Chapter 2

Hierarchical Neutrino Mass Models

2.1 Introduction

The neutrino oscillation experiments [68] provide only the values of mass squared differences $\Delta m^2_{21} = |m_2^2 - m_1^2|$ and $\Delta m^2_{23} = |m_3^2 - m_2^2|$ where $m_i$'s are neutrino mass eigenvalues. This information allows any one of the following neutrino mass schemes, viz., (i) normal hierarchical (ii) inverted hierarchical and (iii) quasi-degenerate. It is now the established fact that the solar neutrino mixing angle ($\theta_{12}$) is large, but not maximal, while the atmospheric neutrino mixing angle ($\theta_{23}$) is nearly maximal with a lower bound [48]. The CHOOZ experiment [41] puts the constraint on the neutrino mixing parameter $|V_{e3}| = \sin \theta_{13}$ to an upper limit.

In this chapter we specifically focus on the hierarchical models of three active neutrino flavor mixing without sterile neutrino, which can accommodate the large mixing angle (LMA) MSW solution to solar neutrino problem and maximal mixing of the atmospheric neutrinos. The left-handed Majorana neutrino mass models $m_{LL}$ are constructed in the framework of the see-saw mechanism which contains both the Dirac neutrino mass matrix $m_{LR}$ and the heavy right-handed Majorana neutrino mass matrix $M_R$. There are several alternative approaches for realisation of the left-handed Majorana neutrino mass models for description of the experimental data in literature.
Most of these attempts employ the non-diagonal texture of the Dirac neutrino mass matrix \( m_{LR} \) and the charged lepton mass matrix \( m_e \). It will be interesting to study the non-zero texture of the right-handed Majorana mass matrix \( M_R \) in a model independent way while keeping the Dirac neutrino and charged lepton mass matrices in the diagonal form. \( M_R \) will be assumed to be fully responsible for large leptonic mixing and also for correct neutrino mass splitting. This phenomenological approach to \( M_R \) may improve our understanding of the right-handed Majorana neutrino (RH) sector [76] which remains almost unexplored. The knowledge of the RH neutrino sector is also important for the study of the origin of baryon asymmetry of the universe.

At the time of preparation of the original paper [77] presented in this chapter, the latest results from the Super-Kamiokande experiment [31] on atmospheric and solar neutrino oscillations, were at hand. According to the analysis of those data [78], the effective two-neutrino mixing parameters in case of the atmospheric neutrino oscillation, were constrained at 90% C.L. in the range \( 1.5 \times 10^{-3} \text{eV}^2 < \Delta m_{23}^2 < 5 \times 10^{-3} \text{eV}^2 \) and \( \sin^2 2\theta_{23} \geq 0.88 \) with the best fit values \( \Delta m_{23}^2 = 3 \times 10^{-3} \text{eV}^2 \) and \( \sin^2 2\theta_{23} = 1 \). On the other hand, the Super-Kamiokande results on solar neutrino oscillation disfavored some of the solutions to solar neutrino problem, viz., the small mixing angle (SMA) MSW solution and Vacuum (VAC) solution. The favourable solution was the LMA MSW solution, although LOW solution was not completely ruled out. A typical best fit value for LMA solution [79, 80, 81] was found to be \( \Delta m_{21}^2 = 4.2 \times 10^{-5} \text{eV}^2 \) with the corresponding mixing angle \( \sin^2 2\theta_{12} = 0.8163 \). With this LMA MSW solution the mass splitting parameter defined by \( \xi = \Delta m_{21}^2/\Delta m_{23}^2 \) is obtained as \( \xi = 0.014 \) [69, 70]. The CHOOZ experiment [41] limits \( \sin^2 2\theta_{13} \) to the range from 0.1 to 0.3 over the SK preferred range of \( \Delta m_{23}^2 \). We exclude the LSND results [40] in the present analysis and confine to three neutrino species only.
2.2 Neutrino mass matrix from a non-zero texture of $M_R$

The chapter is organised as follows. In section 2.2, we present a general form of $M_R$ and then generate the left-handed Majorana neutrino mass matrix, keeping the Dirac neutrino and the charged lepton mass matrices in the diagonal form. In section 2.3, we present the bimaximal mixings for hierarchical models. We conclude with a discussion in section 2.4.

### 2.2 Neutrino mass matrix from a non-zero texture of $M_R$

The left-handed Majorana neutrino mass matrix is given by the see-saw formula [62, 63]

$$m_{LL} = -m_{LR}M_R^{-1}m_{LR}^T$$

(2.1)

where $m_{LR}$ is the Dirac neutrino mass matrix in the left-right (LR) convention. The lepton mixing matrix known as the MNS mixing matrix [19] is defined as

$$V_{MNS} = V_{eL}V_{\nu L}^\dagger$$

(2.2)

where $V_{eL}$ and $V_{\nu L}$ are obtained from the diagonalisation of the charged lepton and left-handed Majorana mass matrices,

$$m_{l_{\text{diag}}} = V_{eL}m_1V_{eR}^\dagger = \text{Diag}(m_e, m_\mu, m_\tau)$$

$$m_{LL}^\text{diag} = V_{\nu L}m_{LL}V_{\nu L}^T = \text{Diag}(m_{\nu 1}, m_{\nu 2}, m_{\nu 3})$$

(2.3)

The neutrino flavour eigenstate $\nu_f$ is related to the mass eigenstate $\nu_i$ by the relation

$$\nu_f = V_{fi}\nu_i$$
where \( f = e, \mu, \tau \) and \( i = 1, 2, 3 \). The mixing matrix \( V_{fi} \) is now defined by the MNS matrix in Eq.(2.2). From the unitary conditions of the MNS matrix elements and its parametrisation by a sequence of three rotations about the 1, 2, 3 axes, the mixing angles are usually expressed in terms of the elements of \( V_{MNS} \) as

\[
\sin^2 \theta_{12} = \frac{4V_{e2}^2 V_{e1}^2}{(V_{e2}^2 + V_{e1}^2)^2},
\]

\[
\sin^2 \theta_{23} = \frac{4V_{\mu3}^2 V_{\tau3}^2}{(V_{\mu3}^2 + V_{\tau3}^2)^2} \tag{2.4}
\]

In the basis where the charged lepton mass matrix is diagonal, the MNS mixing matrix in Eq.(2.2) reduces to \( V_{MNS} = V_{\nu L}^T \), and \( m_{LL} \) in Eq.(2.1) is replaced by \[82\]

\[
m_{LL} = V_{\nu L} m_{LL} V_{\nu L}^T \tag{2.5}
\]

where \( m_{LR} \) is redefined as \((V_{\nu L} m_{LR})\) in the see-saw formula in Eq.(2.1). In order to generate the lepton mixings from the texture of \( M_R \) only, we consider the diagonal form of \( m_i \) and \( m_{LR} \). This leads to \( m'_{LL} = m_{LL} \), and \( V_{MNS} = V_{\nu L}^T \). The origin of \( M_R \) is quite different from those of the Dirac mass matrices \( m_{LR} \) in the underlying grand unified model. Since the Dirac neutrino mass matrices are hierarchical in nature and the CKM mixing angles of the quark sector are relatively small, our choices of the diagonal Dirac neutrino mass matrix and charged lepton mass matrix are partly justified. The large mixings for the solar and the atmospheric neutrino oscillations will now have their origin from the texture of \( M_R \). We parametrise in a model independent way, the most general form of \( M_R \) as

\[
M_R = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} \epsilon_1 & \sigma & \rho \\ \sigma & \epsilon_2 & \mu \\ \rho & \mu & \epsilon_3 \end{pmatrix} h_{33} v_R \tag{2.6}
\]
where $v_R$ is the vacuum expectation value (VEV) of the Higgs field which gives mass to the right-handed Majorana neutrino mass $M_R$ and $h_{33}$ is the Yukawa coupling of the heaviest right-handed Majorana neutrino. For simplicity we take $h_{33} = 1$ in our calculation. But there is an overall scaling freedom and one can scale all the $h_{ij}$'s by a common factor that would affect the prediction of $v_R$. The inverse of $M_R$ is now given by

$$M_R^{-1} = \begin{pmatrix} (\varepsilon_2 \varepsilon_3 - \mu^2) & (\mu \rho - \varepsilon_3 \sigma) & (\sigma \mu - \varepsilon_2 \rho) \\ (\mu \rho - \varepsilon_3 \sigma) & (\varepsilon_1 \varepsilon_3 - \rho^2) & (\sigma \rho - \varepsilon_1 \mu) \\ (\sigma \mu - \varepsilon_2 \rho) & (\sigma \rho - \varepsilon_1 \mu) & (\varepsilon_1 \varepsilon_2 - \sigma^2) \end{pmatrix} \left( Dv_R \right)^{-1} \quad (2.7)$$

where

$$D = \det|M_R| = (\varepsilon_1 \varepsilon_2 \varepsilon_3 + 2 \rho \sigma \mu - \varepsilon_1 \mu^2 - \varepsilon_2 \rho^2 - \varepsilon_3 \sigma^2). \quad (2.8)$$

For simplicity, we maintain the hierarchical structure in $M_R$ by making the substitution $\varepsilon_2 = \varepsilon^2$, $\mu = b \varepsilon$, $\varepsilon_3 = 1$, $\varepsilon_1 < \varepsilon_2$ in Eqs(2.6) and (2.7):

$$M_R = \begin{pmatrix} \varepsilon_1 & \sigma & \rho \\ \sigma & \varepsilon^2 & b \varepsilon \\ \rho & b \varepsilon & 1 \end{pmatrix} v_R, \quad (2.9)$$

$$M_R^{-1} = \begin{pmatrix} \frac{\varepsilon_3^2}{\rho^2} (1 - b^2) & -\frac{1}{\rho^2} (\sigma - b \varepsilon) & -\frac{1}{\rho^2} (\rho \varepsilon^2 - \sigma b \varepsilon) \\ -\frac{1}{\rho^2} (\sigma - b \varepsilon) & -(1 - \varepsilon_1 / \rho^2) & \frac{\varepsilon}{\rho} (1 - b \varepsilon \varepsilon_1 / \rho \sigma) \\ -\frac{1}{\rho^2} (\rho \varepsilon^2 - \sigma b \varepsilon) & \frac{\varepsilon}{\rho} (1 - b \varepsilon \varepsilon_1 / \rho \sigma) & -\frac{\varepsilon^2}{\rho^2} (1 - \varepsilon_1 \varepsilon^2 / \sigma^2) \end{pmatrix} \rho^2 / (Dv_R) \quad (2.10)$$

The Dirac neutrino mass matrix $m_{LR}$ appeared in the see-saw formula in Eq.(2.1), can be either the charged lepton mass matrix $m_t$ (referred to as case (i)) or the up-quark mass matrix $m_{up}$ (referred to as case (ii)) depending on the particular model of the SUSY SO(10) GUT and the content of the Higgs fields [83, 84, 85, 86] employed.
2.2 Neutrino mass matrix from a non-zero texture of $M_R$

Case (i) where $m_{LR} = (\tan \beta) m_l$

Since the charged lepton masses have the ratio $m_\tau : m_\mu : m_e = 1 : \lambda^2 : \lambda^6$, our choice for the diagonal $m_{LR}$ is [74, 75, 87]

$$m_{LR} = \tan \beta \begin{pmatrix} \lambda^6 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_\tau \quad (2.11)$$

where we take the value of the Wolfenstein parameter [88] $\lambda \approx 0.22$. Substituting Eqs (2.10) and (2.11) to (2.1), we have

$$m_{LL} = \begin{pmatrix} \frac{1}{\rho^2} (1 - b^2) \lambda^8 & \frac{1}{\rho^4} (\sigma - \rho \beta e) \lambda^4 & \frac{1}{\rho^4} (\rho e^2 - \sigma \beta e) \lambda^2 \\ \frac{1}{\rho^4} (\sigma - \rho \beta e) \lambda^4 & (1 - \frac{\lambda^4}{\rho^2}) & -\frac{\rho}{\rho^4} (1 - \frac{\sigma \beta e}{\rho^2}) \lambda^{-2} \\ \frac{1}{\rho^4} (\rho e^2 - \sigma \beta e) \lambda^2 & -\frac{\rho}{\rho^4} (1 - \frac{\sigma \beta e}{\rho^2}) \lambda^{-2} & \frac{\rho^2}{\rho^4} (1 - \frac{\sigma \beta e}{\rho^2}) \lambda^{-4} \end{pmatrix} R_1 \quad (2.12)$$

where

$$R_1 = \left( \frac{m_\tau^2 \lambda^4 \rho^2 \tan^2 \beta}{D v_R} \right) \times 10^9,$$

and $m_{LL}$ is expressed in eV, $m_\tau$ and $v_R$ in GeV.

Case (ii) where $m_{LR} = m_{up}$

The up-quark masses have the mass ratio $m_t : m_c : m_u = 1 : \lambda^4 : \lambda^6$ and $m_{LR}$ has the form [74, 75, 87]

$$m_{LR} = \begin{pmatrix} \lambda^6 & 0 & 0 \\ 0 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_t \quad (2.13)$$

and the corresponding left handed Majorana neutrino mass matrix can be calculated as

$$m_{LL} = \begin{pmatrix} \frac{1}{\rho^2} (1 - b^2) \lambda^8 & \frac{1}{\rho^4} (\sigma - \rho \beta e) \lambda^4 & \frac{1}{\rho^4} (\rho e^2 - \sigma \beta e) \\ \frac{1}{\rho^4} (\sigma - \rho \beta e) \lambda^4 & (1 - \frac{\lambda^4}{\rho^2}) & -\frac{\rho}{\rho^4} (1 - \frac{\sigma \beta e}{\rho^2}) \lambda^{-2} \\ \frac{1}{\rho^4} (\rho e^2 - \sigma \beta e) & -\frac{\rho}{\rho^4} (1 - \frac{\sigma \beta e}{\rho^2}) \lambda^{-2} & \frac{\rho^2}{\rho^4} (1 - \frac{\sigma \beta e}{\rho^2}) \lambda^{-4} \end{pmatrix} R_2 \quad (2.14)$$
2.3 Hierarchical neutrino mass matrices

where $R_2 = \left(\frac{m_1^2 \lambda^2 \rho^3}{D v_R}\right) \times 10^9$, and $m_{LL}$ is expressed in eV and $m_\ell$ and $v_R$ in GeV.

In general we can have any structure of hierarchical $m_{LR}$ in the diagonal form and a corresponding $m_{LL}$ can be calculated. $m_{LL}$ in Eqs(2.12) and (2.14) for case (i) and case (ii) can be reduced to a form which can generate neutrino mixings suitable for the explanation of the LMA solution to solar neutrino problem and maximal mixing of the atmospheric neutrino oscillation. We give the numerical demonstration for the hierarchical [77] scheme of neutrino mass patterns in the next section 2.3. The same formalism, we shall employ to deal with the inverted hierarchical and degenerate schemes of neutrino mass patterns [89] in chapter 3.

2.3 Hierarchical neutrino mass matrices

In the hierarchical model we have the neutrino mass pattern $m_{\alpha 3} >> m_{\alpha 2} >> m_{\alpha 1}$ and the splitting parameter $\xi = \Delta m_{12}^2 / \Delta m_{23}^2 \simeq m_{\alpha 2}^2 / m_{\alpha 3}^2$. We analyse $m_{LL}$ for case (i) and case (ii) given in Eqs(2.12) and (2.14) respectively for generating bimaximal mixings[77].

Case (i) where $m_{LR} = (\tan \beta)m_4$:

In Eq.(2.12) we make suitable choices for the parameters in $m_{LL}$ as follows:

(a) The choice $\rho = \lambda^5$, $\sigma = \lambda^7$, $\epsilon = \lambda^3$, $\epsilon_1 = \lambda^{11}$ gives

$$m_{LL} = \begin{pmatrix}
-\lambda^4 (1 - b^2) & \lambda (1 - b \lambda) & \lambda^3 (1 - b \lambda^{-1}) \\
\lambda (1 - b \lambda) & (1 - \lambda) & -(1 - b \lambda^2) \\
\lambda^3 (1 - b \lambda^{-1}) & -(1 - b \lambda^2) & (1 - \lambda^3)
\end{pmatrix} R_1. \quad (2.15)$$

This simplifies to, for $b=0$,

$$m_{LL} = \begin{pmatrix}
-\lambda^4 & \lambda & \lambda^3 \\
\lambda & (1 - \lambda) & -1 \\
\lambda^3 & -1 & (1 - \lambda^3)
\end{pmatrix} R_1, \quad (2.16)$$
2.3 Hierarchical neutrino mass matrices

where $D = \lambda^{14}$ and $R_1$ given in Eq.(2.12). The eigenvalues are $m_{\nu i} = (0.11184, 0.24525, 1.9004)R_1$ in eV. The choice of $R_1 = 0.03$ corresponds to the correct magnitudes of the observed parameters, and this in turn predicts $v_R = 8.92 \times 10^{13}$ GeV. The other predicted values are $\xi = \Delta m_{12}^2/\Delta m_{23}^2 \simeq 0.0134$, $\sin^2 2\theta_{12} = 0.8987$, $\sin^2 2\theta_{23} = 0.9809$, $|V_{e3}| = 0.074$. The corresponding texture of $M_R$ is

$$M_R = \begin{pmatrix} \lambda^{11} & \lambda^7 & \lambda^5 \\ \lambda^7 & \lambda^6 & 0 \\ \lambda^5 & 0 & 1 \end{pmatrix} v_R. \quad (2.17)$$

The three right-handed Majorana neutrino masses given by the diagonalisation of $M_R$ are $M_R^{\text{diag}} = \text{Diag}(5.425 \times 10^{-6}, 1.186 \times 10^{-4}, 1.00)v_R$.

(b) The choice $\rho = \lambda^4$, $\sigma = \lambda^6$, $\epsilon = \lambda^3$, $\epsilon_1 = -\lambda^{10}$, $b = \frac{1}{\lambda^4}$ gives

$$m_{LL} = \begin{pmatrix} \lambda^2 - \lambda^6 & -\lambda + \lambda^2 & -\lambda + \lambda^4 \\ -\lambda + \lambda^2 & 1 + \lambda^2 & -(1 + \lambda) \\ -\lambda + \lambda^4 & -(1 + \lambda) & (1 + \lambda^4) \end{pmatrix} R_1, \quad (2.18)$$

and $D = 2\lambda^{11}$. This predicts $m_{\nu i} = (0.2276, 0.3745, 2.246)R_1$ in eV. With the same input value $R_1 = 0.03$ we get $v_R = 1.0 \times 10^{13} GeV$. The other predictions are $\xi = \Delta m_{12}^2/\Delta m_{23}^2 = 0.018$, $\sin^2 2\theta_{12} = 0.8372$, $\sin^2 2\theta_{23} = 0.9994$, $|V_{e3}| = 0.014$. The corresponding $M_R$ is

$$M_R = \begin{pmatrix} -\lambda^{10} & \lambda^6 & \lambda^4 \\ \lambda^6 & \lambda^6 & \lambda \\ \lambda^4 & \lambda & 1 \end{pmatrix} v_R, \quad (2.19)$$

and $M_R^{\text{diag}} = \text{Diag}(2.4033 \times 10^{-6}, 4.615 \times 10^{-2}, 1.05)v_R$.

Case (ii) where $m_{LR} = m_{\nu p}$

We again examine the texture of $m_{LL}$ in Eq.(2.14) with the following choices of parameters.
2.3 Hierarchical neutrino mass matrices

(c) The choice $\rho = \lambda^7$, $\sigma = \lambda^{11}$, $\epsilon = \lambda^6$, $\epsilon_1 = \lambda^{15}$ leads to the same texture of $m_{LL}$ in Eq.(2.15), and for $b=0$, it again reproduces the same $m_{LL}$ given in Eq.(2.16), but $R_1$ is now replaced by $R_2$ given in Eq.(2.14) and $D = \lambda^{22}$. With the same choice of value $R_2 = 0.03$ we get $v_R = 0.68 \times 10^{13}\text{GeV}$. The corresponding $M_R$ is

$$M_R = \begin{pmatrix} \lambda^{15} & \lambda^{11} & \lambda^7 \\ \lambda^{11} & \lambda^{10} & 0 \\ \lambda^7 & 0 & 1 \end{pmatrix} v_R. \tag{2.20}$$

and $M_R^{\text{diag}} = \text{Diag}(1.260 \times 10^{-8}, 2.771 \times 10^{-7}, 1.0)v_R$.

(d) The choice $\rho = \lambda^7$, $\sigma = \lambda^{11}$, $\epsilon = \lambda^4$, $\epsilon_1 = -\lambda^{15}$, $b = \lambda$ gives

$$m_{LL} = \begin{pmatrix} -\lambda^2(1 - \lambda^2) & \lambda(1 - \lambda) & \lambda(1 - \lambda) \\ \lambda(1 - \lambda) & (1 + \lambda) & -(1 + \lambda^2) \\ \lambda(1 - \lambda) & -(1 + \lambda^2) & (1 + \lambda) \end{pmatrix} R_2. \tag{2.21}$$

This predicts $m_{\nu i} = (0.20319, 0.32874, 2.26844)R_2$ with $D = 2\lambda^{22}$. Taking the input value of $R_2 = 0.03$ we have $v_R = 0.34 \times 10^{13}\text{GeV}$. Other predictions are $\xi = \Delta m_{12}^2/\Delta m_{23}^2 = 0.01326$, $\sin^2 2\theta_{12} = 0.8326$, $\sin^2 2\theta_{23} = 0.99$, $|V_{e3}| \approx 0.0$. The corresponding $M_R$ is

$$M_R = \begin{pmatrix} -\lambda^{15} & \lambda^{11} & \lambda^7 \\ \lambda^{11} & \lambda^{10} & \lambda^8 \\ \lambda^7 & \lambda^8 & \lambda^3 \end{pmatrix} v_R. \tag{2.22}$$

and $M_R^{\text{diag}} = \text{Diag}(1.429 \times 10^{-9}, 5.504 \times 10^{-6}, 1.0)v_R$.

In the above examples, we have confined to the diagonal forms of the Dirac mass matrices with positive eigenvalues as physical masses. The most general form of the right-handed Majorana neutrino mass matrix $M_R$ is complex and symmetric. A general fitting procedure for obtaining the desired $m_{LL}$ would leave a large number of free parameters [90]. In the present work our objective is to derive $m_{LL}$ analytically from
the structure of $M_R$, and it is difficult to carry out analytically if we consider the complex parameters having the same order of $\lambda$ [90]. For simplicity, we have considered the extreme cases with positive and negative real values of the parameters in the above examples. It is evident that all other values of the phases will be some intermediate cases of these two extreme cases. Analysis of such extreme cases is also useful and very common in literature [83, 91].

There are various sources of uncertainties in the prediction of $v_R$ in the above examples. Our choice $h_{33} = 1$ can be relaxed as there is an overall scaling freedom and one can scale all the $h_{ij}$'s by a common factor (viz., from 10 to 0.1) [83, 91]. This would change the predicted value of $v_R$ by a factor of 10 on both sides. The upper bound on the seesaw scale $v_R$ can be pushed up to order of GUT scale or even beyond. A large value of $v_R$, $v_R \geq M_{GUT}$ is naturally expected in some GUTs e.g., SUSY SO(10)GUT if it is broken down to SU(5) close to or slightly below the Planck mass $\sim 10^{19} GeV$ [74, 75]. Our predictions on $v_R$ are within the acceptable bound, and the lightest right-handed Majorana neutrino masses $M_{R1}$ lie above the electroweak scale in all examples.

2.4 Summary and discussion

The bimaximal mixings which explain both LMA MSW solution of the solar neutrino anomaly and the atmospheric neutrino solution, have been generated through the seesaw mechanism using the non-zero texture of the right-handed Majorana neutrino mass matrix $M_R$ and the diagonal forms of the Dirac neutrino mass matrix and the charged lepton mass matrix [77]. The left-handed Majorana neutrino mass matrix $m_{\nu L}$ has been derived for the hierarchical neutrino mass scheme with the proper choices of the values of the parameters in $M_R$. The most general matrix $M_R$ is complex and symmetric. For simplicity of calculation, we have considered the two extreme cases with positive and negative real values of the parameters with the view of all other values of the phases
leading to some intermediate cases of the two special choices. In our analysis we have considered both possibilities where the Dirac neutrino mass matrix $m_{LR}$ is either the charged-lepton or the up-quark mass matrix. For illustration, we have presented four representative examples only.

In analysis, we have made the seesaw scale $v_R$ as a free parameter and its prediction is made for each example by adjusting the correct magnitude of the predictions $\Delta m^2_{ij}$ with experimental data and as expected the corresponding right handed neutrino mass eigenvalues are obtained well above the electroweak scale. The predictions in all examples are found to be in excellent agreement with the best fit values $\xi = 0.014, \sin^2 2\theta_{12} = 0.83, \sin^2 2\theta_{23} = 1$ or with slight deviations. In all examples the CHOOZ angles are found as $|V_{e3}| = 0.0$.

Here the comment on the radiative stability is in order. The predictions of neutrino mass splitting and mixing angles in hierarchical models are found to undergo a very mild deviation on running the RGEs in MSSM model from the unification scale to the top-quark mass scale [92] (see chapter 6).

Though the present work is a model independent analysis without using any underlying symmetry, it would serve as a useful guide to building models in the framework of GUTs with extended flavor $U(1)$ symmetry. In short the present analysis explores the possible origin of the bimaximal neutrino mixings from the non-zero texture of the right handed Majorana neutrino mass matrices.

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