Introduction

A fuzzy set is characterized by a *membership-degree function*, which maps the members of a non empty set into the unit interval $[0,1]$. The value 0 means that the member is not included in the given set, 1 describes a fully included member. The values between 0 and 1 characterize fuzzy members. For a non empty set $X$ and the membership-degree function $f : X \rightarrow [0,1]$, the fuzzy set $A$ is defined as $A = \{(x, f(x)) \mid x \in X\}$. If the membership-degree function takes only two values 0 and 1, then it is the characteristic function on $X$. Thus a fuzzy set generalizes an ordinary set. In 1965, Lotfi Zadeh introduced the above concept of fuzzy set in his groundbreaking paper [75], and it was a new episode towards the development of science and engineering. Lotfi’s concept of fuzzy set theory especially deals with inaccurate and indistinct situation.

In 1967, Goguen [19] generalized the notion of fuzzy set due to Zadeh by taking the membership value in an arbitrary but fixed complete lattice. In the last three decades many researchers are engaged in developing fuzzy situation in various aspects of Mathematics using the concepts due to Zadeh and Goguen. In the trajectory of stupendous growth of fuzzy set theory, fuzzy algebra has become an important area of research. In 1971, A. Rosenfeld used the concept of Zadeh in abstract algebra [61] and opened up a new insight in the field of Mathematical science. Since than many researchers are working on the concepts like fuzzy semigroups, fuzzy groups, fuzzy rings, fuzzy semirings, fuzzy near-rings and so on.

W. Liu [37] initiated the study of fuzzy sub rings and fuzzy ideals of a ring around 1982. Mukherjee and Sen [51] characterized regular rings in terms of fuzzy ideals of rings. Abou-Zaid [2], Dixit [13,14], Kumar [32] and others [36, 38, 50, 74, 76] have carried out extensive
work on fuzzy ideals of rings. Recently, Liu et al. [39] introduced the notion of fuzzy BCI-positive implication ideals and fuzzy BCI-implication ideals of BCI-algebras and investigated different characteristics of these concepts. Meng and Guo [49] elegantly showed the way of generating a fuzzy ideal by a fuzzy set in a BCI-algebra. In [62] Saidi and Jaballah established the method of identifying a fuzzy ideal of a commutative ring with its corresponding chain of level ideals and its range of values.

Mukherjee and Sen [51] studied various aspects of fuzzy prime ideals and determined all fuzzy prime ideals of the ring of integers. Jun [28] obtained the relation between fuzzy prime ideals and invertible fuzzy ideals in BCK-algebras. In [49], Meng and Guo established the prime fuzzy ideal theorem for commutative BCK-algebra. Considerable works on these aspects are being carried out by Abadi, Dixit, Kumar, Kumbhojkar, Liu, Malik, Yue [1, 13, 32, 33, 36, 38, 43, 74]. Dutta et. al. [16, 17] generalized the notion of fuzzy K-ideals and fuzzy prime ideals to semirings.

Near-rings are generalizations of rings. A near-ring is an algebraic system with binary operation of addition and multiplication satisfying all the ring axioms except, possibly, one of the distributive laws and commutativity of addition. A near-ring group (or N-group) over a near-ring N is an algebraic structure with an internal composition called addition and an external composition called scalar multiplication satisfying all the axioms of modules in ring theory except one of the distributive laws. Due to non-ring character of a near-ring and non-module character of a near-ring group the results have their own beauty. Betsh, Beidleman, Ramakotaiah, Ligh, Pilz, Meldrum, Clay, Chaudhury, Satyanarayana and others [60, 47,11, 68, 9, 10, 12, 21, 26, 55, 56] had generalized various concepts of rings and modules to near-rings and near-ring groups. Chowdhury, Saikia [9, 10, 67] carried out extensive work on near-rings and near-ring groups with ascending chain conditions on annihilators.

Abou-Zaid [3] in 1991, introduced the notions of fuzzy ideals (subgroup, subrings) and fuzzy prime ideals in near-rings. Subsequently Dutta et. al. [15], Hong et. al. [25], Kim at. al.
and Kim et al. [30] carried out extensive work on fuzzy ideals and prime ideals in near-rings. Our attempt is to investigate different characteristics of fuzzy N-subgroups and fuzzy ideals of near-rings and near-ring groups.

Malik and Mordeson [42, 44, 45], Mukherjee and Sen [53], Kumbhojkar [36], Liu [38] have studied fuzzy primary ideals, fuzzy maximal ideals and fuzzy radical of an ideal of a ring. In [33, 76], Kumar and Zahedi fuzzified the notion of semiprime ideals of rings. Xie [73] introduced the concepts of prime, quasi-prime and weakly quasi-prime fuzzy left ideals of a semi group and proved related properties. We attempt to study the fuzzy substructures like fuzzy semiprime, fuzzy maximal and fuzzy primary N-subgroups of near-rings and to obtain analogous results in the fuzzy aspects of near-rings [64, 65]

In [34, 71], Kumar and Swami studied fuzzy irreducible ideals of a ring and irreducibility in algebraic fuzzy systems. We deal with the concept of fuzzy irreducible N-subgroups (ideal) of near-rings and fuzzify the results of irreducible ideals of duo near-rings.

Mashour et al. [46] introduced the notion of fuzzy submodules of a module around 1986. Following their initiation Kumar [31], Pan [58, 59], Zahedi [77, 78], Mukherjee and Sen [52], Sidky [70] made a lot of contribution in developing such fuzzy structures. Extending this concept we establish various properties of fuzzy N-subgroups, fuzzy normal N-subgroups, fuzzy prime N-subgroups of an N-group and attempt to study fuzzy factor N-group

In the theory of near-rings the N-subgroups obtained using annihilation, which correspond to prime ideals (or prime N-subgroups), play an important role in the study of decomposition theory, quotient near-ring and Goldie like structures. On the other hand, a specific type of "largeness" is embodied in the notion of essential ideals (N-subgroups) of a near-ring (near-ring group). This notion is helpful in the study of injective N-group, singular N-group and near-ring (near-ring group) of quotients. Our attempt is on to fuzzify the concepts of annihilators and essentiality of near-rings and near-ring groups.
Objective of the work: The objective of our work is to investigate various characteristics of fuzzy $N$-subgroups, fuzzy ideals of a near-ring $N$ and a near-ring group $E$. These fuzzy substructures motivate one to study fuzzy factor $N$-group induced by a fuzzy ideal. Characterization of fuzzy subgroups and fuzzy ideals of a near-ring (or near-ring group) with chain condition plays a significant role in developing different aspects of fuzzy near-ring theory.

Study of fuzzy homomorphism and fuzzy isomorphism between two fuzzy invariant sub near-rings provides essential paths to examine the algebraic nature of fuzzy substructures.

Our aim to study fuzzy substructures like fuzzy prime, fuzzy semiprime, fuzzy primary and fuzzy irreducible $N$-subgroups is to characterize strongly regular near-rings, to obtain some elegant structure theorems and to study near-rings with chain conditions in fuzzy situation.

Annihilator of a subset of $N$ (or $E$) and essential ideals ($n$-subgroups) play a vital role in various structures of near-ring theory. Fuzzification of such crisp sets leads to structures that can be termed as fuzzy prime $N$-subgroup, fuzzy singular $N$-subgroup. The relation between fuzzy prime $N$-groups and fuzzy annihilators helps to study fuzzy associated primes. Fuzzifying the notions of annihilators and essentiality one may expect to study Goldie like structure, near-ring of quotients, singular $N$-subgroups, injective $N$-subgroups in fuzzy situation.

The outcome of our work along the line mentioned above has been described in five chapters.

Chapter I highlights the basic materials, which are available in the standard literature. The concept of fuzzy left (right, invariant) $N$-subset of $N$ is discussed in this chapter.

The main results of chapter II is the outcome of our paper [[63]] published in ‘The Journal of Fuzzy Mathematics’. This chapter deals with the results on fuzzy $N$-subgroups and fuzzy ideals of near-ring $N$ as well as $N$-group $E$. Considering several operations of fuzzy $N$-
subgroups and fuzzy ideals we prove some interesting results. It is shown that the sum of a fuzzy left ideal and a fuzzy left N-subgroup of N is a left N-subgroup of N while sum of two left fuzzy ideals is a left fuzzy ideal of N. The concepts of fuzzy N-subgroups and fuzzy ideals of an N-group E are introduced in this chapter and various results on fuzzy N-subgroups and fuzzy ideals of E are established. We also investigate the algebraic nature of such type of fuzzy substructures under homomorphism. The notion of fuzzy coset of a fuzzy ideal of E plays an important role in various aspects of fuzzy near-ring theory. If μ is a fuzzy ideal of E then the set \( E/\mu \) of all fuzzy cosets of μ is an N-group under certain binary compositions and this motivates to define fuzzy factor N-group of a fuzzy N-subgroup \( \sigma \) of E modulo \( \mu \).

**Chapter III** is devoted to the study of fuzzy homomorphism and fuzzy isomorphism between two fuzzy invariant sub near-rings. If \( \mu \) and \( \theta \) are fuzzy invariant sub near-rings of N, \( \rho(\mu) = \{ x_i : \mu(x) \geq t \in (0, 1) \} \), \( \rho(\theta) = \{ x_i : \theta(x) \geq t \in (0, 1) \} \) and \( \mu^*, \theta^* \) be the support of \( \mu \) and \( \theta \) respectively, then a fuzzy homomorphism(isomorphism) \( F: \rho(\mu) \rightarrow \rho(\theta) \) can be defined by a corresponding ordinary homomorphism(isomorphism) \( f: \mu^* \rightarrow \theta^* \) and an increasing(strictly increasing onto function) \( \pi: (0, \mu(0)] \rightarrow (0, \theta (0)) \). Using the above concept we define \( \pi \)-fuzzy quotient near-ring of a fuzzy invariant sub near-ring with respect to an ideal of N. We establish the fundamental theorem of fuzzy homomorphism and order preserving correspondence theorem between fuzzy ideals of two near-rings. Generalizing these notions of fuzzy homomorphism and fuzzy isomorphism to an N-group, what may be termed as fuzzy N-homomorphism and fuzzy N-isomorphism, analogous results are obtained.

**Chapter IV** is the outcome of our papers [[63, 64, 65]]. This chapter deals with the study of fuzzy substructures like fuzzy prime, fuzzy semiprime, fuzzy maximal, fuzzy primary and fuzzy irreducible N-subgroups of an abelian duo near-ring with unity satisfying the condition that for all \( a, b \in N \), \( abN = baN \) and various results on these substructures are established. It is proved that the intersection of all fuzzy prime N-subgroups of N containing a fuzzy prime N-subgroup is a fuzzy N-subgroup, which is analogous to the fuzzy radical of a fuzzy ideal of a ring, and it is shown that such type of fuzzy N-subgroup is fuzzy semiprime. In some cases, a fuzzy N-subgroup can be expressed as a finite intersection of fuzzy primary N-subgroups. If every prime N-subgroup is a minimal prime then (i) every irreducible N-subgroup is fuzzy primary N-subgroup and (ii) every fuzzy N-subgroup is the intersection of all fuzzy primary N-
subgroups containing it. A primary near-ring is characterized by means of fuzzy nil points of a zero divisor. A necessary and sufficient condition for a fuzzy N-subgroup to be a fuzzy irreducible is obtained. If \( f: N \to K \) is a near-ring homomorphism and \( \mu \) is a fuzzy irreducible N-subgroup of \( N \) then \( f(\mu) \) is a fuzzy irreducible K-subgroup of \( K \) only if \( \mu \) is invariant and \( f^{-1}(\theta) \) may not be a fuzzy irreducible N-subgroup of \( N \) if \( \theta \) is a fuzzy irreducible K-subgroup of \( K \) unless every element of \( K \) is an idempotent. It is also proved that in some special cases a fuzzy N-subgroup \( \mu \) can be expressed as the intersection of all fuzzy irreducible N-subgroups of \( N \) containing \( \mu \).

Most of the results of Chapter V is the outcome of our paper [[66]] published in ‘The Journal of Fuzzy Mathematics’ and some parts form the paper[[4]] which is communicated for publication in IJPAM. In this chapter the notion of fuzzy annihilator of a fuzzy subset and essential N-subgroup of an N-group \( E \) are introduced and various related results are established. It is proved that the fuzzy annihilator of a fuzzy subset is a fuzzy left ideal of \( N \). If the fuzzy subset is a fuzzy N-subgroup then its annihilator is a fuzzy ideal of \( N \). Moreover if a fuzzy N-subgroup \( \mu \) satisfies the supremum condition then the product \( \text{ann}(\mu) \circ \mu \) is the characteristic function of \( (0) \). The concept of fuzzy annihilator motivates us to define fuzzy prime N-subgroup of an N-group \( E \). If \( \mu \) is a fuzzy prime N-subgroup of \( E \) then \( \text{ann}(\mu) \) is a fuzzy completely prime ideal of \( N \). Using the notion of fuzzy completely prime ideal of \( N \) one can characterize the prime N-group \( N/\mu \) induced by fuzzy ideal \( \mu \). If \( \mu \) is an essential N-subgroup of \( _NN(N \text{ as a left N-group}) \) then for any \( a \) in \( N \) there exists a fuzzy essential N-subgroup \( v \) of \( _NN \) such that \( va \) is non zero and \( \mu \) contains \( va \). The concept of essential N-subgroups leads to the notion \( Z_{\alpha}(N) \) which is analogous to the singular ideal of a ring.