Chapter 3

Degenerate and Inverted Hierarchical Neutrino Mass Models

3.1 Introduction

If the result on neutrinoless double beta decay ($0\nu\beta\beta$) experiment [54] is confirmed, then it shows a possible evidence for the non-zero Majorana mass of the electron neutrino in the range of $m_{ee} = (0.05 - 0.86)$ eV at 95% C.L. with the best value of $m_{ee} = 0.4$ eV [54]. There are certain important implications of this result. Firstly, it rules out all models predicting the Dirac neutrino masses, leaving only the option for the Majorana type of neutrinos. It also allows the lepton number violating processes such as leptogenesis in a natural way. Secondly, this result together with the earlier experimental data on the atmospheric [30] and the solar [31, 32] neutrino oscillations, allows only the degenerate and the inverted hierarchical solutions for the three generation left-handed Majorana neutrinos [54, 93, 94].

Now in context of the $0\nu\beta\beta$ experiment, it is very important to construct theoretical models which can predict the degenerate and inverted hierarchical patterns of the Majorana neutrino mass matrices within the framework of the grand unified theories (GUTs) with or without supersymmetry [93, 94, 95]. In this chapter we attempt to generate
3.2 Neutrino mass matrices from see-saw formula

The left-handed Majorana neutrino mass matrix $m_{LL}$ is given by the celebrated see-saw formula [62, 63],

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Values at 90% C.L.</th>
<th>Best fit value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta m_{21}^2 [10^{-5} eV^2]$</td>
<td>$2.2 - 17$</td>
<td>$4.2$</td>
</tr>
<tr>
<td>$\Delta m_{23}^2 [10^{-3} eV^2]$</td>
<td>$1.5 - 5.0$</td>
<td>$3.0$</td>
</tr>
<tr>
<td>$\sin^2 2\theta_{12}$</td>
<td>$\leq 0.98$</td>
<td>$0.81$</td>
</tr>
<tr>
<td>$\sin^2 2\theta_{23}$</td>
<td>$\geq 0.88$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\sin^2 2\theta_{13}$</td>
<td>$0.1 - 0.3$</td>
<td>--</td>
</tr>
</tbody>
</table>

The chapter is organised as follows. In section 3.2 we present the generation of the degenerate as well as inverted hierarchical neutrino mass matrices using the see-saw formula, and their predictions on mass eigenvalues and mixing angles. Section 3.3 is devoted to summary and conclusion.
3.2 Neutrino mass matrices from see-saw formula

Table 3.2: Zeroth-order neutrino mass matrices with texture zeros corresponding to the LMA MSW solution with bimaximal mixings [74, 93, 94].

<table>
<thead>
<tr>
<th>Type</th>
<th>( m_{LL} )</th>
<th>( m_{LL}^{\text{diag}} )</th>
</tr>
</thead>
</table>
| I(A) | \[
\begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2}
\end{pmatrix} m_0 \] | \( \text{Diag}(1, -1, 1) m_0 \) |
| I(B) | \[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix} m_0 \] | \( \text{Diag}(1, 1, 1) m_0 \) |
| I(C) | \[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} m_0 \] | \( \text{Diag}(1, 1, -1) m_0 \) |
| II(A) | \[
\begin{pmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} \\
0 & 1 & 1
\end{pmatrix} m_0 \] | \( \text{Diag}(1, 1, 0) m_0 \) |
| II(B) | \[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} m_0 \] | \( \text{Diag}(1, -1, 0) m_0 \) |

\[
m_{LL} = -m_{LR} M_{RR}^{-1} m_{LR}^T,
\] (3.1)

where \( m_{LR} \) is the Dirac neutrino mass matrix in the left-right (LR) convention [82]. The leptonic (MNS) mixing matrix is now given by \( V_{MNS} = V_{\nu L}^\dagger \) where \( m_{LL}^{\text{diag}} = V_{\nu L} m_{LL} V_{\nu L}^T \). Here both \( m_{LR} \) and the charged lepton mass matrix \( m_\ell \) are taken as diagonal, whereas the right-handed Majorana neutrino mass matrix \( M_{RR} \) as non-diagonal. Using the see-saw formula (1) we generate both patterns of \( m_{LL} \) viz., (I) nearly degenerate and (II) inverted hierarchical neutrino mass models. We concentrate here only on the types of neutrino mass matrices which have bimaximal mixings being listed in Table 3.2.

The Dirac neutrino mass matrix \( m_{LR} \) involved in the see-saw formula Eq.(3.1), can be either the charged lepton mass matrix \( m_\ell \) (case (i)) or the up-quark mass matrix \( m_u \) (case (ii)) depending on the particular SUSY SO(10) GUT model and the contents of
the Higgs fields employed \([83, 84, 85, 86]\):

\[
m_{LR} = \tan \beta \begin{pmatrix} \lambda^6 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_r, \quad (3.2)
\]

and

\[
m_{LR} = \begin{pmatrix} \lambda^8 & 0 & 0 \\ 0 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_t, \quad (3.3)
\]

respectively. The above two forms of \(m_{LR}\) may be written together as

\[
m_{LR} = \begin{pmatrix} \lambda^m & 0 & 0 \\ 0 & \lambda^n & 0 \\ 0 & 0 & 1 \end{pmatrix} m_f, \quad (3.4)
\]

where \(m_f\) corresponds to \((m_r \tan \beta)\) in SUSY models for charged lepton mass matrix in case (i), and \(m_t\) for up-quark mass matrix in case (ii). The pair of the exponents \((m, n)\) are \((6, 2)\) for charged lepton and \((8, 4)\) for up-quark mass matrices respectively. The value of the Wolfenstein parameter is taken as \(\lambda = 0.22\). Using the diagonal form of \(m_{LR}\) in Eq.\((3.4)\) and a suitable choice of the non-diagonal texture of \(M_{RR}\), the following four types of neutrino mass matrices \(m_{LL}\) are calculated.

I(A). Nearly degenerate mass matrix with opposite sign mass eigenvalues

The degenerate mass matrix \(m_{LL}\) having opposite sign mass eigenvalues is now generated through see-saw formula \([62, 63]\) in Eq.\((3.1)\) for the choices of \(m_{LR}\) in Eq.\((3.4)\) and

\[
M_{RR} = \begin{pmatrix} -2\delta_2 \lambda^{2m} & (\frac{1}{\sqrt{2}} + \delta_1)\lambda^{m+n} & (\frac{1}{\sqrt{2}} + \delta_1)\lambda^m \\ (\frac{1}{\sqrt{2}} + \delta_1)\lambda^{m+n} & (\frac{1}{2} + \delta_1 - \delta_2)\lambda^{2n} & (\frac{1}{2} + \delta_1 - \delta_2)\lambda^n \\ (\frac{1}{\sqrt{2}} + \delta_1)\lambda^m & (\frac{1}{2} + \delta_1 - \delta_2)\lambda^n & (\frac{1}{2} + \delta_1 - \delta_2) \end{pmatrix} v_R, \quad (3.5)
\]
leading to a simple form,

\[
m_{LL} = \begin{pmatrix}
  -2\delta_1 + 2\delta_2 & \frac{1}{\sqrt{2}} - \delta_1 & \frac{1}{\sqrt{2}} - \delta_1 \\
  \frac{1}{\sqrt{2}} - \delta_1 & \frac{1}{2} + \delta_2 & -\frac{1}{2} + \delta_2 \\
  \frac{1}{\sqrt{2}} - \delta_1 & -\frac{1}{2} + \delta_2 & \frac{1}{2} + \delta_2
\end{pmatrix} m_0,
\]  

where \( m_0 \) controls the overall magnitude of the masses of the neutrinos whereas \( \delta_1 \) and \( \delta_2 \) give the desired splittings for solar and atmospheric data. When \( \delta_1 = \delta_2 = 0 \), Eq.(3.6) reduces to the zeroth order mass matrix of the type I(A) in Table 3.2, with no splittings [74]. The diagonalisation of \( m_{LL} \) in Eq.(3.6) leads to the following neutrino mass eigenvalues and mixings:

\[
\begin{align*}
  m_{\nu_1} &= [1 + 2\delta_2 - \delta_1(1 + \sqrt{2})]m_0, \\
  m_{\nu_2} &= [-1 + 2\delta_2 - \delta_1(1 - \sqrt{2})]m_0, \\
  m_{\nu_3} &= m_0 \\
  \sin^2 2\theta_{12} &\approx (1 - \delta_1^2/8), \quad \sin^2 2\theta_{23} = 1, \quad \sin^2 2\theta_{13} = 0
\end{align*}
\]

For the choice of the values of the parameters \( m_0 = 0.4 \) eV, \( \delta_1 = 0.0061875 \), \( \delta_2 = 0.0030625 \), Eqs(3.7) to (3.10) lead to the following numerical predictions

**mass eigenvalues:**

\[
m_{\nu_i} = (0.396484, -0.396532, 0.4) \text{ eV}, \quad i = 1, 2, 3; \quad \Delta m_{12}^2 = 3.806 \times 10^{-5} \text{eV}^2 \quad \text{and} \quad \Delta m_{23}^2 = 2.76 \times 10^{-3} \text{eV}^2;
\]

and

**mixing angles:**

\[
\sin^2 2\theta_{12} = 0.999, \quad \sin^2 2\theta_{23} \approx 1.0, \quad |V_{e3}| = 6.124 \times 10^{-9}.
\]
Here the prediction on solar mixing angle is nearly maximal which is not consistent with the LMA MSW solution [31, 32]. The expression for \( m_0 \) in Eq.(3.6) for case (i) is worked out as \( m_0 = m_1^2 \tan^2 \beta / v_R \). For input values of \( m_0 = 0.4 \text{eV}, \tan \beta = 40, m_\tau = 1.7 \text{ GeV} \), the see-saw scale is calculated as \( v_R \approx 10^{13} \text{ GeV} \). This in turn gives masses to the three right-handed Majorana neutrinos. After diagonalisation of \( M_{RR} \) in Eq.(3.5), we have the mass eigenvalues of the heavy right-handed Majorana neutrinos as \( M^\text{diag}_{R} = \text{Diag}(5.0427 \times 10^{12}, 3.0981 \times 10^{8}, 1.9613 \times 10^{7}) \text{ GeV} \). Similarly, in case(ii) we have \( m_0 = m_t^2 / v_R \) in Eq.(3.6), and with the input values \( m_0 = 0.4 \text{ eV}, m_t = 200 \text{ GeV} \), we obtain \( v_R = 10^{14} \text{ GeV} \) and the mass eigenvalues of the right-handed Majorana neutrinos as \( M^\text{diag}_{R} = \text{Diag}(4.5932 \times 10^{15}, 7.2731 \times 10^{6}, 5.005 \times 10^{13}) \text{ GeV} \).

### I(B). Nearly degenerate mass matrix with the same sign mass eigenvalues

The mass matrix \( m_{LL} \) of this type can be realised in the see-saw mechanism in Eq.(3.1) using the general texture of \( m_{LR} \) in Eq.(3.4) and

\[
M_{RR} = \begin{pmatrix}
(1 + 2\delta_1 + 2\delta_2)\lambda^{2\text{m}} & \delta_1\lambda^{\text{m+n}} & \delta_1\lambda^\text{m} \\
\delta_1\lambda^{\text{m+n}} & (1 + \delta_2)\lambda^{2\text{m}} & \delta_2\lambda^n \\
\delta_1\lambda^\text{m} & \delta_2\lambda^n & (1 + \delta_2)
\end{pmatrix} v_R, \tag{3.11}
\]

leading to the nearly degenerate mass matrix,

\[
m_{LL} = \begin{pmatrix}
(1 - 2\delta_1 - 2\delta_2) & -\delta_1 & -\delta_1 \\
-\delta_1 & (1 - \delta_2) & -\delta_2 \\
-\delta_1 & -\delta_2 & (1 - \delta_2)
\end{pmatrix} m_0. \tag{3.12}
\]

The diagonalisation of \( m_{LL} \) in Eq.(3.8) leads to

\[
m_{\nu_1} \simeq (1 - 2\delta_2 - (\sqrt{3} + 1)\delta_1) m_0, \tag{3.13}
\]
\[
m_{\nu_2} \simeq (1 - 2\delta_2 + (\sqrt{3} - 1)\delta_1) m_0, \tag{3.14}
\]
\[
m_{\nu_3} \simeq m_0, \tag{3.15}
\]
3.2 Neutrino mass matrices from see-saw formula

\[
\sin^2 2\theta_{12} = \frac{2}{3}, \quad \sin^2 2\theta_{23} = 1, \quad \sin^2 2\theta_{13} = 0.
\] (3.16)

For the choice of the values of the parameters \(m_0 = 0.4 \text{ eV}, \delta_1 = 3.6 \times 10^{-5}, \delta_2 = 3.9 \times 10^{-3}\), Eqs(3.13) to (3.15) lead to the following numerical predictions:

**mass eigenvalues:**

\[m_{\nu_i} = (0.39684, 0.396892, 0.4) \text{ eV}, \quad i = 1, 2, 3; \text{ leading to } \Delta m^2_{12} = 4.13 \times 10^{-5} \text{eV}^2\]

and \(\Delta m^2_{23} = 2.48 \times 10^{-3} \text{eV}^2\).

The mixing angles which are given by Eq.(3.16), are independent of \(\delta_{1,2}\). The prediction on solar mixing angle is consistent with the LMA MSW solution [31, 32].

For case(i), the expression for \(m_0\) in Eq.(3.12) is again worked out as \(m_0 = m^2_0 \tan^2 \beta/v_R\), and for input values of \(m_0 = 0.4 \text{ eV}, \tan \beta = 40, m_r = 1.7 \text{ GeV}, \) we obtain \(v_R = 1.156 \times 10^{13} \text{ GeV}\). The mass eigenvalues of the heavy right-handed Majorana neutrinos are given by the diagonalisation of \(M_{RR}\) in Eq.(3.11): \(M_R^{\text{diag}} = \text{Diag}(1.15 \times 10^{13}, 2.71 \times 10^{10}, 1.498 \times 10^{5}) \text{ GeV}\). Similarly for case (ii), we have \(m_0 = m^2_0/v_R\) and for input values of \(m_0 = 0.4 \text{ eV} \) and \(m_\tau = 200 \text{ GeV} \), we have \(v_R = 1.0 \times 10^{14} \text{ GeV}\) and \(M_R^{\text{diag}} = (1.0039 \times 10^{14}, 5.5089 \times 10^{8}, 3.035 \times 10^{4}) \text{ GeV}\).

I(C). Nearly degenerate mass matrix with opposite sign mass eigenvalues

We consider another texture for the nearly degenerate mass matrix \(m_{LL}\) with opposite mass eigenvalues [95]. We take \(m_{LR}\) given in Eq.(3.4) and the following right-handed neutrino mass matrix

\[
M_{RR} = \begin{pmatrix}
(1 + 2\delta_1 + 2\delta_2)\lambda^{2n} & \delta_1 \lambda^{m+n} & \delta_1 \lambda^n \\
\delta_1 \lambda^{m+n} & \delta_2 \lambda^{2n} & (1 + \delta_2)\lambda^n \\
\delta_1 \lambda^m & (1 + \delta_2)\lambda^n & \delta_2
\end{pmatrix} v_R.
\] (3.17)

The left-handed neutrino mass matrix \(m_{LL}\) from Eq.(3.1) is obtained as
3.2 Neutrino mass matrices from see-saw formula

\[
M_{\text{LL}} = \begin{pmatrix}
(1 - 2\delta_1 - 2\delta_2) & -\delta_1 & -\delta_1 \\
-\delta_1 & -(1 - \delta_2) & (1 - \delta_2) \\
-\delta_1 & (1 - \delta_2) & -(1 - \delta_2)
\end{pmatrix} m_0, \quad (3.18)
\]

where \(m_0\) controls the overall magnitude of the masses of the neutrinos whereas \(\delta_1\) and \(\delta_2\) give the desired splittings for solar and atmospheric data. When \(\delta_1 = \delta_2 = 0\), Eq. (3.18) reduces to the zeroth order mass matrix of the Type I(C) in Table 3.2, with no splittings [95, 74]. The diagonalisation of \(M_{\text{LL}}\) in Eq. (3.18) leads to the following eigenvalues and mixing angles:

\[
m_{\nu_1} \simeq (1 - 2\delta_2 - (\sqrt{3} + 1)\delta_1)m_0, \quad (3.19)
\]
\[
m_{\nu_2} \simeq (1 - 2\delta_2 + (\sqrt{3} - 1)\delta_1)m_0, \quad (3.20)
\]
\[
m_{\nu_3} \simeq -m_0, \quad (3.21)
\]
\[
\sin^2 2\theta_{12} = \frac{2}{3}, \quad \sin^2 2\theta_{23} = 1, \quad \sin^2 2\theta_{13} = 0. \quad (3.22)
\]

The Eqs (3.19 - 3.21) lead to neutrino mass eigenvalues \(m_{\nu_i} = (0.39684, 0.396892, -0.4)\) eV, \(i = 1, 2, 3\) for the same choices of the input values of \(\delta_{1,2}\) and \(m_0\) as in case of Type I(B). Further, the predictions on the three mixing angles remain the same as in Eq. (3.22), which are independent of \(\delta_{1,2}\).

In case (i), for input values of \(m_0 = 0.4\) eV, \(\tan \beta = 40\), \(m_r = 1.7\) GeV, the see-saw scale is calculated as \(v_R \approx 10^{13}\) GeV. This in turn gives masses to the three right-handed Majorana neutrinos. The mass eigenvalues of the heavy right-handed Majorana neutrinos are obtained by diagonalising \(M_{RR}\): \(M_{RR}^{\text{diag}} = \text{Diag}(4.67 \times 10^{11}, 1.296 \times 10^5, 5.06 \times 10^4)\) GeV. Similarly for case (ii), we have \(m_0 = m_t^2/v_R\) in Eq. (3.18), and with the input values \(m_0 = 0.4\) eV, \(m_t = 200\) GeV, we obtain \(v_R = 10^{14}\) GeV and the mass eigenvalues of the right-handed Majorana neutrinos: \(M_{RR}^{\text{diag}} = \text{Diag}(1.105 \times 10^{11}, 3.035 \times 10^3, 5.005 \times 10^{11})\) GeV.
II(A). Inverted hierarchical mass matrix with same sign mass eigenvalues

The most general form of the inverted hierarchical mass matrix $m_{LL}$ with the same sign mass eigenvalues, can be calculated with the choice of $m_{LR}$ given in Eq.(3.4) and $M_{RR}$ of the following form

$$M_{RR} = \begin{pmatrix} 2a\eta(1 + 2\epsilon)\lambda^{2n} & \eta\epsilon \lambda^{m+n} & \eta\epsilon \lambda^m \\ \eta\epsilon \lambda^{m+n} & \lambda^{2n} & -(a - \eta)\lambda^n \\ \eta\epsilon \lambda^m & -(a - \eta)\lambda^n & a \end{pmatrix} \frac{v_R}{2a\eta},$$

leading to

$$m_{LL} = \begin{pmatrix} (1 - 2\epsilon) & -\epsilon & -\epsilon \\ -\epsilon & a & (a - \eta) \\ -\epsilon & (a - \eta) & a \end{pmatrix} m_0, \tag{3.24}$$

where $a = 0.5$ and $m_0$ is the overall factor for the masses of the neutrinos. The parameters $\epsilon$ and $\eta$ give the desired splittings for solar and atmospheric data. The diagonalisation of $m_{LL}$ in Eq.(3.24) leads to the following mass eigenvalues and mixing angles

$$m_{\nu_1} \simeq (1 - (\sqrt{3} + 1)\epsilon - \frac{\eta}{2} + \frac{\sqrt{\eta}\epsilon}{6})m_0, \tag{3.25}$$

$$m_{\nu_2} \simeq (1 + (\sqrt{3} - 1)\epsilon - \frac{\eta}{2} - \frac{\sqrt{\eta}\epsilon}{6})m_0, \tag{3.26}$$

$$m_{\nu_3} \simeq \eta m_0, \tag{3.27}$$

$$\sin^2 2\theta_{12} = \frac{2}{3}, \quad \sin^2 2\theta_{23} = 1, \quad \sin^2 2\theta_{13} = 0. \tag{3.28}$$

When $\epsilon = \eta = 0$, Eq.(3.24) reduces to the zeroth order mass matrix of the type II(A) in Table 3.2, with no solar splitting [74, 93, 94]. For solution of the LMA MSW solar data and atmospheric neutrino oscillation, we have the choice of the parameters $m_0 = 0.05$ eV, $\epsilon = 0.002$ and $\eta = 0.0001$ leading to the following predictions:
mass eigenvalues:

\[ m_{\nu_i} = (0.05007, 0.04973, 0.000005) \text{eV}, \quad i = 1, 2, 3; \] leading to \( \Delta m^2_{12} = 3.393 \times 10^{-5} \text{eV}^2 \) and \( \Delta m^2_{23} = 2.47 \times 10^{-3} \text{eV}^2 \).

The mixing angles are same as Eq.(3.28), which are independent of \( \epsilon, \eta \).

The expression for \( m_0 \) in Eq.(3.24) for case (i) is given by \( m_0 = m_t^2 \tan^2 \beta / v_R \). For input values of \( m_0 = 0.05 \text{eV}, \tan \beta = 5, m_t = 1.7 \text{GeV} \), we obtain \( v_R = 1.445 \times 10^{12} \text{GeV} \) which leads to mass eigenvalues of the right handed Majorana neutrinos, \( M_R^{\text{diag}} = \text{Diag}(0.742 \times 10^3, \ 2.831 \times 10^4, \ 7.24 \times 10^{15}) \text{GeV} \). Again for case(ii) \( m_0 = m_t^2 / v_R \) in Eq.(3.24). Using the input values \( m_0 = 0.05 \text{eV}, m_t = 200 \text{GeV} \), we have \( v_R = 8 \times 10^{14} \text{GeV} \) and \( M_R^{\text{diag}} = \text{Diag}(2.4 \times 10^4, \ 4 \times 10^{18}, \ 2.4 \times 10^9) \text{GeV} \) where the mass of the heaviest right-handed Majorana neutrino lies above the GUT scale but below the Planck scale [74].

II(B). Inverted hierarchical mass matrix with opposite sign mass eigenvalues

The most general inverted hierarchical mass matrix which is having the eigenvalues with the first two of opposite sign, is of the following form [82, 89, 97, 98, 99, 100]:

\[
\begin{pmatrix}
\epsilon & 1 & 1 \\
1 & \delta_1 & \delta_2 \\
1 & \delta_2 & \delta_1
\end{pmatrix}
\begin{pmatrix}
m_0, & \epsilon, & \delta_1, & \delta_2 << 1.
\end{pmatrix}
\]

For \( \delta_1, \delta_2, \epsilon = 0 \), it leads to the type II(B) in Table 3.2, with no mass splitting [74, 93, 94]. This structure can been successfully generated within the see-saw mechanism from the Dirac neutrino mass matrix in Eq.(3.4) and \( M_{RR} \) of the following form:

\[
M_{RR} = \begin{pmatrix}
-(\delta_1^2 + \delta_1 \delta_2 + \epsilon \delta_1^2 / 2) \lambda^{2n} & \delta_1 \lambda^{m+n} & \delta_1 \lambda^n \\
\delta_1 \lambda^{m+n} & \lambda^{2n} & -(1 - \epsilon \delta_2) \lambda^n \\
\delta_1 \lambda^m & -(1 - \epsilon \delta_2) \lambda^n & 1
\end{pmatrix}
\begin{pmatrix}
v_R \\
2\delta_1
\end{pmatrix}
\]

\[ (3.30) \]
Now the diagonalisation of $m_{LL}$ in Eq.(3.29), leads to the following

**mass eigenvalues:**

$$m_{\nu_1} \simeq (\epsilon + \delta_1 + \delta_2 - 2\sqrt{2})m_0/2, \quad \text{(3.31)}$$

$$m_{\nu_2} \simeq (\epsilon + \delta_1 + \delta_2 + 2\sqrt{2})m_0/2, \quad \text{(3.32)}$$

$$m_{\nu_3} = (\delta_1 - \delta_2)m_0, \quad \text{(3.33)}$$

and

**mixing angles:**

$$\sin^2 2\theta_{12} = \frac{8}{8 + (\epsilon - \delta_1 - \delta_2)^2}, \quad \text{(3.34)}$$

$$\sin^2 2\theta_{23} = 1, \quad \sin^2 2\theta_{13} = 0.$$  

Assigning $\delta_1 = \lambda^3$ and $\epsilon, \delta_2 = 0$ to Eq.(3.29) for numerical demonstration, we have the left-handed Majorana neutrino mass matrix of the form

$$m_{LL} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & \lambda^3 & 0 \\ 1 & 0 & \lambda^3 \end{pmatrix} m_0. \quad \text{(3.35)}$$

This leads to the following predictions

**mixing angles:**

$$\sin^2 2\theta_{12} = 0.99, \quad \sin^2 2\theta_{23} \simeq 1.0, \quad |V_{23}| = 0;$$

**mass eigenvalues:**

$$m_i = (0.0709, -0.0704, 0.0005) \text{eV}, \quad i = 1, 2, 3,$$

leading to $\Delta m_{12}^2 = 7.06 \times 10^{-5} \text{eV}^2$ and $\Delta m_{23}^2 = 5.02 \times 10^{-3} \text{eV}^2$.

The expression for $m_0$ in Eq.(3.29) for case (i) is given by $m_0 = m_r^2 \tan^2 \beta / v_R$. For input values of $m_0 = 0.05 \text{ eV}, \tan \beta = 5, m_r = 1.7 \text{ GeV}$, we obtain $v_R = 1.4 \times 10^{12}$.
3.3 Summary and discussion

GeV which leads to $M_R^{\text{diag}} = \text{diag}(1.6 \times 10^4, -1.6 \times 10^4, 1.4 \times 10^{14})$ GeV. Again for case(ii) $m_0 = m_2^2/v_R$ in Eq.(3.29). Using the input values $m_0 = 0.05\text{eV}$, $m_t = 200\text{GeV}$, we have $v_R = 8 \times 10^{14}$ GeV and $M_R^{\text{diag}} = \text{Diag}(9.1 \times 10^6, -9.1 \times 10^6, 8 \times 10^{15})$ GeV where the mass of the heaviest right-handed Majorana neutrino lies above the GUT scale but below the Planck scale [74].

The solar mixings predicted from $m_{LL}$ in Eqs(3.6) and (3.29) (Types I(A) and II(B)) are above the upper experimental limit [31, 32], $\sin^2 2\theta_{12} \leq 0.98$ with the best fit value $\sin^2 2\theta_{12} = 0.82$. Any fine tuning can hardly improve the value of $\sin^2 2\theta_{12}$. One may expect some spectacular changes if the contribution from the diagonalisation of the charged lepton mass matrix having special entries in the 1-2 block [101], is taken into consideration in the MNS mixing matrix $V_{\text{MNS}} = V_{\text{eL}}V_{\nu L}^\dagger$ which will be discussed in chapter 4.

A few comments on the stability condition under radiative corrections are in order. The nearly degenerate mass matrices $m_{LL}$ in Eqs(3.6),(3.12) and (3.18) and inverted hierarchical mass matrix in Eq.(3.24) are unstable under radiative correction in minimal supersymmetric standard model (MSSM) while the inverted hierarchical mass matrix given in Eq.(3.29) with opposite sign mass eigenvalues, is stable under radiative correction [101, 102, 103]. This aspect of the radiative stability of a neutrino mass matrix will be discussed in chapter 6.

3.3 Summary and discussion

In summary, we generate the textures of the nearly degenerate as well as the inverted hierarchical left-handed Majorana neutrino mass matrices from the see-saw formula using the diagonal form of the Dirac mass matrix and non-diagonal form of the right-handed Majorana neutrino mass matrix. This is, in fact, a continuation of chapter 2,
where bimaximal mixings are generated from the texture of \( M_{RR} \) in case of hierarchical and inverted hierarchical models. The predictions on lepton mixing angles \( \sin^2 2\theta_{12} \approx 0.67, \sin^2 2\theta_{23} \approx 1.0 \) and \( |V_{e3}| \approx 0 \) are in excellent agreement with the experimental values in all cases except for types I(A) and II(B). We also get good predictions for \( \Delta m^2_{12} \) and \( \Delta m^2_{23} \) which are necessary for the 0\( \nu \beta \beta \) decays, LMA MSW solar oscillation and atmospheric oscillation data. In all cases the masses of the right-handed Majorana neutrinos are above the weak scale. Though the present work is a model independent analysis without using any underlying symmetry, it would serve as a useful guide to building models under the framework of grand unified theories with extended flavour symmetry. In short the present analysis explores the possible origin of the bimaximal neutrino mixings from the texture of right-handed Majorana mass matrices.

Appendix

Diagonalisation of the mass matrix \( m_{LL} \)

The diagonalisation of a symmetric left-handed neutrino mass matrix \( m_{LL} \) of the following form which is very much useful in the present work, follows:

\[
m_{LL} = \begin{pmatrix}
a & b & c \\
b & d & e \\
c & e & f \\
\end{pmatrix},
\]

with \( c = -t_{23}b, \ t_{23} = \sin \theta_{23}/\cos \theta_{23} \) and \( f = d + (t_{23}^{-1} - t_{23})e, \)

can be diagonalised by \( V_{MNS} \)

\[
V_{MNS} = \begin{pmatrix}
\cos \theta_{12} & \sin \theta_{12} & 0 \\
-\cos \theta_{23} \sin \theta_{12} & \cos \theta_{23} \cos \theta_{12} & \sin \theta_{23} \\
n \sin \theta_{23} \sin \theta_{12} & -\sin \theta_{23} \cos \theta_{12} & \cos \theta_{23} \\
\end{pmatrix},
\]

where we have taken \( \sin \theta_{13} = 0. \)

This MNS matrix transforms \( |\nu_i > \) with the masses \( (m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) \) into \( |\nu_f > \) via

\[
|\nu_f > = V_{MNS} |\nu_i >, \ f = e, \mu, \tau \text{ and } i = 1, 2, 3.
\]
The mass eigenvalues and $\theta_{12}$ are calculated as

$$m_{\nu_1} = a - \frac{1}{2}\sqrt{b^2 + c^2} (x + \eta \sqrt{x^2 + 8}),$$

$$m_{\nu_2} = (\eta \to -\eta \text{ in } m_{\nu_1}),$$

$$m_{\nu_3} = d + t_{23}^2 (d - a + x \sqrt{\frac{b^2 + c^2}{2}}),$$

$$\sin^2 2\theta_{12} = \frac{8}{8 + x^2},$$

where $x = (a - d + t_{23} e) / (\sqrt{b^2 + c^2})$ and $|m_{\nu_1}| < |m_{\nu_2}|$ is always maintained by adjusting the sign of $\eta(= \pm 1)$. 

* * *