq A \).
Finally if \( \{ A_i : 1 \leq i \leq n \} \subseteq T_1 \cap T_2 \), then
\[
\{ A_i : 1 \leq i \leq n \} \in T_1 \text{ and } \{ A_i : 1 \leq i \leq n \} \in T_2 \text{ and } T_1 \text{ and } T_2 \text{ are p-topologies, } \bigcap_{i=1}^{n} A_i \in T_1 \cap T_2. \]
Hence \( T_1 \cap T_2 \) is a p-topology on \( X \).

**Example 2.2.6** Let \( X = \{a, b, c, d, e\} \) and let
\[
T_1 = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, d\}, X\}
\]
and
\[
T_2 = \{\{a\}, \{a, b\}, \{a, d, e\}, X\}.
\]

\( T_1 \) and \( T_2 \) are compatible p-topologies with \( \{a\} \) as the minimal p-open set on \( X \). Then \( T_1 \cap T_2 = \{\{a\}, \{a, b\}, X\} \) is a p-topology on \( X \).

**Proposition 2.2.3** Union of two compatible p-topologies is a p-topology compatible with the original p-topology.

**Proof:** Let \( T_1 \) and \( T_2 \) be two compatible p-topologies on a set \( X \) with the minimal p-open set \( A_0 \). Then \( X \in T_1 \cup T_2 \) and \( A_0 \in T_1 \cup T_2 \). Let \( A \) be any member of \( T_1 \cup T_2 \), then \( A \in T_1 \) or \( A \in T_2 \) or \( A \) belongs to both \( T_1 \) and \( T_2 \), so that \( A \subseteq A_0 \). Finally, if \( A, B \in T_1 \cup T_2 \), then \( A, B \in T_1 \) or \( A, B \in T_2 \) or \( A, B \) belong to both \( T_1 \) and \( T_2 \); since \( T_1 \) and \( T_2 \) are p-topologies on \( X \), then \( A \cap B \in T_1 \) and \( A \cap B \in T_2 \) and hence \( A \cap B \in T_1 \cup T_2 \). Therefore \( T_1 \cup T_2 \) is a compatible p-topology on \( X \).

**Proposition 2.2.4** The intersection of two p-topologies on a finite set \( X \) is a p-topology on \( X \).

**Proof:** Let \( T_1 \) and \( T_2 \) be two p-topologies on a finite set \( X \). If they have the common minimal element, then it is already seen that \( T_1 \cap T_2 \) is a p-topology compatible with the given two p-topologies.
If $T_1$ and $T_2$ have different minimal elements and $g_0$ is the minimal common member of $T_1$ and $T_2$, then $g_0$ will be the minimal element of $T_1 \cap T_2$. Clearly $X \in T_1 \cap T_2$ and $T_1 \cap T_2$ is closed under formation of finite intersection.

But union of two $p$-topologies is not a $p$-topology. We consider example 2.2.7.

**Example 2.2.7** Let $X = \{1, 2, 3, 4, 5\}$ and we consider two $p$-topologies $T_1$ and $T_2$ on $X$ such that

$T_1 = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, X\}$

and $T_2 = \{\{2\}, \{2, 3\}, \{2, 4, 5\}, X\}$

Then $T_1 \cup T_2 = \{\{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{2, 4, 5\}, X\}$

is not a $p$-topology on $X$ since $T_1 \cup T_2$ has no minimal $p$-open set.

**Comparison of $p$-topologies:**

**Definition 2.2.3** Let $T_1$ and $T_2$ be two $p$-topologies on a set $X$. We say that $T_1$ is coarser (or weaker) than $T_2$ or that $T_2$ is finer (or stronger) than $T_1$ iff $T_1 \subseteq T_2$. If either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$, we say that the $p$-topologies $T_1$ and $T_2$ are comparable.

**Definition 2.2.4** Let $X$ be a non-empty set. Then $I = \{X\}$ is a $p$-topology on $X$. This $p$-topology on $X$ is called the indiscrete $p$-topology on $X$. The pair $(X, I)$ is called an indiscrete $p$-topological space.

**Definition 2.2.5** Let $X$ be a non-empty set. Let $D$ be the collection of all subsets of $X$, then $D$ is a $p$-topology on $X$, called the discrete $p$-topology on $X$. The pair $(X, D)$ is called a discrete $p$-topological space.
Remark 2.2.3 It is seen that the discrete p-topology and the discrete topology on a set are same.

2.3 Pseudo Algebraic Spaces:

We begin with the definition of Pseudo algebraic structure on Pseudo topological space.

Definition 2.3.1 A p-topological space \((X, T)\) is said to have a Pseudo algebraic structure (p-a structure) if there exists a Pseudo algebraic function (p-a function)

\[ \alpha : P^X \times P^X \rightarrow P^X \] (\(P^X\) is the power set of \(X\))

satisfying the following conditions:

i) \[ \alpha(\alpha(A, B), C) = \alpha(\alpha(B, C), A) \], \(A, B, C \in P^X\)

ii) \(\alpha(A, B) \in T\) if \(\alpha(A, B) = \alpha(B, A)\) for \(A, B \in T\)

iii) if \(A_1 \subseteq A, B_1 \subseteq B\), then \(\alpha(A_1, B_1) \subseteq \alpha(A, B)\)

iv) \(\alpha(A_0, A) = \alpha(A, A_0), A \in P^X\) where \(A_0\) is the minimal p-open set.

We say that the triplet \((X, T, \alpha)\) is a p-topological space with a p-a structure \(\alpha\) or simply a Pseudo algebraic space (p-a space).

Example 2.3.1 A topological space \((X, T)\) is a p-a space where \(\alpha(A, B) = A \cup B\).

Let \((X, T)\) be a p-topological space. Then \(T\) is a p-topology. Let us define

\[ \alpha : P^X \times P^X \rightarrow P^X \] by \(\alpha(A, B) = A \cup B\) for all \(A, B \in P^X\).

For \(A, B, C \in T\)
i) $\alpha(\alpha(A, B), C) = \alpha(A \cup B, C)$
   
   $= (A \cup B) \cup C$
   
   $= A \cup (B \cup C)$
   
   $= \alpha(A, \alpha(B, C))$

ii) $\alpha(A, B) = A \cup B$

   $= B \cup A$

   $= \alpha(B, A)$

   $\therefore \alpha(A, B) \in T.$

iii) if $A_i \subseteq A, \ B_i \subseteq B$, then

   $\alpha(A_i, B_i) = A_i \cup B_i$

   $\subseteq A \cup B$

   $= \alpha(A, B)$

iv) $\alpha(A_0, A) = A_0 \cup A, \ A \in P^X$, where $A_0$ is the minimal $p$-open set.

   $= A \cup A_0$

   $= \alpha(A, A_0)$

   $\therefore (X, T, \alpha)$ is a $p$-a space.

**Example 2.3.2** A group $G$ with the usual $p$-topology is a $p$-a space, where $\alpha(A, B) = AB$ ($AB$ means usual product of complexes $A$ and $B$)

Let $\alpha : P^G \times P^G \rightarrow P^G$ be defined by $\alpha(A, B) = AB$ ($AB$ means the usual product of complexes $A$ and $B$).
Then \( \alpha \) satisfies the conditions of a p-a function i.e.,

for \( A, B, C \in T \)

i) \( \alpha(\alpha(A, B), C) = \alpha(AB, C) \)

\[ = (AB)C \]

\[ = A(BC) \]

\[ = \alpha(\alpha(A, \alpha(B, C))) \]

ii) for two subgroups \( A, B \) of \( G \), the product \( AB \) is a subgroup if and only if

\( AB = BA \) i.e. \( A, B \in T \)

\( \alpha(A, B) \in T \) if \( \alpha(A, B) = \alpha(B, A) \)

iii) if \( A, B \leq G \) and \( A_1 \subseteq A, B_1 \subseteq B \), then

\[ A_1B_1 \subseteq AB \] i.e. if \( A, B \in T \) and

\[ A_1 \subseteq A, B_1 \subseteq B, \text{ then } \alpha(A_1, B_1) \subseteq \alpha(A, B) \]

iv) \( \alpha(A_0, A) = A_0A, A_0 \subseteq A, A \in PG \), where \( A_0 \) is the minimal p-open set.

\[ = AA_0 \]

\[ = \alpha(A, A_0) \]

\( \therefore (G, T, \alpha) \) is a p-a space.

2.4 Pseudo Normal Set:

We define p-normal set in a p-a space and discuss its basic properties.
Definition 2.4.1  A subset $A$ in a p-a space $(X, T, \alpha)$ is called a p-normal set if $\alpha(A, Y) = \alpha(Y, A)$ $\forall Y \in \mathcal{P}X$.

Definition 2.4.2  The p-topology $T$ of a p-a space $(X, T, \alpha)$ is said to be p-normal if every p-open set is p-normal and a p-a space is said to be p-normal if its p-topology is p-normal.

Example 2.4.1  (i) Every topological space is a p-normal p-a space where $\alpha(A, B) = A \cup B$

We have shown that $(X, T, \alpha)$ is a p-a space where $(X, T)$ is a p-topological space.

Now we show that it is also p-normal p-a space

$\alpha(A, X) = A \cup X$

$= X \cup A$

$= \alpha(X, A)$ for any $A \in T$ and $X \in \mathcal{P}X$

$\therefore (X, T, \alpha)$ is a p-normal p-a space.

(ii) Every abelian group with the usual p-topology is a p-normal p-a space where $\alpha(A, B) = AB$ (AB means usual product of complexes $A$ and $B$)

We have proved that $(G, T, \alpha)$ is a p-a space where $T$ is the usual p-topology on $G$. Again $(G, T, \alpha)$ is also a p-a space where $G$ is an abelian group and $T$ and $\alpha$ are the usual p-topology and the usual p-a function respectively. Now we show that it is also a p-normal p-a space.

$\alpha(A, G) = AG = GA = \alpha(G, A)$ for any $A \in T$ and $G \in \mathcal{P}G$.

$\therefore (G, T, \alpha)$ is a p-normal p-a space.
iii) Let $G$ be a group and $T$ be the set of all $p$-normal subgroups of $G$. Then $(G, T, \alpha)$ is a $p$-normal $p$-a space where $\alpha(A, B) = AB$. (AB means usual product of complexes $A$ and $B$)

We have proved that a group $G$ with the usual $p$-topology is a $p$-a space where

$\alpha(A, B) = AB$

Since $T$ is the set of all $p$-normal subgroups of $G$, therefore, $T$ is $p$-normal and hence $(G, T, \alpha)$ is a $p$-normal $p$-a space.

**Proposition 2.4.1** Let $(X, T, \alpha)$ be a $p$-a space. Let $A$ be a $p$-open set and $B$ be a $p$-normal $p$-open subset of $X$, then $\alpha(A, B)$ is a $p$-open subset of $X$.

**Proof**: Let $\alpha : P^X \times P^X \rightarrow P^X$.

Since $B$ is a $p$-normal $p$-open subset of $X$,

$\alpha(B, Y) = \alpha(Y, B)$, for any $Y \in T$,

Hence $\alpha(A, B) = \alpha(B, A), A \in T$

Therefore, $\alpha(A, B) \in T$. Hence proved.

**Proposition 2.4.2** Let $(X, T, \alpha)$ be a $p$-a space and $A, B$ be two $p$-normal $p$-open subsets of $X$. Then $\alpha(A, B)$ is a $p$-normal $p$-open subset of $X$.

**Proof**: Let $\alpha : P^X \times P^X \rightarrow P^X$

Since $A, B$ are $p$-normal in $X$,

$\alpha(A, Y) = \alpha(Y, A)$

and $\alpha(B, Y) = \alpha(Y, B)$, for any $Y \in T$
Hence, \( \alpha(A, B) = \alpha(B, A), \ A \in T \)

Therefore, \( \alpha(A, B) \) is a p-open subset of \( X \).

Now \( \alpha(\alpha(A, B), Y) = \alpha(A, \alpha(B, Y)) \)

\[
= \alpha(A, \alpha(Y, B)) \\
= \alpha(\alpha(A, Y), B) \\
= \alpha(\alpha(Y, A), B) \\
= \alpha(Y, \alpha(A, B))
\]

Therefore, \( \alpha(A, B) \) is a p-normal p-open set.

**Proposition 2.4.3**  Let \((X, T, \alpha)\) be a p-a space. Let \( H \) be a p-normal p-open subset of \( X \) and \( K \) is p-open subset of \( X \) such that \( H \subseteq K \subseteq X \), then \( \alpha(H, K) \) is also a p-normal p-open subset of \( X \).

**Proof:** Let \( \alpha : P^X \times P^X \rightarrow P^X \). Since

\( H \) is a p-normal p-open subset of \( X \),

\[
\alpha(H, Y) = \alpha(Y, H), \text{ for any } Y \in T
\]

Hence, \( \alpha(H, K) = \alpha(K, H), \ ( \therefore K \in T \) \)

\( \therefore \alpha(H, K) \) is a p-normal p-open subset of \( X \).

**Proposition 2.4.4**  Let \( H \) and \( K \) be p-normal p-open sets of a p-a space \((X, T, \alpha)\). Then \( HK \) is also a p-normal p-open subset of \( X \).

**Proof:** For p-normal p-open subset \( H \) of \( X \),

\( \alpha(H, Y) = \alpha(Y, H), \text{ for any } Y \in T. \)

Similarly for a p-normal p-open subset \( K \) of \( X \),
\( \alpha(K, Y) = \alpha(Y, K) \), for any \( Y \in T \).

\( \alpha(HK, Y) = \alpha(\alpha(H, K), Y) \), \( Y \in T \) where \( \alpha(H, K) = HK \)

\[ = \alpha(H, \alpha(K, Y)) \]

\[ = \alpha(\alpha(K, Y), H), \quad \text{Since } H \text{ is a p-normal p-open subset of } X \]

\[ = \alpha(\alpha(Y, K), H) \]

\[ = \alpha(Y, \alpha(K, H)), \quad \text{Since } K \text{ is p-normal.} \]

\[ = \alpha(Y, \alpha(H, K)) \]

\[ = \alpha(Y, HK) \]

\( \therefore HK \text{ is a p-normal p-open subset of } X. \)

**Proposition 2.4.5** The intersection of any two p-normal p-open sets of a p-a space \( (X, T, \alpha) \) is a p-normal p-open set.

**Proof**: Let \( H \) and \( K \) be two p-normal p-open sets of a p-a space \( (X, T, \alpha) \)

Therefore for any \( Y \in T \)

\( \alpha(H, Y) = \alpha(Y, H). \)

and \( \alpha(K, Y) = \alpha(Y, K) \)

\( \alpha(H, Y) \cap \alpha(K, Y) = \alpha(Y, H) \cap \alpha(Y, K) \)

\( \Rightarrow \alpha(H \cap K, Y) = \alpha(Y, H \cap K), \; Y \in T \)

\( \Rightarrow H \cap K \text{ is a p-normal p-open subset of } X. \)

**Proposition 2.4.6** Let \( B_1 \) and \( B_2 \) be p-normal p-open sets of a p-a space \( (X, T, \alpha) \) and \( \alpha(A, B_1 \cap B_2) = \alpha(A, B_1) \cap \alpha(A, B_2) \) for
any $A \in T$. Then $B_1 \cap B_2$ is a p-normal p-open set.

**Proof**: Since $B_1$ and $B_2$ are p-normal p-open sets, it follows that

$\alpha(A, B_1)$ and $\alpha(A, B_2)$ are p-open sets.

So, $\alpha(A, B_1) \cap \alpha(A, B_2)$ is p-open and hence $\alpha(A, B_1 \cap B_2)$ is p-open

i.e., $\alpha(A, B_1 \cap B_2) = \alpha(B_1 \cap B_2, A), A \in T$

i.e., $B_1 \cap B_2$ is p-normal p-open set.

### 2.5 Pseudo Continuity:

The concept of Pseudo continuity of a function between two Pseudo topological spaces is introduced following the concept of continuity of a function between two topological spaces. We define

**Definition 2.5.1** Let $(X, T)$ and $(Y, T^*)$ be two p-topological spaces. A function $f : X \to Y$ is said to be p-continuous if $f^!(A^*) \in T$ whenever $A^* \in T^*$ and is called p-open if $f(A) \in T^*$ whenever $A \in T$.

**Proposition 2.5.1** Let $(X, T)$ and $(Y, T^*)$ be two p-topological spaces with the minimal p-open sets $A_o$ and $A_o^*$ respectively.

Then (i) $f$ is p-continuous $\Rightarrow f(A_o) \subseteq A_o^*$

(ii) $f$ is p-open and p-continuous $\Rightarrow f(A_o) = A_o^*$

**Proof**: (i) If $f$ is p-continuous, then $f^!(A_o^*)$ is

a p-open subset of $X$ and so $A_o \subseteq f^!(A_o^*)$

and hence $f(A_o) \subseteq f f^!(A_o^*) \subseteq A_o^*$.

ii) If in addition $f$ is p-open, $f(A_o)$ is a p-open subset of $Y$ and
so, \( A_0^* \subseteq f(A_0) \). Also \( f(A_0) \subseteq A_0^* \) \[\text{(from (i)}\]

This proves that \( f(A_0) = A_0^* \)

**Example 2.5.1** Let \( X = \{1, 2, 3\} \) and \( Y = \{a\} \) and

\[\text{let } T_1 = \{\{1\}, \{1, 2\}, X\} \text{ and } T_2 = \{\{a\}\}\]

\[\text{let } f : X \to Y \text{ be defined by}\]

\[f(1) = f(2) = f(3) = a\]

Then \( f \) is both \( p \)-open and \( p \)-continuous.

\[f(\{1\}) = \{a\}, \quad f(\{2\}) = \{a\}, \quad f(\{3\}) = \{a\}\]

Thus the image of every \( p \)-open set in \( T_1 \) is \( p \)-open in \( T_2 \).

Hence, \( f \) is \( p \)-open.

Further,

\[f^{-1}(\{a\}) = \{1, 2, 3\} \in T_1\]

So, \( f \) is \( p \)-continuous.

This shows that \( f \) is both \( p \)-open and \( p \)-continuous.

**Example 2.5.2** Let \( X = \{a, b, c, d\} \) and \( Y = \{x, y, z, w\} \)

\[\text{let } T = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, X\}\]

and \( T^* = \{\{x\}, \{x, y\}, \{x, z\}, \{x, w\}, Y\} \) be two \( p \)-topologies on \( X \) and \( Y \) respectively.

\[\text{Let } f : X \to Y \text{ defined by}\]

\[a \to x\]
\[b \to y\]
\[c \to w\]
\[d \to z\]
Here \( f(\{a\}) = \{x\} \in T^* \), \( f(\{a, b\}) = \{x, y\} \in T^* \), \( f(\{a, c\}) = \{x, w\} \in T^* \), \( f(x) = y \in T^* \)

This shows that \( f \) is p-open,

\[
\begin{align*}
  f^*(\{x\}) &= \{a\}, & f^*(\{x, y\}) &= \{a, b\}, & f^*(\{x, z\}) &= \{a, d\}, \\
  f^*(\{x, w\}) &= \{a, c\}, & f^*(Y) &= X. & \text{Hence, } f \text{ is p-continuous.}
\end{align*}
\]

**Proposition 2.5.2** Let \((X, T_1)\), \((Y, T_2)\) and \((Z, T_3)\) be p-topological spaces and the mapping \( f : X \to Y \) and \( g : Y \to Z \) be p-continuous. Then the composition map \( g \circ f : X \to Z \) is p-continuous.

**Proof:** Let \( H \) be any p-open subset of \( Z \). Since \( g \) is p-continuous \( g^{-1}(H) \) is p-open subset of \( Y \). Again since \( f \) is p-continuous

\[
  f^{-1}[g^{-1}(H)] \text{ is p-open subset of } X.
\]

But

\[
  f^{-1}[g^{-1}(H)] = (f^{-1} \circ g^{-1})(H) = (g \circ f)^{-1}(H)
\]

Thus the inverse image under \( g \circ f \) of a p-open subset of \( Z \) is a p-open subset of \( X \) and therefore \( g \circ f \) is p-continuous.

**Remark 2.5.1** If there is a p-continuous and p-open function

\[
f : (X, T) \to (Y, T^*)\]

i) if \( X \) has a non-empty minimal p-open set, then \( Y \) must also have a non-empty minimal p-open set. Moreover, for any \( B \in T^* \), \( f^{-1}(B) \) can not be empty, thus there exists an element \( b \) in every \( B \in T^* \) such that we have an \( a \) in some \( A \in T \) with \( f(a) = b \).

ii) if \( Y \) has an empty minimal p-open set, then \( X \) must also have an empty minimal p-open set.
2.6 Pseudo Algebraic Homomorphism:

A p-a homomorphism is simply a one-one correspondence between the p-open sets in X and the p-open sets in Y where X and Y are two p-a spaces. Now we begin with the definition of p-a homomorphism.

Definition 2.6.1 Let (X, T, α) and (Y, T*, β) be two p-a spaces. A function f : X → Y is called p-a homomorphism if it is such that

i) f is both p-open and p-continuous

ii) f(α(A, B)) = β(f(A), f(B)), A, B ∈ Px

and α(f*(A*), f*(B*)) = f*(β(A*, B*)) for A*, B* ∈ Py

In this case, Y is said to be a p-a homomorphic image of X.

Proposition 2.6.1 An onto p-a homomorphism maps the minimal p-open set onto the minimal p-open set.

Proof: It follows immediately from proposition 2.5.1 as a p-a homomorphism is both p-open and p-continuous.

Proposition 2.6.2 A p-a homomorphism maps a p-normal p-open set onto a p-normal p-open set.

Proof: Let f : (X, T, α) → (Y, T*, β) be a p-a homomorphism from one p-a space (X, T, α) to another p-a space (Y, T*, β). Let A be p-normal p-open subset of X. Then f(A) is a p-normal p-open subset of Y. Let N be any p-open subset of Y. Then from the p-continuity of f, it follows that f*(N) is a p-open subset of X.

Since A is a p-normal p-open subset of X,
\[ \alpha(A, f^{-1}(N)) = \alpha(f^{-1}(N), A) \] .......(i)

Since \( f \) is a \( p \)-a homomorphism

\[ f(\alpha(A, f^{-1}(N))) = \beta(f(A), f f^{-1}(N)) \]

\[ = \beta(f(A), N) \] (\( f \) is onto, \( f f^{-1}(N) = N \))

\[ f(\alpha(f^{-1}(N), A)) = \beta(f f^{-1}(N), f(A)) \]

\[ = \beta(N, f(A)) \]

By (i) it follows that

\[ \beta(f(A), N) = \beta(N, f(A)) \]

But \( N \) is arbitrary, so that

\[ \beta(f(A), N) = \beta(N, f(A)) \text{ for any } N \in T^* \]

Thus \( f(A) \) is a \( p \)-normal \( p \)-open subset of \( Y \).

**Example 2.6.1**

Let \( T_1 = \langle \{q\}, \{p, q\}, \{q, r\}, \{q, s\}, \{p, q, r\}, \{p, q, s\}, \{q, r, s\}, X \rangle \) be a \( p \)-topology on \( X = \{p, q, r, s\} \) where

\[ P^X = \{\varnothing, \{p\}, \{q\}, \{r\}, \{s\}, \{p, q\}, \{p, r\}, \{p, s\} \]

\[ \{q, r\}, \{q, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}, X \}

and let \( T_2 = \langle \{a\}, \{a, b\}, \{a, c\}, Y \rangle \) be

a \( p \)-topology on \( Y = \{a, b, c\} \) where

\[ P^Y = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, Y \}

The mapping defined as follows is a \( p \)-a homomorphism.
\( f : X \rightarrow Y \)

- \( f(p) = c, \quad f(q) = a, \quad f(r) = b, \quad f(s) = b \)

i) \( f \) is onto since every element in \( Y \) is the image of at least one element in \( X \).

ii) \( f(\{q\}) = \{a\}, \quad f(\{p, q\}) = \{a, c\}, \quad f(\{q, r\}) = \{a, b\} \)

- \( f(\{q, s\}) = \{a, b\}, \quad f(\{p, q, r\}) = \{a, b, c\}, \quad f(\{p, q, s\}) = \{a, b, c\} \)

Thus the image of every \( p \)-open set in \( T_1 \) is \( p \)-open set in \( T_2 \).

\( \therefore f \) is \( p \)-open mapping.

- \( f^-[\{a\}] = \{q\}, \quad f^-[\{a, b\}] = \{q, r, s\}, \quad f^-[\{a, c\}] = \{p, q\}, \quad f^-[Y] = X \)

Thus the inverse image of every \( p \)-open set in \( T_2 \) is a \( p \)-open set in \( T_1 \).

\( \therefore f \) is \( p \)-continuous

Hence \( f \) is both \( p \)-open and \( p \)-continuous.

iii) Let \( A = \{p, q\}, \quad B = \{q, r\}, \quad A, B \in T_1 \)

- \( f(\alpha(A, B)) = f(\{p, q, r\}) \) where \( \alpha(A, B) = A \cup B \)

- \( = \{a, b, c\} \)

- \( = Y \)
\[ \beta(f(A), f(B)) = \beta([a, c], [a, b]) \]

\[ = [a, b, c] \quad \text{where } \beta(A, B) = A \cup B \]

\[ = Y \]

\[ \therefore f(\alpha(A, B)) = \beta(f(A), f(B)), \; A, B \in T_i \]

Similarly, we can verify this property for other elements of \( T_i \).

\[ \alpha(f^t(A^*), f^t(B^*)) = \alpha(f([a, b], [a, c])) \]

where \( A^* = [a, b] \), \( B^* = [a, c] \)

\[ = \alpha([q, r, s], \{p, q\}) \]

\[ = \{p, q, r, s\} \]

\[ = X. \]

\[ f^t(\beta(A^*, B^*)) = f^t(\beta([a, b], [a, c])) \]

\[ = f^t([a, b, c]) \quad \text{where } \beta(A, B) = A \cup B \]

\[ = [p, q, r, s] \]

\[ = X. \]

\[ \therefore \alpha(f^t(A^*), f^t(B^*)) = f^t(\beta(A^*, B^*)) \quad A^*, B^* \in T_i \]

Similarly, we can verify this property for other elements of \( T_i \).

\[ \therefore f \text{ is a } p\text{-a homomorphism.} \]

**Example 2.6.2** Let \( T_i = \{[a], [a, c], X\} \) be a

\( p \)-topology on \( X = \{a, b, c\} \) and

\[ P^X = \{\emptyset, [a], [b], [c], [a, b], [a, c], [b, c], X\} \]

Let \( T_2 = \{[p], Y\} \) be a \( p \)-topology on \( Y = \{p, q\} \) and
The mapping defined as follows is a p-a homomorphism.

\[ f : X \rightarrow Y \]

\[ a \rightarrow p \]

\[ b \rightarrow q \]

\[ c \]

\[ f(a) = p, \ f(b) = q, \ f(c) = p. \]

i) \( f \) is onto since every element in \( Y \) is the image of at least one element in \( X \).

ii) \( f([a]) = \{p\}, \ f([a, c]) = \{p\}, \ f[X] = Y \)

\[ \therefore f \text{ is p-open.} \]

\[ f^{-1}[\{p\}] = \{a, c\}, \ f^{-1}[Y] = X \]

\[ \therefore f \text{ is p-continuous.} \]

\[ \therefore f \text{ is both p-open and p-continuous.} \]

iii) Let \( A = \{a\}, \ B = \{a, c\} \)

\[ f(\alpha(A, B)) = f(\alpha([a], \{a, c\}) = f([a, c]) = \{p\} \]

where \( \alpha(A, B) = A \cup B \).

\[ \beta(f(A), f(B)) = \beta(f([a]), f([a, c])) \]

\[ = \beta([p], \{p\}) \]

\[ = \{p\} \]

where \( \beta(A, B) = A \cup B \)

\[ \therefore f(\alpha(A, B)) = \beta(f(A), f(B)), \ A, B \in T_1 \]
\[
\alpha(f^\text{I}(\{p\}), f^\text{I}(Y)) = \alpha([a, c], X) = X
\]
\[
f^\text{I}(\beta(\{p\}, Y)) = f^\text{I}(Y) = X
\]
\[
\therefore \alpha(f^\text{I}(\{p\}), f^\text{I}(Y)) = f^\text{I}(\beta(\{p\}, Y))
\]
\[
\therefore f \text{ is a } p-a \text{ homomorphism.}
\]

**Definition 2.6.2** Let p-a homomorphism
\[
f : (X, T, a) \rightarrow (Y, T^*, p)
\]
induce a mapping
\[
f_o : T \rightarrow T^* \text{ such that } f_o(A_o) = A_o^*,
\]
\[
f_o(X) = Y \text{ where } A_o, A_o^* \text{ are the minimal } p\text{-open subsets of } X \text{ and } Y \text{ respectively.}
\]

If \( f_o \) is onto, we call \( f \) is a p-a epimorphism. If \( f_o \) is one-one, we call \( f \) is a p-a monomorphism. If \( f_o \) is both one-one and onto, we call \( f \) is a p-a isomorphism.

**Remark 2.6.1** If \( f : (X, T, a) \rightarrow (Y, T^*, \beta) \) is a p-a homomorphism and \( Y_o \) is a p-normal p-open subset of \( Y \), then \( f^\text{I}(Y_o) \) is not necessarily a p-normal p-open subset of \( X \). [Example 2.6.4.]

If \( f \) is one-one, then \( f^\text{I}(Y_o) \) is a p-normal p-open subset of \( X \). This is proved in the following proposition.

**Proposition 2.6.3** If the p-a homomorphism \( f \) induces a one-one correspondence between the p-open subsets of \( X \) and the p-open subsets of \( Y \), then for any p-normal p-open subset \( Y_o \) of \( Y \), \( f^\text{I}(Y_o) \) is p-normal in \( X \).

**Proof:** Under the given condition, for any
\[
A \in T \text{ there exists an } A^* \text{ in } T^* \text{ such that }
\]
\[
A = f^\text{I}(A^*)
\]
Then \( \beta(Y_0, A^*) = \beta(A^*, Y_0) \) for all \( A^* \in T^* \).

\( f^1(Y_0) \) is a p-open subset of \( X \) and for any \( A \in T \), we have

\[
\alpha(f^1(Y_0), A) = \alpha(f^1(Y_0), f^1(A^*)) = f^1(\beta(Y_0, A^*))
\]

\[
= f^1(\beta(A^*, Y_0)), \quad \text{since } Y_0 \text{ is p-normal}
\]

\[
= \alpha(f^1(A^*), f^1(Y_0))
\]

\[
= \alpha(A, f^1(Y_0))
\]

Therefore \( f^1(Y_0) \) is p-normal in \( (X, T, \alpha) \).

**Example 2.6.4** Let \( T_1 = \{\{a\}, \{a, c\}, X\} \) be a p-topology on \( X = \{a, b, c\} \) and \( P^X = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \).

Let \( T_2 = \{\{p\}, Y\} \) be a p-topology on \( Y = \{p, q\} \) and \( P^Y = \{\varnothing, \{p\}, \{q\}, Y\} \).

The mapping defined as follows is a p-a homomorphism. (Example 2.6.2)

\[
f : X \to Y, \quad f_0 : T_1 \to T_2
\]

\[
a \mapsto p
\]

\[
b \mapsto q
\]

\[
c
\]

Clearly \( f_0 \) is not one-one since \( f_0(\{a\}) = f_0(\{a, c\}) = \{p\} \in T_2 \). Now we see that whether p-a homomorphism \( f \) pulls back every p-normal p-open set to a p-normal p-open set or not with the help of the following table - 1 and table - 2.
In table 1, \( \{a\} \) is p-normal p-open set but \( f^{-1}(\{a\}) = \{a, c\} \) which is p-open in \( T \), but \( \{a, c\} \) is not p-normal p-open set in \( T \) since \( \alpha(\{a, c\}, [b]) \neq \alpha([b], \{a, c\}) \).

Therefore it is seen that a p-a homomorphism \( f \) does not pull back a p-normal p-open set to a p-normal p-open set unless its induced map is one-one.

**Table - 1**

Let \( T_1 = \{\{a\}, \{a, c\}, X\} \) be a p-topology on \( X = \{a, b, c\} \) and

\[
p^X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}
\]

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2.7 Pseudo Kernel of a p-a Homomorphism:

We define p-Kernel of a p-a homomorphism from one p-a space to another p-a space as follows:

**Definition 2.7.1** Let \( f : (X, T, \alpha) \rightarrow (Y, T^*, \beta) \) be a p-a homomorphism from one p-a space \((X, T, \alpha)\) to another p-a space \((Y, T^*, \beta)\). Let \( A^*_0 \) be the minimal p-open set in \( Y \). Then \( f^!(A^*_0) \) is called the p-Kernel of \( f \).

**Example 2.7.1** Let \( T_1 = \{\{p\}, \{p, q\}, X_1\} \) be a p-topology on \( X_1 = \{p, q, r\} \) where \( P^{X_1} = \{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, X_1\} \), and let \( T_2 = \{\{a\}, X_2\} \) be a p-topology on \( X_2 = \{a, b\} \) where \( P^{X_2} = \{\emptyset, \{a\}, \{b\}, X_2\} \).

The mapping defined as follows is a p-a homomorphism.
$f : X_1 \rightarrow X_2$

\[ \begin{array}{ccc}
 p & \rightarrow & a \\
 & \nearrow & \\
 q & \rightarrow & a \\
 & \searrow & \\
 r & \rightarrow & b \\
\end{array} \]

i.e., $f(p) = a$, $f(q) = a$, $f(r) = b$

i) $f$ is onto since every element in $X_2$ is the image of at least one element in $X_1$.

ii) $f([p]) = \{a\}$, $f([p, q]) = \{a\}$, $f(X_1) = X_2$

Thus the image of every $p$-open set in $T_1$ is a $p$-open set in $T_2$.

.: $f$ is a $p$-open mapping.

$f^{-1}[\{a\}] = \{p, q\}$, $f^{-1}[\{b\}] = \{r\}$,

$f^{-1}[\{a, b\}] = f^{-1}(X_2) = X_1$

Thus the inverse image of every $p$-open set in $T_2$ is a $p$-open set in $T_1$.

.: $f$ is $p$-continuous.

Hence $f$ is both $p$-open and $p$-continuous.

iii) Let $A = \{p\}$, $B = \{p, q\}$, $A, B \in T_1$

$f(\alpha(A, B)) = f(\alpha(\{p\}, \{p, q\}))$

\[
= f([p, q]) \quad \text{where } \alpha(A, B) = A \cup B
\]

\[
= \{a\}
\]

$\beta(f(A), f(B)) = \beta(f(\{p\}), f(\{p, q\}))$
\[ p(\{a\}, \{a\}) = \{a\} \] where \( \beta(A, B) = A \cup B \)

\[ \therefore f(\alpha(A, B)) = \beta(f(A), f(B)), \ A, B \in T_1 \]

Similarly, this holds for other elements of \( T_1 \).

\[ \alpha(f^{-1}(A^*), f^{-1}(B^*)) = \alpha(f^{-1}(\{a\}), f^{-1}(X_2)) \] where \( A^* = \{a\}, B^* = X_2 \)

\[ = \alpha([p, q], X_1) \]

\[ = X_1 \]

\[ f^{-1}(\beta(A^*, B^*)) = f^{-1}(\beta([a], X_2)) \]

\[ = f^{-1}(X_2) \]

\[ = X_1 \]

\[ \therefore \alpha(f^{-1}(A^*), f^{-1}(B^*)) = f^{-1}(\beta(A^*, B^*)), A^*, B^* \in T_2 \]

\[ \therefore f \] is a \( p \)-\( a \) homomorphism.

Again \( f^{-1}(A_0^*) = f^{-1}(\{a\}) = \{p, q\} \) is the \( p \)-Kernel

of \( f \) where \( A_0^* = \{a\} \in T_2 \).

**Proposition 2.7.1** If \( f : (X, T, \alpha) \to (Y, T^*, \beta) \) is a \( p \)-\( a \) homomorphism

from one \( p \)-\( a \) space \( (X, T, \alpha) \) to another \( p \)-\( a \) space \( (Y, T^*, \beta) \)

and \( A \) is its \( p \)-Kernel, then \( A \) is \( p \)-normal in \( X \) if \( f \) is one-to-one.

**Proof:** From proposition 2.6.2, we know that

\[ \beta(A_0^*, N) = \beta(N, A_0^*) \quad \forall N \subseteq Y, \text{ where } A_0^* \text{ is the minimal } \]

\[ \text{p-open subset of } Y \text{ and } A = f^{-1}(A_0^*) \]

Let \( M \subseteq X \), now

\[ f^{-1}(\beta(A_0^*, N)) = f^{-1}(\beta(N, A_0^*)), N = f(M) \]

\[ \Rightarrow \alpha(f^{-1}(A_0^*), f^{-1}(N)) = \alpha(f^{-1}(N), f^{-1}(A_0^*)) \]

\[ \Rightarrow \alpha(A, f^{-1}(f(M)) = \alpha(f^{-1}(f(M)), A) \]

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But if is one-one implies \( M = f^{-1}(f(M)) \)
\[ \Rightarrow \alpha(A, M) = \alpha(M, A) \]
\[ \therefore A \text{ is } p \text{ normal in } X. \]

### 2.8 Pseudo Inverse:

#### Definition 2.8.1
Let \((X, T)\) be a \(p\)-topological space and let \(\alpha\) be a pseudo algebraic function such that

\[ \alpha : P^X \times P^X \to P^X \text{ (} P^X \text{ is the power set of } X). \]

\(B \in P^X\) is said to be \(p\)-inverse of \(A\) where \(A \in P^X\)

if \(\alpha(A, B) = A_0 = \alpha(B, A)\) where \(A_0\) is the minimal \(p\)-open set.

Then we write \(B = A^1\).

#### Proposition 2.8.1
Let \((X, T, \alpha)\) and \((Y, T^*, \beta)\) be two \(p\)-a spaces.

Let \(f : X \to Y\) be an onto \(p\)-a homomorphism.

Then \(f(A^1) = (f(A))^{-1}\).

#### Proof:
We prove that

\[ \beta(f(A^1), f(A)) = A_0^* = \beta(f(A), f(A^{-1})) \]
\[ \beta(f(A^1), f(A)) = \beta(f(A, A^{-1})) = \beta(A_0) = A_0^* \]
\[ \beta(f(A), f(A^{-1})) = \beta(\alpha(A, A^{-1})) = \beta(A_0^*) = A_0^* \]

\[ \therefore \beta(f(A^1), f(A)) = \beta(f(A), f(A^{-1})) \]
\[ \therefore f(A^1) = (f(A))^{-1}. \text{ This completes the proof.} \]

We conclude this chapter with the following extension of

\[ (ab)^{-1} = b^{-1}a^{-1}. \]
Proposition 2.8.2  Let \((X, T)\) be a \(p\)-topological space and let

\[
\alpha : P^X \times P^X \rightarrow P^X \quad (P^X \text{ is the power set of } X)
\]

be a function such that \(\alpha(A_0, A_n) = A_0\).

Let \(A, B \in P^X\) and \(A^{-1}\) and \(B^{-1}\) be the \(p\)-inverses of \(A\) and \(B\) respectively. Then \((\alpha(A, B))^{-1} = \alpha(B^{-1}, A^{-1})\).

Proof:

\[
\begin{align*}
\alpha(\alpha(A, B), \alpha(B^{-1}, A^{-1})) & = \alpha(\alpha(A, B), B^{-1}), A^{-1}) \\
& = \alpha(\alpha(A, \alpha(B, B^{-1})), A^{-1}) \\
& = \alpha(\alpha(A, A_0), A^{-1}) \\
& = \alpha(\alpha(A_0, A), A^{-1}) \\
& = \alpha(A_0, \alpha(A, A^{-1})) \\
& = \alpha(A_0, A_0) \\
& = A_0
\end{align*}
\]

This proves that

\([ \alpha(A, B)]^{-1} = \alpha(B^{-1}, A^{-1})\).