Chapter I

Preliminaries

1.1. Fuzzy Subset.

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1.3. Fuzzy subset and fuzzy N-subset of \( N \).

1.4. Fuzzy subset of an N-group \( E \).
Chapter I

Preliminaries

In this chapter, we present the necessary basic definitions and results used in our work. The chapter is divided into four sections. The first section highlights the fundamental properties of a fuzzy subset of a set. The second one deals with fuzzy subgroups, fuzzy normal subgroups of a group and related basic results on them. The third section contains the notion of fuzzy subsets of a near-ring and their different operations like \( \mu + \theta, -\mu, \mu \circ \theta, \mu \theta \) where \( \mu \) and \( \theta \) are fuzzy subsets of a near-ring \( N \). We also introduce the notion of fuzzy \( N \)-subset of \( N \) with some basic properties. In the last section some properties on fuzzy subsets of a near-ring group \( E \) are presented. This section also contains various results pertaining to the operations such as \( \sigma \circ \mu, \sigma \mu, n \mu \), where \( \sigma \) is a fuzzy subset of a near-ring \( N \), \( \mu \) is a fuzzy subset of a near-ring group \( E \) and \( n \in \mathbb{N} \).

1.1 Fuzzy subset

1.1.1. Definition: [75] Let \( A \) be a non-empty set. A function \( \mu : A \rightarrow [0,1] \) is called a fuzzy subset of \( A \). A fuzzy subset is characterized by its membership function \( \mu \).

1.1.2. Example: Let \( X \) be a non-empty set.

(i) Then the function \( \mu : X \rightarrow [0,1] \) defined by \( \mu(x) = 0 \), for all \( x \in X \) is a fuzzy subset of \( X \). We denote this fuzzy subset by the symbol \( \hat{0} \).

(ii) The function \( \mu : X \rightarrow [0,1] \) defined by \( \mu(x) = 1 \), for all \( x \in X \) is a fuzzy subset of \( X \). We denote this fuzzy subset by the symbol \( \hat{1} \).

(iii) Let \( A \subseteq X \). Then the function \( \mu : X \rightarrow [0,1] \) defined by
\( \mu(x) = 1 \) when \( x \in A \) and 
\( = 0, \) otherwise
is a fuzzy subset of \( X \) which is the characteristic function \( \chi_A \) of \( A. \)

1.13. Definition:[27] If \( \mu \) is a fuzzy subset of a set \( X \) then the set \( \{ x \in X : \mu(x) > 0 \} \) is said to be support of \( \mu \) and is denoted by \( \mu^*. \)

1.14. Definition: Two fuzzy subsets \( \mu, \sigma \) of a non empty set \( X \) are said to be equal if \( \mu(x) = \sigma(x) \), for all \( x \in X. \)

1.15 Definition:[75] Let \( \mu \) and \( \theta \) be fuzzy subsets of \( X. \) Then the union and intersection of \( \mu \) and \( \theta \) are denoted by \( \mu \cup \theta \) and \( \mu \cap \theta \) respectively and defined by

\[
(\mu \cup \theta)(x) = \mu(x) \lor \theta(x), \text{ for all } x \in X. \quad \text{and}
\]

\[
(\mu \cap \theta)(x) = \mu(x) \land \theta(x), \text{ for all } x \in X.
\]

More generally, if \( \{ \mu_i : i \in \Delta \} \) is any family of fuzzy subsets of \( X, \) where \( \Delta \) is a non empty index set, then , for \( x \in X,
\]
\[
(\bigcup_{i} \mu_i)(x) = \lor_{i \in \Delta} \mu_i(x) \quad \text{and}
\]
\[
(\bigcap_{i} \mu_i)(x) = \land_{i \in \Delta} \mu_i(x)
\]

The proof of the following lemma is straightforward.

1.16. Lemma: Let \( \mu \) be a fuzzy subset of \( X. \) Then

(i) \( \mu \cap \hat{\nu} = \hat{\nu} \)

(ii) \( \mu \cup \hat{\nu} = \mu \)

(iii) \( \mu \cap \hat{1} = \mu \)

(iv) \( \mu \cup \hat{1} = \hat{1} \)

1.17. Definition: Let \( \mu \) be a fuzzy subset of \( X. \) Then the set \( \{ \mu(x) : x \in X \} \) is called the image set of \( \mu \) denoted by \( \text{Im} \mu. \) Also \( |\text{Im} \mu| \) will mean the cardinality of \( \text{Im} \mu. \)

1.18. Definition:[75] Let \( \mu \) be a fuzzy subset of \( X. \) Then the complement of \( \mu \) is denoted by
\[ \mu^c(x) = 1 - \mu(x), \text{ for all } x \in X. \]

1.1.9. Lemma: Let \( \mu \) and \( \theta \) be fuzzy subsets of \( X \). Then the following are true:

(i) \( (\mu \cup \theta)^c = \mu^c \cap \theta^c \)

(ii) \( (\mu \cap \theta)^c = \mu^c \cup \theta^c \)

1.1.10. Remark: In fuzzy set theory, if \( \mu \) is a fuzzy subset of \( X \), then in general

\[ \mu \cap \mu^c \neq \emptyset \text{ and } \mu \cup \mu^c \neq \hat{1}. \]

1.1.11. Definition:[75] Let \( \mu \) be a fuzzy subset of \( X \) and \( t \in [0,1] \). Then the set

\[ \mu_t = \{ x \in X : \mu(x) \geq t \} \]

is called \textit{t-level or level subset} of \( \mu \).

1.1.12. Definition: Let \( \mu \) and \( \theta \) be fuzzy subsets of \( X \). If \( \mu(x) \geq \theta(x) \), for all \( x \in X \), then \( \mu \) is said to be \textit{subset of the fuzzy subset} \( \theta \) and is denoted by \( \mu \subseteq \theta \). If \( \mu \subseteq \theta \) and \( \mu \neq \theta \), then \( \mu \) is called \textit{proper subset of the fuzzy subset} \( \theta \) and we write \( \mu \subset \theta \).

1.1.13. Lemma: Let \( \mu \) and \( \theta \) be fuzzy subsets of \( X \). Then the following hold:

(i) \( \mu \subseteq \theta \Rightarrow \mu_t \subseteq \theta_t \) where \( t \in [0,1] \).

(ii) \( t \leq s, \Rightarrow \mu_t \subseteq \mu_s \) where \( t, s \in [0,1] \)

(iii) \( \mu = \theta \) if and only if \( \mu_t \equiv \theta_t \), \( \forall t \in [0,1] \)

1.1.14. Lemma: Let \( \{\mu_i : i \in \Delta\} \) be any family of fuzzy subsets of \( X \), where \( \Delta \) is a non-empty index set, then

(i) \( \bigcup_{i \in \Delta} (\mu_i) \subseteq (\bigcup_{i \in \Delta} \mu_i) \), where \( i \in \Delta \) and \( t \in [0,1] \).

(ii) \( \bigcap_{i \in \Delta} (\mu_i) = (\bigcap_{i \in \Delta} \mu_i) \) where \( i \in \Delta \) and \( t \in [0,1] \)

1.1.15. Definition:[61] A fuzzy subset \( \mu \) of \( X \) is said to have \textit{supremum property} if for any subset \( A \) of \( X \), there exists \( x \in A \) such that
\[ \mu(x) = \bigvee_{a \in A} \mu(a) \]

1.1.16. **Definition:** [70] A fuzzy subset \( \mu \) of \( X \) is said to have **infimum property** if for any subset \( A \) of \( X \), there exists \( x \in A \) such that
\[ \mu(x) = \bigwedge_{a \in A} \mu(a) \]

1.1.17. **Definition:** [61] Let \( f \) be any function from a set \( S \) to a set \( T \). Let \( \mu \) and \( \theta \) be fuzzy subsets of \( S \) and \( T \) respectively. Then the **image of \( \mu \)** under \( f \) denoted by \( f(\mu) \) is a fuzzy subset of \( T \) and is defined as
\[ f(\mu)(y) = \bigvee_{f(x) = y} \mu(x) \quad \text{if} \quad f^{-1}(y) \neq \emptyset \]
\[ = 0 \quad \text{otherwise.} \]

Also the **pre-image** of \( \theta \) under \( f \) is denoted by \( f^{-1}(\theta) \) is a fuzzy subset of \( S \) and is defined as
\[ [f^{-1}(\theta)](x) = \theta(f(x)) \quad \text{for all} \quad x \in S \]

1.1.18. **Definition:** [61] Let \( f \) be any function from a set \( S \) to a set \( T \). Let \( \mu \) be a fuzzy subset of \( S \). Then \( \mu \) is called \( f \)-**invariant** if
\[ f(x) = f(y) \quad \Rightarrow \quad \mu(x) = \mu(y) \quad \text{for} \quad x, y \in S. \]

1.1.19. **Lemma:** Let \( S \) and \( T \) be two sets and \( f \) be a mapping from \( S \) into \( T \). Let \( \mu \) and \( \theta \) be fuzzy subsets of \( S \) and \( \zeta \) and \( \xi \) be fuzzy subsets of \( T \). Then the following are true:

(i) \[ [f(\mu)]^c \subseteq f([\mu]^{c}) \]

(ii) \[ f^1(\xi^c) = [f^{-1}(\xi)]^c \]

(iii) \[ \zeta \subseteq \xi \quad \Rightarrow \quad f^{-1}(\zeta) \subseteq f^{-1}(\xi) \]

(iv) \[ \mu \subseteq \theta \quad \Rightarrow \quad f(\mu) \subseteq f(\theta) \]

(v) \[ f[f^{-1}(\zeta)] \subseteq \zeta \]

(vi) \[ f[f^{-1}(\zeta)] = \zeta \quad \text{if} \quad f \text{ is onto} \]

(vii) $\mu \subseteq f^{-1}[f(\mu)]$

(viii) $\mu = f^{-1}[f(\mu)]$ if $\mu$ is $f$-invariant.

1.1.20. Lemma: Let $S$ and $T$ be two sets and $f$ be a mapping from $S$ into $T$. Let

$\{\mu_i : i \in \Delta\}$ and $\{\zeta_i : i \in \Delta\}$ be any family of fuzzy subsets of $S$ and $T$ respectively. Then

(i) $f(\bigcap_{i \in \Delta} \mu_i) \subseteq \bigcap_{i \in \Delta} f(\mu_i)$

(ii) $f(\bigcap_{i \in \Delta} \mu_i) = \bigcap_{i \in \Delta} f(\mu_i)$ if $f$ is one-one.

(iii) $f(\bigcup_{i \in \Delta} \mu_i) = \bigcup_{i \in \Delta} f(\mu_i)$

(iv) $f^{-1}(\bigcup_{i \in \Delta} \zeta_i) = \bigcup_{i \in \Delta} f^{-1}(\zeta_i)$

(v) $f^{-1}\left(\bigcap_{i \in \Delta} \zeta_i\right) = \bigcap_{i \in \Delta} f^{-1}(\zeta_i)$

Proof: (i) For each $y \in T$,

$$f(\bigcap_{i \in \Delta} \mu_i)(y) = \vee_{f(x)=y} \left[\wedge, \{\mu_i(x)\}\right]$$

$$\leq \wedge, \left[\vee_{f(x)=y} \{\mu_i(x)\}\right]$$

$$= \bigcap_{i \in \Delta} f(\mu_i)(y).$$

(ii) For each $y \in T$,

$$f(\bigcap_{i \in \Delta} \mu_i)(y) = \vee_{f(x)=y} \left[\wedge, \{\mu_i(x)\}\right]$$

$$= \wedge, \mu_i(x) \quad \text{as there exists only one } x \in S \text{ such that } f(x) = y$$

$$=[\bigcap_i f(\mu_i)](y).$$

Similarly, the remaining proofs can be obtained.
1.1.21. **Lemma:** Let $f: S \rightarrow T$ and $g: T \rightarrow R$ be two mappings and let $\mu$ and $\theta$ be fuzzy subsets of $S$ and $R$ respectively. Then the following are true:

(i) $(g \circ f)(\mu) = g(f(\mu))$

(ii) $[g \circ f]^{-1}(\theta) = f^{-1}(g^{-1}(\theta))$ where $g \circ f$ is composition of $f$ and $g$.

1.1.22. **Definition:** Let $X$ be a non empty set and $A \subseteq X$. Let $t \in (0,1]$. A fuzzy subset $t_A$ of $X$ is defined as follows:

$$t_A(x) = t, \text{ if } x \in A$$

$$= 0, \text{ if } x \in X \setminus A$$

If $A$ is a singleton set, say $\{x\}$ then we write $t_{\{x\}}$ as $x$, which is often called *fuzzy point* or *fuzzy singleton*.

1.1.23. **Lemma:** Let $\mu$ be a fuzzy subset of $X$. Let $x \in X$ and $t \in (0,1]$. Then $x \in \mu_t$ if and only if $x, \subseteq \mu$.

1.1.24. **Remark:** If $x, \mu$ is a fuzzy point of $X$ and $\mu$ is a fuzzy subset of $X$ then we say that $x \in \mu_t$ if and only if $x, \in \mu$.

### 1.2 Fuzzy subgroup

In the following discussion $G$ denotes an arbitrary group with an additive binary operation and identity $0$.

1.2.1. **Definition:**[27] Let $\mu$ and $\theta$ be two fuzzy subsets of a group $G$. Then the *addition* $\mu + \theta$ of $\mu$ and $\theta$ and the *inverse* $-\mu$ of $\mu$ are defined as follows:

$$(\mu + \theta)(x) = \vee_{y, y' \in G} \{\mu(y) \land \theta(z) : y, z \in G\} \text{ for all } x \in G$$

$$(-\mu)(x) = \mu(-x) \text{ for all } x \in G$$
1.2.2. Lemma: Let $\mu$ and $\theta$ be fuzzy subsets of a group $G$. Then the following statements are true:

(i) $-(-\mu) = \mu$

(ii) $\mu \subseteq -\mu \Leftrightarrow -\mu \subseteq \mu$

(iii) $\mu \subseteq \theta \Leftrightarrow (-\mu) \subseteq (-\theta)$

(iv) $-(\mu \cup \theta) = (-\mu) \cup (-\theta)$

(v) $-(\mu \cap \theta) = (-\mu) \cap (-\theta)$

(vi) $-(\mu + \theta) = (-\theta) + (-\mu)$

1.2.3. Definition [61]: Let $\mu$ be a fuzzy subset of a group $G$. Then $\mu$ is called a fuzzy subgroup of $G$ if for all $x, y \in G$,

(i) $\mu(x + y) \geq \mu(x) \land \mu(y)$

(ii) $\mu(-x) \geq \mu(x)$

1.2.4. Example: Let $G = \{0, a, b, c\}$ Then $(G, +)$ is a group, where $+$ is defined as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
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<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
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We consider the fuzzy subset $\mu : G \to [0, 1]$ such that $\mu(0) = \mu(b) = t$ and $\mu(a) = \mu(c) = s$ where $t, s \in [0, 1], t > s$. Then $\mu$ is a fuzzy subgroup of $G$.

1.2.5. Remarks: If $\mu$ is a fuzzy subset of $G$ then

$$\mu(-x) \geq \mu(x), \forall x \in G \Leftrightarrow \mu(-x) = \mu(x), \forall x \in G.$$  

1.2.6. Lemma: Let $\mu$ be a fuzzy subgroup of $G$. Then for all $x, y \in G$

(i) $\mu(0) \geq \mu(x)$
(ii) \( \mu(x) = \mu(-x) \)

(iii) \( \mu(x - y) = \mu(0) \Rightarrow \mu(x) = \mu(y) \).

**Proof:** Let \( x \in G \).

(i) \( \mu(0) = \mu(x - x) \geq \mu(x) \wedge \mu(-x) = \mu(x) \).

(ii) follows from 1.2.5.

(iii) \( \mu(x) = \mu(x - y + y) \geq \mu(x - y) \wedge \mu(y) \)

\[ = \mu(y) \quad [\text{by (i)}] \]

Similarly, we have \( \mu(y) \geq \mu(x) \) and consequently the result follows. \( \square \)

1.2.7. **Lemma:** A fuzzy sub set \( \mu \) of \( G \) is a fuzzy subgroup of \( G \) if and only if

\[ \mu(x - y) \geq \mu(x) \wedge \mu(y), \forall x \in G. \]

1.2.8. **Lemma:** Let \( \mu (\neq \hat{0}) \) be a fuzzy subgroup of \( G \). Then \( \mu^* \) is also a subgroup of \( G \).

**Proof:** Since \( \mu \neq \hat{0} \), \( \mu^* \) is non empty. Let \( x, y \in \mu^* \). Then \( \mu(x) > 0 \) and \( \mu(y) > 0 \). Since \( \mu \) is a fuzzy subgroup of \( G \), \( \mu(x - y) \geq \mu(x) \wedge \mu(y) > 0 \). Hence \( x - y \in \mu^* \) and consequently \( \mu^* \) is subgroup of \( G \). \( \square \)

1.2.9. **Lemma:** A fuzzy subset \( \mu \) of \( G \) is a fuzzy subgroup of \( G \) if and only if \( \mu_t \), for all \( t \in \text{Im} \mu \) is a subgroup of \( G \).

**Proof:** Let \( \mu \) be a fuzzy subgroup of \( G \) and \( t \in \text{Im} \mu \). Clearly \( \mu_t \) is non empty as \( 0 \in \mu_t \). Let \( x, y \in \mu_t \). Since \( \mu \) is a fuzzy subgroup, we have,

\[ \mu(x - y) \geq \mu(x) \wedge \mu(y) \]

\[ \geq t \quad \text{as } x, y \in \mu_t. \]

Thus \( x - y \in \mu_t \) and so \( \mu_t \) is a subgroup of \( G \).

Conversely, let \( \mu_t \) for all \( t \in \text{Im} \mu \) be a subgroup of \( G \). Let \( x, y \in G \) and let \( \mu(x) = t \) and \( \mu(y) = s \). Then \( x, y \in \mu_{t \wedge s} \). Since \( t \wedge s \in \text{Im} \mu \), by assumption \( x - y \in \mu_{t \wedge s} \). Thus
\[
\mu(x - y) \geq t \land s = \mu(x) \land \mu(y)
\]

This gives \( \mu \) is a fuzzy subgroup of \( G \).

1.2.10. **Lemma**: Let \( A \) be a non empty sub-set of \( G \). Then \( A \) is a fuzzy subgroup of \( G \) if and only if the characteristic function \( \chi_A \) of \( A \) is fuzzy subgroup of \( G \).

1.2.11. **Lemma**: A fuzzy sub set \( \mu \) of \( G \) is a fuzzy subgroup of \( G \) if and only if

\[
\begin{align*}
\text{(i)} & \quad \mu + \mu \subseteq \mu \\
\text{(ii)} & \quad -\mu \subseteq \mu \quad \text{(or} \quad \mu = -\mu) 
\end{align*}
\]

1.2.12. **Remark**: If a fuzzy subset \( \mu \) of \( G \) is such that \( \mu(x + y) \geq \mu(x) \land \mu(y), \forall x \in G \), then

\[
\mu(nx) \geq \mu(x), \forall x \in G, \text{ where } n \in \mathbb{Z}^+. 
\]

1.2.13. **Lemma**: Let \( \mu \) and \( \theta \) be fuzzy subgroups of \( G \), then \( \mu + \theta \) is a fuzzy subgroup of \( G \) if and only if \( \mu + \theta = \theta + \mu \).

**Proof**: Let \( \mu + \theta \) be a fuzzy subgroup of \( G \). Then

\[
\mu + \theta = -(\mu + \theta) \\
= (-\theta) + (-\mu) \quad [1.2.2(vi)] \\
= \theta + \mu 
\]

Conversely, let \( \mu + \theta = \theta + \mu \). Then

\[
\begin{align*}
(\mu + \theta) + (\mu + \theta) \\
= \mu + (\theta + \mu) + \theta \\
= \mu + (\mu + \theta) + \theta \\
= (\mu + \mu) + (\theta + \theta) \\
\subseteq \mu + \theta 
\end{align*}
\]

Also \( -(\mu + \theta) = -(\theta + \mu) = (-\mu) + (-\theta) = \mu + \theta \)

Hence \( \mu + \theta \) is a fuzzy subgroup of \( G \).
1.2.14. **Theorem:** Let \( G \) and \( G' \) be two groups and \( f : G \to G' \) be a homomorphism. Let \( \mu \) and \( \theta \) be fuzzy subgroups of \( G \) and \( G' \) respectively. Then

(i) \( f(\mu) \) is fuzzy subgroup of \( G' \) if \( f \) is onto.

(ii) \( f^{-1}(\theta) \) is a fuzzy subgroup of \( G \).

1.2.15. **Theorem:** The intersection of a family of fuzzy subgroups of a group \( G \) is a fuzzy subgroup of \( G \).

But in case of fuzzy subgroups, there exist fuzzy subgroups whose union is a fuzzy subgroup while none is contained in the other. This can be seen in the following example:

1.2.16. **Example:** Let \( G \neq \{0\} \), and let \( t_0, t_1, t_2, t_3 \in (0,1] \) such that \( t_0 > t_1 > t_2 > t_3 \).

Fuzzy subsets \( \mu, \theta \) and \( \sigma \) of \( G \) are defined as

\[
\mu(x) = t_2, \text{ if } x \neq 0 \text{ and } \mu(0) = t_0;
\]

\[
\theta(x) = t_2, \text{ if } x \neq 0 \text{ and } \theta(0) = t_1
\]

and \( \sigma(x) = t_3, \text{ if } x \neq 0 \text{ and } \sigma(0) = t_0 \).

Then it is clear that \( \mu, \theta, \sigma \) are fuzzy subgroups of \( G \) and \( \mu = \theta \cup \sigma \), while \( \theta \preceq \sigma \) and \( \sigma \preceq \theta \).

1.2.17. **Lemma:** Let \( \mu \) be any fuzzy subgroup of \( G \) with \( |\text{Im } \mu| = 2, 0 \in \text{Im } \mu \). Then if \( \mu = \theta \cup \sigma \) for some fuzzy subgroups \( \theta \) and \( \sigma \) of \( G \), then

either \( \theta \subseteq \sigma \) or \( \sigma \subseteq \theta \).

1.2.18. **Lemma:** Let \( \mu \) be a fuzzy subset of a group \( G \). Then the following are equivalent:

(i) \( \mu(x + y) = \mu(y + x) \), \quad \forall x, y \in G

(ii) \( \mu(x + y - x) = \mu(y) \), \quad \forall x, y \in G

(iii) \( \mu(x + y - x) \geq \mu(y) \), \quad \forall x, y \in G

(iv) \( \mu(x + y - x) \leq \mu(y) \), \quad \forall x, y \in G
(v) \( \mu + \theta = \theta + \mu, \forall \text{ fuzzy subset } \theta \text{ of } G. \)

1.2.19. Definition [72]: A fuzzy subgroup \( \mu \) of \( G \) is called \textit{fuzzy normal subgroup} of \( G \) if
\[ \mu(x + y) = \mu(y + x), \text{ for all } x, y \in G. \]

1.2.20. Lemma: A fuzzy subgroup \( \mu \) of \( G \) is fuzzy normal subgroup if and only if \( \mu_t \), for all \( t \in \text{Im} \mu \) is a normal subgroup of \( G \).

1.2.21. Lemma: A non empty subset \( A \) of \( G \) is a normal subgroup of \( G \) if and only if the characteristic function \( \chi_A \) is a fuzzy normal subgroup of \( G \).

1.2.22. Lemma: The intersection of two fuzzy normal subgroups of \( G \) is a fuzzy normal subgroup of \( G \).

1.2.23. Lemma: Let \( f : G \rightarrow G' \) be a group homomorphism and \( \mu \) and \( \theta \) be fuzzy normal subgroups of \( G \) and \( G' \) respectively. Then

\begin{enumerate}
\item \( f(\mu) \) is fuzzy normal subgroup of \( G' \) if \( f \) is onto.
\item \( f^{-1}(\theta) \) is a fuzzy normal subgroup of \( G \).
\end{enumerate}

1.3 Fuzzy subset and fuzzy N-subset of \( N \)

This section deals with the basic definitions and various properties of a near-ring and fuzzy subsets of a near-ring \( N \). Near-rings are generalization of rings. A near-ring is an algebraic system with two binary operations addition and multiplication satisfying all the ring axioms except, possibly, one of the distributive laws and commutativity of addition.

1.3.1. Definitions: A \textit{right near-ring} is a triple \((N, +, \cdot)\) consisting of a set \( N \) with two binary operations \(+\) and \(\cdot\) such that

\begin{enumerate}
\item \((N, +)\) is a group
\item \((N, \cdot)\) is a semi group.
\item \((x + y)z = xz + yz\) for all \(x, y, z \in N\).
\end{enumerate}
If the condition (iii) is replaced by the condition

\[ x(y + z) = xy + xz, \text{for all } x, y, z \in \mathbb{N} \]

then \( \mathbb{N} \) is called a **left near-ring**.

Every ring is a trivial example of a right as well as a left near-ring. We shall restrict ourselves to right near-rings and we shall call a right near-ring a near-ring. The identity of the additive group \((\mathbb{N}, +)\) in the near-ring \((\mathbb{N}, +, \cdot)\) will be denoted by 0. If the semi group \((\mathbb{N}, \cdot)\) has an element 1 such that \(a \cdot 1 = 1 \cdot a = a\), for all \(a \in \mathbb{N}\), then 1 is called identity or unity of \(\mathbb{N}\).

If \(\mathbb{N}\) is any near-ring then \(0 \cdot a = 0\) for every \(a \in \mathbb{N}\). Since the near-ring \(\mathbb{N}\) does not satisfy the left distributive property, \(a \cdot 0 \neq 0\) in general. If \(\mathbb{N}\) is such that \(a \cdot 0 = 0\) for all \(a \in \mathbb{N}\), then \(\mathbb{N}\) is called a **zero symmetric** near-ring.

In our entire work, unless otherwise specified \(\mathbb{N}\) will denote a zero symmetric near-ring with unity and \(ab\) will mean \(a \cdot b\) for \(a, b \in \mathbb{N}\).

1.3.2. **Definition**: An element \(a \in \mathbb{N}\) is called a **distributive element** if \(a(x + y) = ax + by\), for all \(x, y \in \mathbb{N}\).

1.3.3. **Lemma**: If \(a\) is a distributive element of \(\mathbb{N}\) then for all \(x \in \mathbb{N}\)

\[ a0 = 0, a(-x) = -(ax) \text{ and } (-a)(-x) = ax. \]

1.3.4. **Definition**: If \(S\) is a multiplicative subsemigroup of distributive element of a near-ring \(\mathbb{N}\) and \(S\) generates the additive group \(\mathbb{N}\) then \(\mathbb{N}\) is called a **distributively generated near-ring** (dgnr) and \(S\) is called a distributively generated set of it.

1.3.5. **Definition**: A non empty subset \(A\) of \(\mathbb{N}\) is called

(i) a **right \(\mathbb{N}\)-subset** of \(\mathbb{N}\) if \(\mathbb{N}A \subseteq A\)

(ii) a **left \(\mathbb{N}\)-subset** of \(\mathbb{N}\) if \(A\mathbb{N} \subseteq A\)

(iii) an **invariant subset** of \(\mathbb{N}\) if \(\mathbb{N} \subseteq A, NA \subseteq A\)

1.3.6. **Lemma**: Intersection of a family of left(right) \(\mathbb{N}\)-subsets of \(\mathbb{N}\) is again a left(right) \(\mathbb{N}\)-subset of \(\mathbb{N}\).
1.3.7. **Lemma**: Intersection of a family of invariant subsets of \( N \) is again an invariant subset of \( N \).

1.3.8. **Definition**: Let \( N \) and \( K \) be two near-rings. A mapping \( f : N \rightarrow K \) is called a **homomorphism** if for all \( a, b \in N \),

1. (i) \( f(a + b) = f(a) + f(b) \)

and

2. (ii) \( f(ab) = f(a)f(b) \)

**Kernel of homomorphism** \( f \) is the set \( \text{Ker} f = \{ x \in N : f(x) = 0 \} \). A homomorphism \( f \) is called a **monomorphism** if \( f \) is one-one. We note that \( f \) is a **monomorphism** if and only if \( \text{Ker} f = \{ 0 \} \). A homomorphism \( f \) is called an **epimorphism** if \( f \) is surjective and \( f \) is called an **isomorphism** if \( f \) is bijective.

1.3.9. **Definition**: Let \( \mu \) and \( \theta \) be fuzzy subset of a near-ring \( N \). We define fuzzy subsets \( \mu + \theta, -\mu, \mu \circ \theta \) and \( \mu \theta \) as follows:

\[
(\mu + \theta)(x) = \lor_{a+b} \{ \mu(a) \land \theta(b) : a, b \in N \}
\]

\[
(-\mu)(x) = \mu(-x)
\]

\[
(\mu \circ \theta)(x) = \lor_{a \cdot b} \{ \mu(a) \land \theta(b) : a, b \in N \}
\]

and

\[
(\mu \theta)(x) = \lor_{\sum_{a, b}} \{ \land_i \{ \mu(a_i) \land \theta(b_i) \} : a_i, b_i \in N \}
\]

1.3.10. **Lemma**: Let \( \mu \) and \( \theta \) be fuzzy subsets of a near-ring \( N \). Then

\[
\mu_i + \theta_i \subseteq (\mu + \theta)_t, \forall t \in [0,1]
\]

**Proof**: Let \( x \in \mu_i + \theta_i \). Then there exists \( y \in \mu_i \) and \( z \in \theta_i \) such that \( x = y + z \).

Hence \( \mu(y) \geq t \) and \( \theta(z) \geq t \) with \( x = y + z \).

Now \( (\mu + \theta)(x) \geq \mu(y) \land \theta(z) \geq t \) which gives \( x \in (\mu + \theta)_t \).

Thus \( \mu_i + \theta_i \subseteq (\mu + \theta)_t \). \( \blacksquare \)
1.3.11. Theorem: Let \( \mu \) and \( \theta \) be fuzzy subsets of a near-ring \( N \). Then the following hold:

(i) \( \mu \circ \theta \subseteq \mu \theta \)

(ii) \( \mu \subseteq \theta \Rightarrow \sigma \mu \subseteq \sigma \theta \) (also \( \mu \sigma \subseteq \theta \sigma \)) for every fuzzy subset \( \sigma \) of \( N \).

Proof: (ii) Let \( x \in N \). Then we have,

\[
(\sigma \mu)(x) = \vee_{\sum_{i=1}^{n} a_i b_i} \left[ (\sigma(a_i) \land \mu(b_i)) : a_i, b_i \in N \right]
\]

\[
\leq \vee_{\sum_{i=1}^{n} a_i b_i} \left[ (\sigma(a_i) \land \theta(b_i)) : a_i, b_i \in N \right]
\]

\[
= (\sigma \theta)(x)
\]

Thus \( \sigma \mu \subseteq \sigma \theta \). \( \blacksquare \)

1.3.12. Theorem: Let \( \mu \), \( \theta \) and \( \sigma \) be fuzzy subsets of a near-ring \( N \). Then

\( (\mu + \theta) \circ \sigma \subseteq \mu \circ \sigma + \theta \circ \sigma \).

Proof: Let \( x \in N \). Then,

\[
[(\mu + \theta) \circ \sigma](x) = \vee_{z=ab} [(\mu + \theta)(a) \land \sigma(b) : a, b \in N]
\]

\[
= \vee_{z=ab} \left[ [\vee_{a+c+d} (\mu(c) \land \theta(d)) \land \sigma(b) : c, d \in N] \right] \ldots \quad \text{(i)}
\]

Now,

\[
[\vee_{a+c+d} (\mu(c) \land \theta(d)) \land \sigma(b)] = [\vee_{a+c+d} (\mu(c) \land \sigma(b)) \land (\theta(d) \land \sigma(b))]
\]

\[
\leq \vee_{a+b+c+d} (\mu(c) \land \sigma(b)) \land (\theta(d) \land \sigma(b))
\]

\[
\leq [\vee_{a+b+c+d} (\mu \circ \sigma)(cb) \land (\theta \circ \sigma)(db)]
\]

\[
\leq [\mu \circ \sigma + (\theta \circ \sigma)](x)
\]

Hence (i) gives,

\[
((\mu + \theta) \circ \sigma)(x) = \vee_{z=ab} \left[ [\vee_{a+c+d} (\mu(c) \land \theta(d)) \land \sigma(b) : a, b, c, d \in N] \right]
\]
\[ \leq ((\mu \circ \sigma) + (\theta \circ \sigma))(x) \]

Thus \((\mu + \theta) \circ \sigma \subseteq \mu \circ \sigma + \theta \circ \sigma.\]

1.3.13. **Lemma**: If \(x, y\) are two fuzzy points of \(N\) where \(x, y \in N\) and \(t, s \in (0,1]\), then
\[ x + y = (x + y)_{t,s} \quad \text{and} \quad xy = (xy)_{t,s} \]

1.3.14. **Definition**: Let \(\mu\) be a fuzzy subset of \(N\) and \(a \in N\). A fuzzy subset \(a\mu\) of \(N\) is defined as follows:
\[(a\mu)(x) = \vee_{y \in N} \{\mu(y) : y \in N\} \]
\[= 0 \quad \text{if} \quad x \not\in aN. \]

1.3.15. **Lemma**: If \(x_i\) is any fuzzy point of \(N\) where \(x \in N\) and \(t \in (0,1]\), and \(a \in N\). Then

(i) \(a x_i = (ax)_t\)

(ii) \((-x_i) = (-x)_t\)

**Proof**: (i) Let \(y \in N\) and \((ax_i)(y) = u\). Then clearly \(u = t\) or 0.

If \(u = t\) then \(y = ax\) and consequently \((ax)_t(y) = t\). Also if \(u = 0\) then \(y \neq ax\) which implies that \((ax)_t(y) = 0\). Therefore \(a x_i = (ax)_t\).

(ii) the proof is straightforward. \(\Box\)

1.3.16. **Lemma**: Let \(\mu, \theta\) be two fuzzy subsets of \(N\). Then

(i) \((-\mu) = \mu\)

(ii) \(\mu \subseteq -\mu \Leftrightarrow \mu \supseteq -\mu \Leftrightarrow \mu = -\mu\)

(iii) \(\mu \subseteq \theta \Rightarrow -\mu \subseteq -\theta\)

(iv) \((- (\mu + \theta)) = (-\theta) + (-\mu)\)

**Proof**: Let \(x \in N\).

(i) \((-(-\mu))(x) = (-\mu)(-x) = \mu(-(-x)) = \mu(x)\). This proves the result.

(ii) Let \(\mu \subseteq -\mu\). Now,
\((-\mu)(x) = \mu(-x) \leq (-\mu)(-x) = \mu(-(-x)) = \mu(x) .
\)

Similarly, the other parts can be proved..

(iii) \((-\mu)(x) = \mu(-x) \leq \theta(-x) = (\theta)(x) . \) Thus \(-\mu \subseteq -\theta .
\)

(iv) \((- (\mu + \theta)(x) = (\mu + \theta)(-x)
\)

\[ = \lor_{x+y=b} \{\mu(a) \land \theta(b)\} \]

\[ = \lor_{x+y=b} \{\mu(-a) \land \theta(-b)\} \]

\[ = \lor_{x-y=a} \{-\mu(a) \land (\theta)(b)\} \]

\[ = \{(-\theta) + (-\mu)(x)\} . \]

Hence the result follows. \(\blacksquare\)

1.3.17. Definition: A fuzzy subset \(\mu\) of \(N\) is called

(i) a \textbf{fuzzy right N-subset} of \(N\) if \(\mu(xy) \geq \mu(x), \forall x, y \in N .\)

(ii) a \textbf{fuzzy left N-subset} of \(N\) if \(\mu(xy) \geq \mu(y), \forall x, y \in N .\)

(iii) an \textbf{fuzzy invariant subset} of \(N\) if \(\mu\) is both fuzzy right and left \(N\)-subset of \(N .\)

1.3.18. Example: We consider the near-ring \(( S ,+, \cdot) , S = \{0, a, b, c\}\) where + and \(\cdot\) are defined as follows:

\[
\begin{array}{c|ccc}
+ & 0 & a & b & c \\
\hline
0 & 0 & a & b & c \\
a & a & 0 & c & b \\
b & b & c & 0 & a \\
c & c & b & a & 0 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\cdot & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & a & a \\
b & 0 & a & b & b \\
c & 0 & a & c & c \\
\end{array}
\]
Fuzzy subsets $\mu$ and $\theta$ of $S$ are defined such that $\mu(0) > \mu(a) = \mu(b) > \mu(c)$ and $\theta(0) > \theta(a) = \theta(c) > \theta(b)$. Then it is seen that $\mu$ is a fuzzy right $N$-subset of $S$ while $\theta$ is an fuzzy invariant subset of $S$.

1.3.19. Definition: Let $\mu$ be a fuzzy right (left or invariant) $N$-subset of $N$. Then $\mu_0$ is defined as $\mu_0 = \{x \in N : \mu(x) = \mu(0)\}$

1.3.20. Theorem: Let $\mu$ be a fuzzy subset of $N$. Then $\mu$ is fuzzy right (left) $N$-subset of $N$ if and only if $\mu_t$, for all $t \in \text{Im}\mu$ is a right (left) $N$-subset of $N$.

Proof: Let $\mu$ be a fuzzy right $N$-subset of $N$. Let $x \in \mu_tN$ where $t \in \text{Im}\mu$. Then $\exists y \in \mu_t, z \in N$ such that $x = yz$. But $\mu$ being a fuzzy right $N$-subset of $N$, we have, $\mu(x) = \mu(yz) = \mu(y) \geq t$, which gives $x \in \mu_t$.

Hence $\mu_t$ is right $N$-subset of $N$.

Conversely, let $\mu_t$, $t \in \text{Im}\mu$ be right $N$-subset of $N$.

Hence $\mu_tN \subseteq \mu_t$, $t \in \text{Im}\mu$. Now let $x, y \in N$ such that $\mu(x) = s$.

Then $x \in \mu_s$ and hence $\mu_sN \subseteq \mu_s$. This gives $xy \in \mu_sN \subseteq \mu_s$.

Thus $\mu(xy) \geq s = \mu(x)$. Therefore $\mu$ is fuzzy right $N$-subset of $N$. 

The proof of the following theorem is straightforward.

1.3.21. Theorem: Let $A$ be a nonempty subset of $N$. Then $A$ is a right (left) $N$-subset of $N$ if and only if $\chi_A$ is a fuzzy right (left) $N$-subset of $N$.

1.3.22. Theorem: Let $\mu$ be a fuzzy right (left) $N$-subset of $N$. Then

(i) $\mu(0) \geq \mu(x), \forall x \in N$

(ii) $\mu(0) \leq \mu(x), \forall x \in N$

(iii) $\mu^*$ is a right(left) $N$-subset of $N$.

Proof: Let $x \in N$. 
(i) If \( \mu \) is fuzzy right \( N \)-subset of \( N \), then we have, \( \mu(0) = \mu(x0) \geq \mu(x) \) and the result follows. Similarly if \( \mu \) is fuzzy left \( N \)-subset of \( N \), \( \mu(0) = \mu(0x) \geq \mu(x) \).

(ii) If \( \mu \) is fuzzy right \( N \)-subset of \( N \), then we have \( \mu(x) = \mu(1x) \geq \mu(1) \). Also if \( \mu \) is fuzzy left \( N \)-subset of \( N \), \( \mu(x) = \mu(x1) \geq \mu(1) \).

(iii) Let \( x \in \mu^*N \). Then there exists \( a \in \mu^* \) and \( b \in N \) such that \( x = ab \). \( \mu \) being a fuzzy right \( N \)-subset of \( N \) and \( \mu(a) > 0 \), we have \( \mu(ab) \geq \mu(a) > 0 \). Hence \( x = ab \in \mu^* \). Thus \( \mu^* \) is a right \( N \)-subset of \( N \).

1.3.23. Theorem: Let \( \mu \) be a fuzzy left \( N \)-subset of \( N \). Then \( \mu(-x) = \mu(x) \forall x \in N \).

The result also follows for fuzzy right \( N \)-subset if \( N \) is d.g. near-ring.

Proof: Let \( x \in N \). If \( \mu \) is a fuzzy left \( N \)-subset of \( N \) then \( \mu(-x) = \mu((-1)x) \geq \mu(x) \). This gives us \( \mu(x) = \mu((-\overline{x})) \geq \mu(-x) \). Thus the result follows.

Next let \( \mu \) be a fuzzy right \( N \)-subset of \( N \) where \( N \) is d.g. near-ring. Then \( \mu(-x) = \mu(x(-1)) \geq \mu(x) \) and as in the first part we get the desired result.

1.3.24. Lemma: Let \( \mu \) be a fuzzy right (left) \( N \)-subset of \( N \). Then if \( \mu(xy) = \mu(y) \) for all \( y \in N \) then \( \mu(x) = \mu(0) \) where \( x \in N \).

Proof is straightforward.

1.3.25. Theorem: Intersection of a family of fuzzy right (left) \( N \)-subsets of \( N \) is a fuzzy right (left) \( N \)-subset of \( N \).

Proof: Let \( \{\mu_i : i \in \Delta\} \) be any collection of fuzzy right \( N \)-subsets of \( N \). Let \( x, y \in N \). Then,

\[
\left( \bigcap_{i \in \Delta} \mu_i \right)(xy) = \bigwedge_{i \in \Delta} \mu_i(xy).
\]

\[
\geq \bigwedge_{i \in \Delta} \mu_i(x)
\]

\[
= \left( \bigcap_{i \in \Delta} \mu_i \right)(x)
\]
Thus \( \bigcap_{i \in \mathbb{N}} \mu_i \) is a fuzzy right \( \mathbb{N} \)-subset of \( \mathbb{N} \).

## 1.4 Fuzzy subset of an \( \mathbb{N} \)-group \( E \)

In this section we consider several operations of fuzzy subsets of an \( \mathbb{N} \)-group \( E \) that has several applications in our future discussion.

### 1.4.1 Definition

Let \( N \) be a near-ring and \( E \) an additive group. \( E \) is said to be a near-ring group or an \( \mathbb{N} \)-group if there exists a mapping \( \mathbb{N} \times E \rightarrow E, (n, e) \rightarrow ne \) such that

(i) \( (n + m)e = ne + me \)

(ii) \( (nm)e = n(me) \)

(iii) \( 1e = e \)

for all \( n, m \in \mathbb{N}, e \in E \).

Unless otherwise stated we denote the zero element of \( E \) by 0. We note that \( \mathbb{N} \) can be considered as an \( \mathbb{N} \)-group denoted by \( N \).

### 1.4.2 Example

In 1.3.18, \( \mu \) and \( \theta \) are fuzzy subsets of the near-ring group \( S \).

### 1.4.3 Definition

Let \( E \) and \( F \) be two \( \mathbb{N} \)-groups. Then a mapping \( f: E \rightarrow F \) is called an \( \mathbb{N} \)-homomorphism if

(i) \( f(x + y) = f(x) + f(y) \)

(ii) \( f(nx) = nf(x), \) for all \( n \in \mathbb{N}, \) and \( x, y \in E \).

### 1.4.4 Definition

Let \( \mu \) and \( \theta \) be two fuzzy subsets of \( E \) and \( \sigma \) be a fuzzy subset of \( \mathbb{N} \). We define fuzzy subsets \( \mu + \theta, \sigma \circ \mu, \sigma \mu \) of \( E \) as follows:

\[
(\mu + \theta)(x) = \bigvee_{a + b = x} \{\mu(a) \land \theta(b) : a, b \in E\}
\]

\[
(\sigma \circ \mu)(x) = \bigvee_{a = n} \{\sigma(n) \land \mu(a) : n \in \mathbb{N}, a \in E\}
\]

\[
(\sigma \mu)(x) = \bigvee_{\sum_{i=1}^{\infty} n_i = x} \{\lambda : \{\sigma(n_i) \land \mu(a_i) : n_i \in \mathbb{N}, a_i \in E\}
\]
1.4.5. Definition: Let \( n \in \mathbb{N} \), and \( \mu \) be a fuzzy subset of an \( \mathbb{N} \)-group \( E \). A fuzzy subset \( n\mu \) of \( E \) is defined as follows:

\[
(n\mu)(x) = \vee_{y \in n\cdot x} \{\mu(y) : y \in E\}
\]

\[= 0 \quad \text{if} \ x \not\in nE \]

1.4.6. Theorem: Let \( m, n \in \mathbb{N} \) and \( \mu \) and \( \theta \) be fuzzy subsets of an \( \mathbb{N} \)-group \( E \). Then the following hold:

(i) \[1\mu = \mu\]

(ii) \( \mu \subseteq \theta \Rightarrow n\mu \subseteq n\theta \)

(iii) \[m(n\mu) = (mn)\mu\]

(iv) \[(m\mu)(mx) \geq \mu(x), \forall x \in E\]

(v) \[(m+n)\mu \subseteq m\mu + n\mu\]

(vi) \[\mu(nx) \geq \theta(x), \forall x \in E \quad \text{if and only if} \ n\theta \subseteq \mu.\]

(vii) \[(m\mu + n\theta)(mx + ny) \geq \mu(x) \wedge \theta(y), \forall x, y \in E\]

(viii) \[\sigma(mx + ny) \geq \mu(x) \wedge \theta(y), \forall x, y \in E \quad \text{if and only if} \]

\[m\mu + n\theta \subseteq \sigma \] where \( \sigma \) is a fuzzy subset of \( E \).

Proof: (i) The result is trivial.

(ii) Let \( x \in E \). If \( x \not\in nE \) then obviously \( n\mu(x) = n\theta(x) \). So let \( x \in nE \).

Then, \( (n\theta)(x) = \vee_{y \in n\cdot x} \{\theta(y) : y \in E\} \).

\[\geq \vee_{y \in n\cdot x} \{\mu(y) : y \in E\} \]

\[= (n\mu)(x)\]

Hence \( n\mu \subseteq n\theta \).

(iii) Let \( x \in E \). If \( x \not\in mE \) then \( x \not\in m(nE) = mnE \) and so \( [m(n\mu)](x) = [(mn)\mu](x) \). So let \( x \in mE \).
Now \( \{m(n\mu)\}(x) = \vee_{x=my} \{(n\mu)(y): y \in E\} \)

\[= \vee_{x=my} \{\vee_{y=ny} \mu(z): y, z \in E\}, \]

0, if \( y \in nE \) where \( x = my \).

\[= \vee_{x=m(n\mu)} \{\mu(z): z \in E\}, \]

0, if \( x \notin m(nE) \)

\[= \vee_{x=m(n\mu)x} \{\mu(z): z \in E\}, \]

0, if \( x \notin (mn)E \)

\[= \{(mn)\mu\}(x) \]

Thus the result follows.

(iv) Let \( x \in E \) Then \( (m\mu)(mx) = \vee_{x=my} \{\mu(y): y \in E\} \geq \mu(x) \).

Hence \( (m\mu)(mx) \geq \mu(x), \forall x \in E \).

(v) Let \( x \in E \) s.t. \( x \in (m+n)E \). Then \( x = my + ny \) for some \( y \in E \). Then we have,

\[ [(m+n)\mu](x) = \vee_{x=(m+n)y} \{\mu(y): y \in E\} \]

\[= \vee_{x=my+ny} \{\mu(y): y \in E\} \]

\[\leq \vee_{x=my+ny} \{(m\mu)(my) \land (n\mu)(ny): y \in E\} \quad \text{[by (iv)]} \]

\[\leq (m\mu + n\mu)(x) \]

For the other choice of \( x \) it is obvious that \( [(m+n)\mu](x) \leq (m\mu + n\mu)(x) \).

So \( (m+n)\mu \leq m\mu + n\mu \).

(vi) Let \( \mu(nx) \geq \theta(x), \forall x \in E \). If \( x \notin nE \) then obviously \( (n\theta)(x) \leq \mu(x) \).

Hence choose \( x \in E \) s.t. \( x \in nE \). Then we have,

\[ (n\theta)(x) = \vee_{x=ny} \{\theta(y): y \in E\} . \]

\[\leq \vee_{x=ny} \{\mu(ny): y \in E\} \]

\[= \mu(x) \]
Thus \( n\theta \subseteq \mu \).

Conversely, let \( n\theta \subseteq \mu \). Then,

\[ \mu(nx) \geq (n\theta)(nx) \geq \theta(x). \]

Hence the required result follows.

(vii) Let \( x, y \in E \) Then,

\[ (m\mu + n\theta)(mx + ny) \geq (m\mu)(mx) \wedge (n\theta)(ny) \]

\[ \geq \mu(x) \wedge \theta(y) \quad [\text{by (iv)}] \]

(viii) Let \( \sigma(mx + ny) \geq \mu(x) \wedge \theta(y), \forall x, y \in E \). Let \( z \in E \). Then we have,

\[ (m\mu + n\theta)(z) = \bigvee_{a+b} \{ (m\mu)(a) \wedge (n\theta)(b) : a, b \in E \} \]

\[ = \bigvee_{a+b} \left( (m\mu)(a) \wedge (n\theta)(b) \right), \]

\[ 0, \quad \text{if } a \not\in mE \text{ or } b \not\in nE \]

\[ = \bigvee_{z=mx+ny} (\mu(x) \wedge \theta(y)), \]

\[ 0, \quad \text{if } z \not\in mE + nE \]

\[ \leq \bigvee_{z=mx+ny} \sigma(mx + ny) \]

\[ = \sigma(z) \]

Conversely \( \sigma(mx + ny) \geq (m\mu + n\theta)(mx + ny) \geq \mu(x) \wedge \theta(y). \)

1.4.7. Theorem: For any fuzzy subset \( \mu \) of \( E \)

(i) \( m\mu \subseteq \mu \Leftrightarrow \mu(mx) \geq \mu(x), \forall x \in E, m \in N. \)

(ii) \( m\mu + n\mu \subseteq \mu \Leftrightarrow \mu(mx + ny) \geq \mu(x) \wedge \mu(y), \forall x, y \in E, m, n \in N \)

Proof: (i) Let \( m\mu \subseteq \mu, \forall m \in N. \) Now \( \mu(mx) \geq (m\mu)(mx) \geq \mu(x). \)

Conversely let \( \mu(mx) \geq \mu(x), \forall x \in E. \) Then for any \( z \in E \) we have,
Thus the result follows.

(ii) By (viii) of 1.4.5., we get the required result.

1.4.8. Theorem: Let $m, n \in \mathbb{N}$ and $\mu, \theta$ be fuzzy subsets of $E$. Let $f: E \to F$ be an $N$-epimorphism where $F$ is an $N$-group. Then,

(i) $f(\mu + \theta) = f(\mu) + f(\theta)$

(ii) $f(n\mu) = nf(\mu)$

(iii) $f(m\mu + n\theta) = mf(\mu) + nf(\theta)$

Proof: (i) Let $y \in F$. Let $\frac{f(\mu + \theta)}{y} = t$. Then for $\varepsilon > 0$ there exists $x_i \in E$ such that $f(x_i) = y$ and

$t - \varepsilon < f(a_i) \land \theta(b_i)$ for some $a_i, b_i \in E$ such that $x_i = a_i + b_i$

Then $\left[f(\mu) + f(\theta)\right](y) = \bigvee_{y_1, y_2} \left\{f(\mu)(y_1) \land f(\theta)(y_2)\right\}$

$\geq f(\mu)(f(a_i)) \land f(\theta)(f(b_i))$, as $y = f(x_i) = f(a_i) + f(b_i)$.

$= f^{-1}(f(\mu))(a_i) \land f^{-1}(f(\theta))(b_i)$

$> t - \varepsilon$

Thus, $f(\mu + \theta)(y) \leq \left[f(\mu) + f(\theta)\right](y)$

Similarly it can be shown that $f(\mu + \theta)(y) \geq \left[f(\mu) + f(\theta)\right](y)$, for all $y \in F$.

Thus $f(\mu + \theta) = f(\mu) + f(\theta)$.

Proofs of (ii) and (iii) are routine matter of checking and so we omit it.