Chapter II

Fuzzy N-subgroups and fuzzy ideals

2.1. Prerequisites.

2.2. Fuzzy N-subgroup and fuzzy ideal of N.

2.3. Fuzzy N-subgroup and fuzzy ideal of an N-group E.

2.4. Fuzzy factor N-group.
Chapter II

Fuzzy N-subgroups and fuzzy Ideals

The concept of fuzzy N-subgroups and fuzzy ideals of a Near-ring N plays an important role in the area of abstract algebraic geometry. S. Abou Zaid [3] in 1991 introduced the notion of fuzzy sub-near-ring, fuzzy ideal and fuzzy prime ideal of a near-ring N. Kim and Jun considered the fuzzification of N-subgroups in a near-ring N. The concept of normalization of fuzzy ideals are applied to BCK-algebra [24] and Gamma ring [23]. In [29], this concept is generalized to N-subgroups of near-rings. In this chapter we study and investigate various characteristics of fuzzy ideals and fuzzy N-subgroups of a near-ring N and a near-ring group E. Using these concepts we finally define fuzzy factor N-group. The main results of this chapter have appeared in our paper [[63]].

This chapter is divided into four sections. The preliminary definitions and results of near-ring theory are presented in the first section. The second section contains some important properties of fuzzy N-subgroups and fuzzy ideals of N. It is shown that sum of a fuzzy left ideal and a fuzzy left N-subgroup of N is a fuzzy left N-subgroup of N while the sum of two fuzzy left ideals of N is a fuzzy left ideal of N. In section 2.3, the notions of fuzzy N-subgroups and fuzzy ideals of a near-ring group E are introduced, and various properties of fuzzy N-subgroups and fuzzy ideals of E are established. The last section contains the notion of fuzzy factor N-group. If μ is a fuzzy ideal of E then the set $E/\mu$ of all fuzzy cosets of μ is an N-group under certain binary compositions and this leads to the notion of fuzzy factor N-group of a fuzzy N-subgroup σ of E modulo μ.
Throughout our discussion, unless otherwise specified, N denotes a zero symmetric near-ring with unity and E a near-ring group.

2.1. Prerequisites

2.1.1. Definition: Let H be a non empty subset of a near-ring (N, +, .). If H is subgroup of (N, +), then H is called

(i) a right N-subgroup of N if HN ≤ H

(ii) a left N-subgroup of N if NH ≤ H

(iii) a sub near-ring of N if HH ≤ H

and (iv) an invariant sub near-ring of N if HN ≤ H and NH ≤ H.

2.1.2. Definition: A subset A of a near-ring group E is called an N-subgroup of E if A is a subgroup of (E, +) and NA ⊆ A.

2.1.3. Remarks: A left N-subgroup of N is an N-subgroup of N and vice-versa.

2.1.4. Definition: A sub near-ring A of N is called distributively generated if there is a multiplicatively sub semi group S of distributive elements of N and if the additive group (A, +) is generated by S.

2.1.5. Lemma: Intersection of two N-subgroups of E is an N-subgroup of E.

2.1.6. Lemma: Intersection of two left (right) N-subgroups of N is a left (right) N-subgroup of N.

Hence, intersection of two invariant sub near-rings of N is an invariant sub near-ring of N.

2.1.7. Lemma: If A is an N-subgroup of E and I is an left N-subgroup of N then for any x ∈ A, lx is an N-subgroup of A.
2.1.8. Definition: A normal subgroup $A$ of $E$ is called an **ideal** of $E$ if $n(a + e) - ne \in A$ for all $n \in \mathbb{N}$, $a \in A$, $e \in E$.

2.1.9. Definition: Let $I$ be an additive normal subgroup of $N$. Then $I$ is called,

(i) a **right ideal** of $N$ if $xn \in I$, for all $x \in I$, $n \in \mathbb{N}$.

(ii) a **left ideal** of $N$ if $n(m + a) - nm \in A$, for all $n, m \in \mathbb{N}$, $a \in I$.

( equivalently if $n(a + m) - nm \in A$, for all $n, m \in \mathbb{N}$, $a \in I$)

(iii) an **ideal** of $N$ if $I$ is both right as well as left ideal of $N$.

2.1.10. Lemma: If $A$ is a right $N$-subgroup of $N$ and $x \in N$, then $xA$ is a right $N$-subset of $N$. Thus $xN$ is a right $N$-subset of $N$.

2.1.11. Lemma: If $A$ is a left $N$-subgroup of $N$ and $B$ is a left ideal of $N$ then $A + B$ is a left $N$-subgroup of $N$.

2.1.12. Lemma: If $A$ and $B$ are left ideals of $N$ then $A + B$ is also a left ideal of $N$.

2.1.13. Lemma: If $A$ and $B$ are ideals of $E$ then $A + B$ is also an ideal of $E$.


2.1.15. Lemma: If $A$ is an ideal of $E$ and $B$ is an $N$-subgroup of $E$ such that $A \subseteq B$, then $A$ is an ideal of $B$.

2.1.16. Lemma: If $A$ is an $N$-subgroup of $E$ and $B$ is an ideal of $E$ then $A \cap B$ is an ideal of $A$.

2.1.17. Definition: If $A$ is an ideal of $E$ then the set $E/A = \{ e + A : e \in E \}$ forms an $N$-group under the followings operations:

(i) $(a + A) + (b + A) = (a + b) + A$, where $a, b \in E$.

(ii) $n(a + A) = na + A$, where $n \in \mathbb{N}$, $a \in E$. 
This N-group \( E/A \) is called a **factor N-group**. If \( E = N \), \( A \) is an ideal of \( N \) and the condition (ii) is replaced by the condition \((a + A)(b + A) = (ab) + A\), where \( a, b \in N \), then \( N/A \) is called a **factor near-ring**. If \( A \) is a left ideal of \( N \) then \( N/A \) is an N-group denoted by \( _N(N/A) \).

2.1.18. **Lemma**: If \( A \) and \( B \) are ideals of \( E \) such that \( A \subseteq B \) then \( B/A \) is an ideal of \( E/A \).

2.1.19. **Lemma**: Let \( A \) be an ideal of \( E \) such that \( A \subseteq B \). Then \( B/A \) is an N-subgroup of \( E/A \) if and only if \( B \) is an N-subgroup of \( E \).

### 2.2 Fuzzy N-subgroup and fuzzy ideal of \( N \).

2.2.1. **Definition** [3]: Let \( \mu \) be a fuzzy subset of a near-ring \( N \). Then \( \mu \) is called

(i) a **fuzzy left N-subgroup** of \( N \) if \( \mu \) is a fuzzy subgroup of \((N, +)\) and \( \mu \) is a fuzzy left N-subset of \( N \) i.e. for \( x, y \in N \),

\[
\mu(x - y) \geq \wedge \{ \mu(x), \mu(y) \}
\]

and

\[
\mu(xy) \geq \mu(y)
\]

(ii) a **fuzzy right N-subgroup** of \( N \) if \( \mu \) is a fuzzy subgroup of \((N, +)\) and \( \mu \) is a fuzzy right N-subset of \( N \) i.e. for \( x, y \in N \),

(i) \( \mu(x - y) \geq \wedge \{ \mu(x), \mu(y) \} \)

(ii) \( \mu(xy) \geq \mu(x) \)

(iii) an **invariant sub-near-ring** if \( \mu \) is both left and fuzzy right N-subgroup of \( N \). A fuzzy invariant sub near-ring is also called a **fuzzy N-subgroup** of \( N \).

2.2.2. **Definition**: Let \( \mu \) be a fuzzy left (right) N-subgroup of \( N \). Then we write,
2.2.3. Definition: A fuzzy left(right or invariant) $N$-subgroup $\mu$ of $N$ is called proper if $\mu_0 \neq N$.

2.2.4. Definition: A fuzzy left(right or invariant) $N$-subgroup $\mu$ of $N$ is called non zero if $\mu_t$, for all $t \in \text{Im} \mu$ is non zero.

2.2.5. Example: We consider the set $S = \{0, a, b, c\}$. Let $\cdot$ and $\cdot$ on $S$ be defined as follows:

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Then $(S, +, \cdot)$ is a near-ring. We define $\mu : S \rightarrow [0,1]$ such that $\mu_\cdot(o) > \mu_\cdot(a) > \mu_\cdot(b) = \mu_\cdot(c)$. Then $\mu(x - y) \geq \wedge\{\mu_\cdot(x), \mu_\cdot(y)\}$ and $\mu_\cdot(xy) \geq \mu_\cdot(y)$ for all $x, y \in S$. Hence $\mu$ is a fuzzy left $S$-subgroup of $S$.

But $\mu_\cdot(ab) \npreceq \mu_\cdot(a)$ and hence $\mu$ is not a fuzzy right $S$-subgroup of $S$. Again considering the addition as above and $\cdot$ as follows:

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we get $(S, +, \cdot)$ is a near-ring. If $\theta : S \rightarrow [0,1]$ is defined as $\theta(0) > \theta(a) > \theta(b) = \theta(c)$ then $\theta$ is an invariant fuzzy sub-near-ring of $S$.

2.2.6. Lemma: Let $\mu$ be a fuzzy left (right) $N$-subgroup of $N$. Then the following assertions hold:
\(\begin{align*}
(i) & \quad \mu(0) \geq \mu(x), \forall x \in N. \\
(ii) & \quad \mu(x) \geq \mu(1), \forall x \in N \\
(iii) & \quad \mu(x) = \mu(-x), \forall x \in N \\
(iv) & \quad \mu(x - y) = \mu(0) \implies \mu(x) = \mu(y), \forall x, y \in N. \\
(v) & \quad \mu^* \text{ is a left(right) N-subgroup of } N.
\end{align*}\)

**Proof:** (i) and (ii) follow from theorem 1.3.22.

(iii) Let \(\mu\) be a fuzzy left N-subgroup of \(N\). Then we have,

\[
\mu(-x) = \mu(0 - x) \\
\geq \mu(0) \land \mu(x) \\
= \mu(x), \forall x \in N \\
\implies \mu(x) \geq \mu(-x).
\]

Thus \(\mu(-x) = \mu(x), \forall x \in N\)

(iv) Let \(\mu(x - y) = \mu(0)\), where \(x, y \in N\).

Then \(\mu(x) = \mu(x - y + y)\)

\[
\geq \mu(x - y) \land \mu(y) \\
= \mu(y) \quad \text{[by (i)]}
\]

and \(\mu(y) = \mu(y - x + x).\)

\[
\geq \mu(y - x) \land \mu(x) \\
= \mu(x - y) \land \mu(x) \\
= \mu(x)
\]

Hence \(\mu(x) = \mu(y)\).

The proof of (v) is a routine matter of checking. \(\blacksquare\)
2.2.7. Remark: The converse of the result 2.2.6(iv) is not true. Considering the example (i) in 2.2.5., we get µ is a fuzzy left S-subgroup of S and µ(b) = µ(c) but µ(b - c) = µ(b + c) = µ(a) < µ(0).

2.2.8. Lemma: Let µ and θ be fuzzy left(right) N-subgroups of N. Then the following are true:

(i) µ ♯ θ ≤ (µ ♯ θ)₀

(ii) µ ♯ θ = (µ ♯ θ)₀ if µ(0) = θ(0)

Proof: (i) Let x ∈ µ ♯ θ. Then µ(x) = µ(0) and θ(x) = θ(0). Hence (µ ♯ θ)(x) = (µ ♯ θ)(0) and this gives x ∈ (µ ♯ θ)₀ and thus µ ♯ θ ≤ (µ ♯ θ)₀.

(ii) Let x ∈ (µ ♯ θ)₀.

Then (µ ♯ θ)(x) = (µ ♯ θ)(0)

⇒ µ(x) ♯ θ(x) = µ(0) ♯ θ(0)

⇒ µ(x) = µ(0) and θ(x) = θ(0).

⇒ x ∈ µ ♯ θ and consequently (µ ♯ θ)₀ ⊆ µ ♯ θ. Thus the result follows.

2.2.9. Lemma: Let µ, θ and σ be fuzzy left(right) N-subgroups of N. Then µ ♯ σ ⊆ θ ⇔ µσ ⊆ θ.

Proof: Let µ ♯ σ ⊆ θ. Then for any x ∈ N, we have,

θ(x) = θ \left( \sum_{i=1}^{n} y_i z_i \right), \text{ where } x = \sum_{i=1}^{n} y_i z_i, y_i, z_i ∈ N, n ∈ \mathbb{Z}_+.

≥ \land_i (θ(y_i z_i) : y_i, z_i ∈ N)

≥ \land_i (µ(y_i) \land σ(z_i) : y_i, z_i ∈ N) \text{ for } µ ♯ σ ⊆ θ.

Hence θ(x) ≥ \land_i (µ(y_i) \land σ(z_i) : y_i, z_i ∈ N) \text{ for any } x = \sum_{i=1}^{n} y_i z_i : y_i, z_i ∈ N.
Therefore

\[ \theta(x) \geq \bigvee_{x \in \bigoplus_{i=1}^{n} y_i z_i} \{ \bigwedge_{i=1}^{n} (\mu(y_i) \land \sigma(z_i)) \} : y_i, z_i \in N \}, i = 1, 2, \ldots, n, n \in \mathbb{Z}_+.\]

\[ = \mu \sigma(x).\]

Thus \( \mu \sigma \subseteq \theta \). The converse part follows from 1.3.11(i).

**2.2.10. Theorem:** Let \( \mu \) be a fuzzy \( N \)-subgroup of \( N \). Then for any fuzzy subset \( \theta \) of \( N \), \( \mu \circ \theta \subseteq \mu \).

**Proof:** Let \( x \in N \) be such that \( x = ab \), where \( a, b \in N \). Then \( \mu \) being a fuzzy \( N \)-subgroup of \( N \), we have,

\[ \mu(x) = \mu(ab) \geq \mu(a) \lor \mu(b).\]

Now if \( \mu(a) \lor \mu(b) = \mu(a) \) then \( \mu(a) \geq \mu(a) \land \theta(b) \)

and if \( \mu(a) \lor \mu(b) = \mu(b) \) then we have \( \mu(b) \geq \mu(a) \land \theta(b) \).

Therefore for any \( x = ab \) with \( a, b \in N \),

\[ \mu(x) = \mu(ab) \geq \mu(a) \lor \mu(b) \geq \mu(a) \land \theta(b).\]

Thus,

\[ \mu(x) \geq \bigvee_{x=ab} \{ \mu(a) \land \theta(b) : a, b \in N \} = (\mu \circ \theta)(x).\]

Hence \( \mu \circ \theta \subseteq \mu \).

**2.2.11. Theorem:** Let \( \mu \) and \( \theta \) be two fuzzy \( N \)-subgroups of \( N \). Then \( \mu \circ \theta \subseteq \mu \cap \theta \).

**Proof:** Let \( x \in N \) be such that \( x = ab \), where \( a, b \in N \). Then,

\[ (\mu \cap \theta)(x) = \mu(x) \land \theta(x) \geq \mu(a) \land \theta(b).\]

Hence \( (\mu \cap \theta)(x) \geq \bigvee_{x=ab} \{ \mu(a) \land \theta(b) : a, b \in N \} = (\mu \circ \theta)(x).\)

Thus \( \mu \circ \theta \subseteq \mu \cap \theta \).
We note that the above theorem also holds when $\mu$ is a fuzzy right $N$-subgroup and $\theta$ is a fuzzy left $N$-subgroup of $N$.

The proof of the following lemma is a routine matter of verification and so we omit it.

2.2.12. Lemma: Let $\mu$ and $\theta$ be two left (right) fuzzy $N$-subgroups of $N$. Then $\mu \cap \theta$ is also fuzzy left (right) $N$-subgroup of $N$.

2.2.13. Theorem: Let $\mu$ and $\theta$ be two fuzzy $N$-subgroups of $N$. Then $\mu \theta \subseteq \mu \cap \theta$.

Proof: From theorem 2.2.11. we have, $\mu \circ \theta \subseteq \mu \cap \theta$. Hence by theorem 2.2.9. $\mu \theta \subseteq \mu \cap \theta$.

2.2.14. Lemma: Let $\mu$ and $\theta$ be two fuzzy left (right) $N$-subgroups of $N$ such that $\mu(0) = \theta(0)$. Then $\mu \subseteq \mu + \theta$ and $\theta \subseteq \mu + \theta$.

Proof: Let $x \in N$. Then $(\mu + \theta)(x) \geq \mu(x) \land \theta(0) = \mu(x)$ and consequently $\mu \subseteq \mu + \theta$.

Also $(\mu + \theta)(x) \geq \mu(0) \land \theta(x) = \theta(x)$ which gives $\theta \subseteq \mu + \theta$.

2.2.15. Lemma: Let $\mu$ be a fuzzy left (right) $N$-subgroup of $N$ satisfying supremum condition. Then $\mu + \mu = \mu$.

Proof: By 2.2.14., $\mu \subseteq \mu + \mu$. Let $x \in N$. Then,

$$(\mu + \mu)(x) = \vee_{a+b \in N} \{\mu(a) \land \mu(b) : a, b \in N\}$$

$$= \mu(c) \land \mu(d)$$

for some $x = c + d$.

$$\leq \mu(x).$$

Hence $\mu + \mu \subseteq \mu$ and this proves the lemma.

2.2.16. Lemma: Let $\mu$, $\theta$ and $\sigma$ be fuzzy $N$-subgroups of $N$. Then $\mu \subseteq \theta, \sigma \subseteq \theta$ imply $\mu + \sigma \subseteq \theta$.

Proof: Let $x \in N$. Then,

$$(\mu + \sigma)(x) = \vee_{a+b \in N} \{\mu(a) \land \sigma(b) : a, b \in N\}$$
Thus the result follows. 

2.2.17. Theorem: Let \( \mu, \theta \) and \( \sigma \) be fuzzy \( N \)-subgroups of \( N \) such that \( \mu(0) = \theta(0) \).

Then \( (\mu + \theta)\sigma = \mu\sigma + \theta\sigma \).

Proof: By 2.2.14, \( \mu \subseteq \mu + \theta \) and \( \theta \subseteq \mu + \theta \).

Again, by 1.3.11.(ii), \( \mu\sigma \subseteq (\mu + \theta)\sigma \) and \( \theta\sigma \subseteq (\mu + \theta)\sigma \).

Thus by 2.2.16. we get \( \mu\sigma + \theta\sigma \subseteq (\mu + \theta)\sigma \).

Conversely, let \( x \in N \). Then,

\[
((\mu + \theta)\sigma)(x) = \bigvee_{i=1}^{n} \{ (\mu + \theta)(y_i) \wedge \sigma(z_i) : y_i, z_i \in N, i = 1, 2, \ldots, n \in Z_+ \}
\]

\[
= \bigvee_{i=1}^{n} \{ (\vee (\mu(p_i) \wedge \theta(q_i)) : p_i, q_i, y_i = p_i + q_i \wedge \sigma(z_i)) \}
\]

\[
: y_i, z_i \in N, i = 1, 2, \ldots, n \in Z_+ \}
\]

\[
= \bigvee_{i=1}^{n} \{ (\mu(p_i) \wedge \theta(q_i)) \wedge \sigma(z_i) : p_i, q_i, z_i \in N, i = 1, 2, \ldots, n \}
\]

\[
\leq \bigvee_{i=1}^{n} \{ (\wedge (\mu(p_i) \wedge \sigma(z_i'))) \wedge (\wedge (\theta(q_i') \wedge \sigma(z_k''))
\}

: p_i', q_i', z_i', z_k'' \in N, i = 1, 2, \ldots, k = 1, 2, \ldots, s \}
\]

\[
= \bigvee_{i=1}^{n} \{ (\mu\sigma)(u) \wedge (\theta\sigma)(v) : u, v \in N \}
\]

\[
= (\mu\sigma + \theta\sigma)(x)
\]

Thus \( (\mu + \theta)\sigma \subseteq \mu\sigma + \theta\sigma \) and hence the result follows.

Using 1.2.9. and 1.3.20. we get the following:
2.2.18. **Lemma:** A fuzzy subset \( \mu \) of \( N \) is fuzzy left (right) \( N \)-subgroup of \( N \) if and only if \( \mu_t \), for all \( t \in \text{Im} \mu \), is a left (right) \( N \)-subgroup of \( N \).

2.2.19. **Lemma:** Let \( A \) be a non-empty subset of a near-ring \( N \). Then \( A \) is left (right) \( N \)-subgroup of \( N \) if and only if \( x_A \) is fuzzy left (right) \( N \)-subgroup of \( N \).

**Proof:** The proof follows from 1.2.10. and 1.3.21.

2.2.20. **Theorem:** The intersection of a non-empty family of fuzzy left (right) \( N \)-subgroups of \( N \) is a fuzzy left (right) \( N \)-subgroup of \( N \).

**Proof:** Let \( \{ \mu_i : i \in \Delta \} \) be an arbitrary collection of fuzzy left \( N \)-subgroups of a near-ring \( N \). Let \( x, y \in N \). Then we have,

\[
\left( \bigcap_{i \in \Delta} \mu_i \right) (x - y) = \wedge, \{ \mu_i (x - y) \} \geq \wedge, \{ \mu_i (x) \wedge \mu_i (y) \} \\
= (\wedge, \mu_i (x)) \wedge (\wedge, \mu_i (y)) \\
= \left( \bigcap_{i \in \Delta} \mu_i \right) (x) \wedge \left( \bigcap_{i \in \Delta} \mu_i \right) (y)
\]

Also \( \left( \bigcap_{i \in \Delta} \mu_i \right) (xy) = \wedge, \mu_i (xy) \geq \wedge, \mu_i (y) = \left( \bigcap_{i \in \Delta} \mu_i \right) (y) \). Thus we get the required result.

2.2.21. **Theorem:**[[63]] Let \( N \) and \( K \) be two near-rings. Let \( f : N \to K \) be a homomorphism. If \( \mu \) is fuzzy left (right) \( N \)-subgroup of \( N \) and \( \theta \) is fuzzy left (right) \( K \)-subgroup of \( K \) then

(i) \( f(\mu) \) is a fuzzy left (right) \( K \)-subgroup of \( K \) if \( f \) is onto.

and (ii) \( f^{-1}(\theta) \) is a fuzzy left (right) \( N \)-subgroup of \( N \).

**Proof:** (i) Let \( p, q \in K \). Then there exist \( x, y \in N \) such that \( f(x) = p \) and \( f(y) = q \).

So, \( f(x + y) = f(x) + f(y) = p + q \) and \( f(xy) = f(x)f(y) = pq \).

Now, \( [f(\mu)](p - q) = \bigvee_{f(z) = p - q} \mu(z) \)
If \( \mu \) is a fuzzy left \( N \)-subgroup of \( N \) then

\[
[f(\mu)](pq) = \bigvee_{f(x)=p, f(y)=q} \mu(x) \\
\geq \bigvee_{f(x)=p, f(y)=q} \mu(xy) \\
\geq \bigvee_{f(y)=q} \mu(y) \\
= [f(\mu)](q).
\]

Thus \( f(\mu) \) is a fuzzy left \( K \)-subgroup of \( K \).

If \( \mu \) is a fuzzy right \( N \)-subgroup of \( N \) then similarly it can be shown that

\[
[f(\mu)](pq) \geq [f(\mu)](p).
\]

Thus the result follows.

(ii) Let \( x, y \in N \) and \( \theta \) be a fuzzy left \( K \)-subgroup of \( K \). Then we have,

\[
[f^{-1}(\theta)](x+y) = \theta(f(x+y)) \\
= \theta(f(x) + f(y)) \\
\geq \theta(f(x)) \wedge \theta(f(y)) \\
= f^{-1}(\theta)(x) \wedge f^{-1}(\theta)(y)
\]

Also

\[
[f^{-1}(\theta)](xy) = \theta(f(xy)) \\
= \theta(f(x)f(y)) \\
= \theta(f(y))
\]
Thus $f^{-1}(\theta)$ is a fuzzy left $N$-subgroup of $N$. 

An alternative proof of the above theorem 2.2.21.(i) by means of level set is established in [63].

### 2.2.22. Theorem:
Let $N$ and $K$ be two near-rings and $f: N \rightarrow K$ be a homomorphism. Let $\mu$ be a fuzzy $N$-subgroup of $N$ and $\theta$ be a fuzzy $K$-subgroup of $K$. Then,

(i) $[\mathcal{f}(\mu)](0') = \mu(0)$ and $f^{-1}(\theta)(0) = \theta(0')$ where $0$ and $0'$ are zero elements of $N$ and $K$ respectively.

(ii) $f(\mu) \subseteq [f(\mu)]_0$ and the equality holds if $\mu$ has sup. property.

(iii) $[f(\mu)](x) = \mu(x)$ $\forall x \in N$ if $\mu$ is $f$-invariant.

(iv) $f^{-1}(\theta)_0 = [f^{-1}(\theta)]_0$

(v) $f^{-1}(\theta)$ is $f$-invariant.

(vi) $(f^{-1} \circ f)(\mu) = \mu$, if $\mu$ is $f$-invariant.

(vii) $(f \circ f^{-1})(\theta) = \theta$, if $f$ is onto.

### Proof:
(i) $[f(\mu)](0') = \vee_{x \in N} \mu(x) = \mu(0)$, by 2.2.6(i). Similarly $f^{-1}(\theta)(0) = \theta(0')$.

(ii) Let $y \in f(\mu)_0$. Then there exists $x_0 \in \mu_0$ such that $f(x_0) = y$. So, we get

$[f(\mu)](y) = \vee_{(x,y) \in \mu}[\mu(x) = \mu(0)]$, as $\mu(x_0) = \mu(0)$ and $f(x_0) = y$. Thus $y \in [f(\mu)]_0$.

Also let $\mu$ be with sup property. Let $y \in [f(\mu)]_0$. Then $[f(\mu)](y) = f(\mu)(0') = \mu(0)$. Thus

$\vee_{x \in N} \mu(x) = \mu(0)$, where $x \in N$.

By assumption, there exists $x_0 \in N$, such that $f(x_0) = y$ and $\mu(x_0) = \mu(0)$.

It follows that, $x_0 \in \mu_0$ and hence $y = f(x_0) \in f(\mu)_0$. 

Thus $f^{-1}(\theta)$ is a fuzzy left $N$-subgroup of $N$. 

An alternative proof of the above theorem 2.2.21.(i) by means of level set is established in [63].

### 2.2.22. Theorem:
Let $N$ and $K$ be two near-rings and $f: N \rightarrow K$ be a homomorphism. Let $\mu$ be a fuzzy $N$-subgroup of $N$ and $\theta$ be a fuzzy $K$-subgroup of $K$. Then,

(i) $[\mathcal{f}(\mu)](0') = \mu(0)$ and $f^{-1}(\theta)(0) = \theta(0')$ where $0$ and $0'$ are zero elements of $N$ and $K$ respectively.

(ii) $f(\mu) \subseteq [f(\mu)]_0$ and the equality holds if $\mu$ has sup. property.

(iii) $[f(\mu)](x) = \mu(x)$ $\forall x \in N$ if $\mu$ is $f$-invariant.

(iv) $f^{-1}(\theta)_0 = [f^{-1}(\theta)]_0$

(v) $f^{-1}(\theta)$ is $f$-invariant.

(vi) $(f^{-1} \circ f)(\mu) = \mu$, if $\mu$ is $f$-invariant.

(vii) $(f \circ f^{-1})(\theta) = \theta$, if $f$ is onto.

### Proof:
(i) $[f(\mu)](0') = \vee_{(x,y) \in \mu}[\mu(x) = \mu(0)]$, by 2.2.6(i). Similarly $f^{-1}(\theta)(0) = \theta(0')$.

(ii) Let $y \in f(\mu)_0$. Then there exists $x_0 \in \mu_0$ such that $f(x_0) = y$. So, we get

$[f(\mu)](y) = \vee_{x \in N} \mu(x) = \mu(0)$, as $\mu(x_0) = \mu(0)$ and $f(x_0) = y$. Thus $y \in [f(\mu)]_0$.

Also let $\mu$ be with sup property. Let $y \in [f(\mu)]_0$. Then $[f(\mu)](y) = f(\mu)(0') = \mu(0)$. Thus

$\vee_{x \in N} \mu(x) = \mu(0)$, where $x \in N$.

By assumption, there exists $x_0 \in N$, such that $f(x_0) = y$ and $\mu(x_0) = \mu(0)$.

It follows that, $x_0 \in \mu_0$ and hence $y = f(x_0) \in f(\mu)_0$. 

Thus $f^{-1}(\theta)$ is a fuzzy left $N$-subgroup of $N$. 

An alternative proof of the above theorem 2.2.21.(i) by means of level set is established in [63].
(iii) If $x \in N$, then, \[ f(\mu)(f(x)) = \bigvee_{f(z) = f(x)} \mu(z) = \mu(x), \text{ as } \mu \text{ is } f\text{-invariant.} \]

(iv) Let $x \in f^{-1}(\theta_0)$. Then $f(x) \in \theta_0 \iff \theta(f(x)) = \theta(0') \iff \theta(f(x)) = f^{-1}(\theta)(0)$

$\iff f^{-1}(\theta)(x) = f^{-1}(\theta)(0) \iff x \in [f^{-1}(\theta)]_0$.

Thus $f^{-1}(\theta_0) = [f^{-1}(\theta)]_0$.

(v) Let $x, y \in N$ be such that $f(x) = f(y)$.

Then $[f^{-1}(\theta)](x) = \theta(f(x)) = \theta(f(y)) = [f^{-1}(\theta)](y)$.

Consequently, $f^{-1}(\theta)$ is $f$-invariant.

(vi) Let $x \in N$. Then

\[ [(f^{-1} \circ f)(\mu)](x) = [f^{-1}(f(\mu))](x) \]

\[ = f(\mu)(f(x)) \]

\[ = \mu(x) \text{ by (iii)} \]

(vii) Let $y \in K$. Then

\[ [(f \circ f^{-1})(\theta)](y) = [f(f^{-1}(\theta))](y) \]

\[ = [f(f^{-1}(\theta))](x), \text{ for some } x \in N \text{ such that } f(x) = y. \]

\[ = f^{-1}(\theta)(x) \]

\[ = \theta(y). \]

Thus the result follows. $\blacksquare$

2.2.23. Definition[3] A fuzzy subset $\mu$ of a near-ring $N$ is called a fuzzy left ideal of $N$ if $\mu$ satisfies the following axioms:

(i) $\mu(x - y) \geq \land\{\mu(x), \mu(y)\}$

(ii) $\mu(x + y) = \mu(y + x)$
(iii) \( \mu[a(b + x) - a b] \geq \mu(x) \), \( \forall \ x, y, a, b \) in \( N \).

If the fuzzy subset \( \mu \) of \( N \) is a fuzzy normal subgroup of \( (N,+) \) and satisfies the following condition, then \( \mu \) is called a \textit{fuzzy right ideal} of \( N \):

\[
\mu(xy) \geq \mu(x), \quad \text{for all } x, y \in N.
\]

If \( \mu \) is both right as well as left ideal of \( N \), then we say that \( \mu \) is a \textit{fuzzy ideal} of \( N \).

2.2.24. Example: We consider the near-ring \( S = \{0, a, b, c\} \) under the addition defined in the example 3.18. and multiplication as in the following table:

<table>
<thead>
<tr>
<th>.</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
</tr>
</tbody>
</table>

and \( \mu: S \rightarrow [0,1] \) is defined such that \( \mu(0) > \mu(a) > \mu(b) = \mu(c) \). Then \( \mu \) is a fuzzy ideal of \( S \).

2.2.25. Remarks: If \( \mu \) is a fuzzy left (right) ideal of \( N \) then the following are immediate:

(i) \( \mu(x) \leq \mu(0), \forall x \in N \)

(ii) \( \mu(x) = \mu(-x), \forall x \in N \)

(iii) \( \mu(x - y) = \mu(0) \Rightarrow \mu(x) = \mu(y). \)

(iv) \( \mu(1) \leq \mu(x), \forall x \in N. \)

2.2.26. Lemma:[3] Let \( a \in N \) and \( \langle a \rangle \) be the intersection of all ideals in \( N \) containing \( a \). If \( \mu \) is a fuzzy ideal of \( N \) then \( \mu(x) \geq \mu(a) \) for all \( x \in \langle a \rangle \).

2.2.27. Lemma: [3] Let \( A \) be a non-empty subset of a near-ring \( N \). Then \( A \) is a left
(right) ideal of N if and only if the characteristic function \( \chi_A \) is a fuzzy left (right) ideal of N.

2.2.28. Lemma: [15] Let \( \mu \) be a fuzzy subset of N. Then \( \mu \) is a fuzzy left (right) ideal of N if and only if \( \mu_t \), for all \( t \in \text{Im} \mu \), is a left (right) ideal of N.

2.2.29. Lemma: If \( \mu \) is a fuzzy left ideal of N then for all \( x, y, a, b \in N \),

\[
\mu[a(x+b)-ab] \geq \mu(x).
\]

Proof: Let \( x \in N \) such that \( \mu(x) = t \).

Then \( \mu[a(b+x)-ab] \geq \mu(x) = t \)

\( \Rightarrow a(b+x)-ab \in \mu_t \)

By 2.2.28, \( \mu_t \) is a left ideal of N. So, \( a(x+b)-ab \in \mu_t \)

Hence \( \mu[a(x+b)-ab] \geq t = \mu(x) \).

2.2.30. Lemma: The intersection of a non-empty family of fuzzy left (right) ideals of a nearring N is a fuzzy left (right) ideal of N.

Proof: Let \( \{ \mu_\lambda : \lambda \in \Delta \} \) be an arbitrary collection of fuzzy left ideals of N. Let \( x, y \in N \).

Now, \( \left( \bigcap_{\lambda \in \Delta} \mu_\lambda \right)(x-y) = \bigwedge \{ \mu_\lambda (x-y) : \lambda \in \Delta \} \)

\( \geq \bigwedge \{ \mu_\lambda (x) \wedge \mu_\lambda (y) : \lambda \in \Delta \} \)

\( = [\bigwedge \mu_\lambda (x) : \lambda \in \Delta] \wedge [\bigwedge \mu_\lambda (y) : \lambda \in \Delta] \)

\( = \left( \bigcap_{\lambda \in \Delta} \mu_\lambda \right)(x) \wedge \left( \bigcap_{\lambda \in \Delta} \mu_\lambda \right)(y) \)

\( \left( \bigcap_{\lambda \in \Delta} \mu_\lambda \right)(x+y) = \bigwedge \{ \mu_\lambda (x+y) : \lambda \in \Delta \} \)

\( = \bigwedge \{ \mu_\lambda (y+x) : \lambda \in \Delta \} \)
Also for any \( a, b \in N \), we have,
\[
\left( \bigcap_{\lambda \in \Delta} \mu_{\lambda} \right) [a(b + x) - ab] = \bigwedge_{\lambda} \{ \mu_{\lambda} [a(b + x) - ab] : \lambda \in \Delta \}
\geq \left[ \bigwedge_{\lambda} \mu_{\lambda}(x) : \lambda \in \Delta \right]
= \left( \bigcap_{\lambda \in \Delta} \mu_{\lambda} \right)(x)
\]
Thus \( \left( \bigcap_{\lambda \in \Delta} \mu_{\lambda} \right) \) is a fuzzy left ideal of \( N \).

**2.2.31. Lemma [3]:** If \( \mu \) is a fuzzy right ideal and \( \sigma \) is a fuzzy left ideal of a near-ring \( N \), then \( \mu \circ \sigma \subseteq \mu \cap \sigma \)

**2.2.32. Lemma [15]:** Let \( A \) be an ideal of \( N \) and \( t < s (\neq 0), t, s \in [0,1] \). Then the fuzzy subset \( \mu \) of \( N \) defined by
\[
\mu(x) = s, \quad \text{if } x \in A
\]
\[
= t, \quad \text{otherwise}
\]
is a fuzzy ideal of \( N \).

In our following discussion if \( \mu \) is a fuzzy left (right) ideal of \( N \) then \( \mu_0 \) is defined as follows:
\[
\mu_0 = \{ x \in N : \mu(x) = \mu(0) \}.
\]

**2.2.33. Theorem:** If \( \mu, \theta, \sigma \) are any fuzzy left (right) ideals of \( N \), then
\begin{itemize}
\item[(i)] \( \mu_0 \cap \theta_0 \subseteq (\mu \cap \theta)_0 \)
\item[(ii)] \( \mu_0 \cap \theta_0 = (\mu \cap \theta)_0 \) if \( \mu(0) = \theta(0) \)
\end{itemize}
(iii) \( \mu \circ \theta \leq \sigma \iff \mu \theta \leq \sigma \)

(iv) \( \mu^* \) is a left (or right) ideal of \( N \)

2.2.34. Theorem: If \( \mu \) is a fuzzy right ideal and \( \theta \) is a fuzzy left ideal of \( N \) then

\[ \mu \theta \leq \mu \cap \theta \]

Proof: Let \( x \in N \) and let \( x = \sum_{i=1}^{n} a_i b_i \), \( a_i, b_i \in N \).

Then we have,

\[ \mu(x) = \mu(\sum_{i=1}^{n} a_i b_i) \]

\[ \geq \land, \mu(a_i, b_i) \]

\[ \geq \land, \mu(a_i) \]

and \( \theta(x) = \theta(\sum_{i=1}^{n} a_i b_i) \)

\[ \geq \land, \theta(a_i, b_i) \]

\[ = \land, \theta(a_i(0+b_i)-a,0), \text{ as N is zero symmetric.} \]

\[ \geq \land, \theta(b_i) \]

Thus for every \( x = \sum_{i=1}^{n} a_i b_i \), \( a_i, b_i \in N \), we have

\[ \mu(x) \land \theta(x) \geq \{ \land, \mu(a_i) \} \land \{ \land, \theta(b_i) \} = \land, \{ \mu(a_i) \land \theta(b_i) \} \]

Thus, \( (\mu \cap \theta)(x) \geq \lor, \{ \mu(a_i) \land \theta(b_i) \} = (\mu \theta)(x) \).

As a consequence we get the required result. \( \Box \)

2.2.35. Theorem: If \( \mu \) is a fuzzy ideal of \( N \), then for any fuzzy subset \( \theta \) of \( N \) \( \mu \circ \theta \leq \mu \).

Proof: Let \( x \in N \) be such that \( x = ab \), where \( a, b \in N \).

As \( \mu \) is a fuzzy right ideal of \( N \), we have \( \mu(x) = \mu(ab) \geq \mu(a) \).
Next, since \( \mu \) is a fuzzy left ideal of \( N \) and \( N \) is zero symmetric, it follows that
\[
\mu(x) = \mu(ab) = \mu[\alpha(0 + b) - a0] \geq \mu(b).
\]

Hence as in 2.2.10. we get \( \mu(x) \geq (\mu \circ \theta)(x), \forall x \in N \). Thus the result follows. \( \blacksquare \)

2.2.36. Theorem:[[63]] If \( \mu \) is a fuzzy left ideal and \( \sigma \) is a fuzzy left \( N \)-subgroup of \( N \) then \( \mu + \sigma \) is a fuzzy left \( N \)-subgroup of \( N \).

**Proof:** Let \( x, y \in N \) and let \( (\mu + \sigma)(x) \land (\mu + \sigma)(y) = (\mu + \sigma)(x) = k \).

Then for \( \varepsilon > 0 \), there exist \( p, q, r, s \in N \) with \( x = p + q \) and \( y = r + s \) such that
\[
k - \varepsilon < \mu(p) \land \sigma(q) \quad \text{and} \quad k - \varepsilon < \mu(r) \land \sigma(s)
\]

Hence \( k - \varepsilon < \mu(r), k - \varepsilon < \sigma(s), k - \varepsilon < \mu(p) \) and \( k - \varepsilon < \sigma(q) \).

Also \( \mu \) being fuzzy left ideal of \( N \), we have,
\[
\mu(p) \land \mu(q + r - q) \geq \mu(p) \land \mu(r) > k - \varepsilon
\]

Thus, \( \mu(p + (q + r - q)) > k - \varepsilon \).

\[
\Rightarrow \mu((p + q) + (r - q)) > k - \varepsilon
\]

\[
\Rightarrow \mu(x + r - q) > k - \varepsilon \quad \text{......... (1)}
\]

Also, \( k - \varepsilon < \sigma(s) \) and \( k - \varepsilon < \sigma(q) \) \( \Rightarrow \sigma(q + s) > k - \varepsilon \). \( \ität{.................(2)} \)

By (1) and (2) we have, \( \mu(x + r - q) > k - \varepsilon, \sigma(q + s) > k - \varepsilon \).

Now, \((\mu + \sigma)(x + y) = \vee_{x+y} \{\mu(u) \land \sigma(v)\} \)
\[
\geq \mu(c) \land \sigma(d)
\]
\[
\geq k - \varepsilon
\]

where \( c = x + r - q, d = q + s \) and \( c + d = x + y \).

Thus \((\mu + \sigma)(x + y) \geq k \).

Now we claim that \((\mu + \sigma)(-x) \geq (\mu + \sigma)(x) \).
Let \((\mu + \sigma)(x) = t\) and \(\varepsilon > 0\). Then there exist \(c, d \in N\) with \(x = c + d\) such that
\[
t - \varepsilon < \mu(c) \wedge \sigma(d)
\]
\[
= \mu(x - d) \wedge \sigma(-d)
\]
\[
= \mu(-d + x) \wedge \sigma(-d)
\]
\[
= \mu((-x + d)) \wedge \sigma(-d)
\]
\[
= \mu(-x + d) \wedge \sigma(-d)
\]
\[
\leq \vee_{x \in \mathbb{R}} \left\{ \left[ \{ \mu(a) \wedge \sigma(b) \} : a, b \in N \right] \right\}
\]
\[
= (\mu + \sigma)(-x)
\]

Thus we get \((\mu + \sigma)(-x) \geq (\mu + \sigma)(x)\).

Next, let \((\mu + \sigma)(y) = m\). So for \(\delta > 0\) there exist \(f, g \in N\) with \(y = f + g\) such that,
\[
m - \delta < \mu(f) \wedge \sigma(g) < \mu(x(g + f) - xg) \wedge \sigma(xg)
\]
\[
= \mu(x(g + f) - xg) \wedge \sigma(xg), \text{ using } 2.2.29
\]
\[
= \mu(t) \wedge \sigma(w), \text{ where } x(f + g) = t + w.
\]

Hence, \((\mu + \sigma)(xy) \geq (\mu + \sigma)(y)\), for all \(x, y \in N\).

Therefore \(\mu + \sigma\) is a fuzzy left \(N\)-subgroup of \(N\). \(\Box\)

2.2.37. Theorem:[[63]] If \(\mu\) and \(\sigma\) are two fuzzy left ideals of \(N\) then \(\mu + \sigma\) is also a fuzzy left ideal of \(N\).

Proof: By 2.2.36. \(\mu + \sigma\) is a fuzzy subgroup of \((N, +)\).

Let \(y \in (\mu + \sigma)_\mathcal{I}\), where \(t \in \text{Im}(\mu + \sigma)\). Then \(t \in \text{Im}\mu\) or \(t \in \text{Im}\sigma\). If \(t \in \text{Im}\mu\) then there exist \(m, n \in N\) such that \(\mu(m) = t\) and \(y = m + n\) and \(\mu(m) < \sigma(n)\). Now if \(x \in N\), then,
\[
(\mu + \sigma)(x + y - x) = (\mu + \sigma)(x + m + n - x)
\]
\[
= (\mu + \sigma)(m' + x + n - x)\]
\[(\mu + \sigma)(x + y - x) = (\mu + \sigma)(m' + n')\]
\[= \bigvee_{m' + n' = x + y'} \{\mu(x') \land \sigma(y')\}\]
\[\geq \mu(m') \land \sigma(n')\]
\[\geq t\]

Thus \(x + y - x \in (\mu + \sigma)\), which implies \(\mu + \sigma\) is a fuzzy normal subgroup of \((N, +)\).

Also if possible let \(a(b + y) - ab \notin (\mu + \sigma)\), for some \(a, b \in N\).

Hence \((\mu + \sigma)[a(b + y) - ab] < t\)
\[
\Rightarrow \bigvee_{a(b+y) - ab = r+s} \{\mu(r) \land \sigma(s)\} < t
\]
\[
\Rightarrow \mu(r) \land \sigma(s) < t, \text{ for all } a(b + y) - ab = r + s, \text{.................(A)}
\]
\[
\Rightarrow \{\mu(a(b + y) - ab) \land \sigma(0)\} < t
\]
\[
\Rightarrow \mu(y) \land \sigma(0) < t
\]
\[
\Rightarrow \mu(y) < t, \text{ since } \sigma \text{ is a fuzzy left ideal } \sigma(0) < t \text{ is not possible.}
\]
\[
\Rightarrow y \notin \mu_i
\]

From (A), it can be proved that \(y \notin \sigma_i\). Thus \(y = m + n \notin \mu_i + \sigma_i\), which is a contradiction as \(m \in \mu_i\) and \(n \in \sigma_i\). Hence \(a(b + y) - ab \in (\mu + \sigma)\), for every \(a, b \in N\).

Therefore \(\mu + \sigma\) is a fuzzy left ideal of \(N\). \(\Box\)

**2.2.38: Theorem:**[15] Let \(N\) and \(K\) be two near-rings and \(f: N \to K\) be a homomorphism. If \(\mu\) and \(\theta\) are fuzzy left ideals of \(N\) and \(K\) respectively then

(i) \(f(\mu)\) is a fuzzy left ideal of \(K\) if \(f\) is onto.
(ii) $f^{-1}(\emptyset)$ is fuzzy left ideal of N.

2.2.39. **Theorem:**[[63]] Let N and K be two near-rings and $f: N \rightarrow K$ be an epimorphism If $\mu$ and $\sigma$ are two fuzzy ideals of N then the following hold:

(i) $f(\mu + \sigma) = f(\mu) + f(\sigma)$

(ii) $f(\mu \circ \sigma) = f(\mu) \circ f(\sigma)$

(iii) $f(\mu \cap \sigma) \subseteq f(\mu) \cap f(\sigma)$

**Proof:** Let $y \in K$.

(i) $f(\mu + \sigma)(y) = \vee_{x \in N} \{f(x) + f(\mu + \sigma)(x)\}$

And for $\varepsilon > 0$ there exists $x_1 \in N$ with $f(x_1) = y$ such that

$$f(\mu + \sigma)(y) < (\mu + \sigma)(x_1) + \varepsilon/2.$$  

For $\varepsilon > 0$ there exist $a_1, b_1 \in N$ with $x_1 = a_1 + b_1$ such that

$$(\mu + \sigma)(x_1) - \varepsilon/2 < \mu(a_1) \land \sigma(b_1).$$

Thus

$$f(\mu + \sigma)(y) < \mu(a_1) \land \sigma(b_1) + \varepsilon.$$  

Next,

$$f(\mu + f(\sigma))(y) = \vee_{y = y_1 + y_2} \{f(\mu)(y_1) \land f(\sigma)(y_2)\}$$

$$\geq f(\mu)(f(a_1)) \land f(\sigma)(f(b_1)), y = y_1 + y_2 \text{ for some}$$

$$y_1, y_2 \in K \text{ and } y = f(x_1) = f(a_1) + f(b_1)$$

$$= f^{-1}(f(\mu))(a_1) \land f^{-1}(f(\sigma))(b_1)$$

$$\geq \mu(a_1) \land \sigma(b_1)$$

$$> f(\mu + \sigma)(y) - \varepsilon,$$ using (1)

Thus

$$f(\mu + f(\sigma))(y) \geq f(\mu + \sigma)(y), \forall y \in K.$$  In a similar manner it can be proved that

$$f(\mu + f(\sigma))(y) \leq f(\mu + \sigma)(y), \forall y \in K.$$
Hence $f(\mu + \sigma) = f(\mu) + f(\sigma)$.

(ii) $f(\mu \circ \sigma)(y) = \vee_{f(\circ)} f(\mu \circ \sigma)(x)$

$$< (\mu \circ \sigma)(x_1) + \varepsilon/2, \text{for some } x_1 \in N \text{ such that } f(x_1) = y$$

$$< \{\mu(a_1) \land \sigma(b_1)\} + \varepsilon \text{ for some } a_1, b_1 \in N \text{ such that } x_1 = a_1 b_1$$

$$\leq f^{-1}(f(\mu)(a_1) \land f^{-1}(f(\sigma))(b_1) + \varepsilon$$

$$= f(\mu)(f(a_1) \land f(\sigma)(f(b_1))) + \varepsilon$$

$$\leq (f(\mu) \circ f(\sigma))(f(a_1) f(b_1)) + \varepsilon$$

$$= (f(\mu) \circ f(\sigma))(y) + \varepsilon$$

Hence $f(\mu \circ \sigma)(y) \leq (f(\mu) \circ f(\sigma))(y)$. Similarly the reverse inequality can be proved. Thus

$$f(\mu \circ \sigma) = f(\mu) \circ f(\sigma).$$

(iii) Since $\mu \cap \sigma \subseteq \mu$ and $\mu \cap \sigma \subseteq \sigma$, we have $f(\mu \cap \sigma) \subseteq f(\mu) \cap f(\sigma)$.

2.2.40. Theorem: Let $N$ and $K$ be two near-rings and $f: N \to K$ be an epimorphism. Let $\mu$ and $\sigma$ be two fuzzy ideals of $N$. Then $f(\mu \cap \sigma) = f(\mu) \cap f(\sigma)$ if $\mu$ or $\sigma$ is $f$-invariant.

Proof: We need to prove $f(\mu \cap \sigma) \supseteq f(\mu) \cap f(\sigma)$. Let $y \in K$ and $\mu$ be $f$-invariant.

Let $[f(\mu) \cap f(\sigma)](y) = t$.

Then for any $\varepsilon > 0$,

$$t - \varepsilon < f(\mu)(y) \land f(\sigma)(y) = f(\mu)(y) \land (\vee_{f(\circ)} \sigma(x)).$$

$$\Rightarrow t - \varepsilon/2 < f(\mu)(y) \text{ and } t - \varepsilon/2 < (\vee_{f(\circ)} \sigma(x))$$

For $\varepsilon > 0$ there exists $x_1 \in N$ with $f(x_1) = y$ such that

$$\left(\vee_{f(\circ)} \sigma(x)\right) - \varepsilon/2 < \sigma(x_1)$$

This gives us $t - \varepsilon < f(\mu)(y)$ and $t - \varepsilon < \sigma(x_1)$
Thus \( t - \varepsilon < f(\mu(x_i)) \) and \( t - \varepsilon < \sigma(x_i) \). Consequently, \( t - \varepsilon < \mu(x_i) \) and \( t - \varepsilon < \sigma(x_i) \) as \( \mu \) is \( f \)-invariant.

Thus \( t - \varepsilon < \mu(x_i) \land \sigma(x_i) = (\mu \land \sigma)(x_i) \) where \( f(x_i) = y \).

\[
\leq \vee_{f(x)=y} (\mu \land \sigma)(x)
\]

\[
= f(\mu \land \sigma)(y)
\]

Hence \( f(\mu \land \sigma) \supseteq f(\mu) \land f(\sigma) \) and from 2.2.39(iii), the result follows. \qed

2.2.41. Theorem: Let \( f \) be a homomorphism from a near-ring \( N \) onto a near-ring \( K \). Let \( \mu \) and \( 0 \) be two fuzzy ideals of \( K \). Then, \( f^{-1}(\mu \circ \theta) \subseteq f^{-1}(\mu \circ \theta) \).

Proof: Let \( x \in N \) and \( \left[ f^{-1}(\mu) \circ f^{-1}(\theta) \right](x) = t \).

Now \( \left[ f^{-1}(\mu) \circ f^{-1}(\theta) \right](x) = \vee_{x=a,b} \left\{ f^{-1}(\mu)(a) \land f^{-1}(\theta)(b) \right\} = \vee_{x=a,b} \left\{ \mu(f(a)) \land \theta(f(b)) \right\} \)

Then for any \( \varepsilon > 0 \) there exist \( a_i, b_i \in N \) with \( x = a_i b_i \) such that

\[
t - \varepsilon < \mu(f(a_i)) \land \theta(f(b_i)),
\]

\[
\leq (\mu \circ \theta)(f(a_i b_i))
\]

\[
= f^{-1}(\mu \circ \theta)(x)
\]

Hence \( f^{-1}(\mu) \circ f^{-1}(\theta) \subseteq f^{-1}(\mu \circ \theta) \). \qed

2.2.42. Theorem: Let \( f \) be a homomorphism from a near-ring \( N \) onto a near-ring \( K \). Let \( \mu \) and \( \sigma \) be two fuzzy ideals of \( N \). Then \( f(\mu \sigma) = f(\mu) f(\sigma) \).

Proof: Let \( y \in K \) and \( \varepsilon > 0 \). Let \( f(\mu \sigma)(y) = t \). Then there exists \( x_i \in N \) with \( f(x_i) = y \)

such that

\[
t - \varepsilon / 2 < (\mu \sigma)(x_i)
\]

\[
= \vee_{x_i} [\land \{ \mu(a_i) \land \sigma(b_i) \}] \text{ where } a_i, b_i \in N, x_i = \sum_{i=1}^{a} a_i b_i.
\]
Also for \( \varepsilon > 0 \) there exist \( u, v, \in N, x_1 = \sum_{i=1}^{n} u_i v_i \) such that

\[
(\mu \sigma)(x_1) - \varepsilon / 2 < \left[ \bigwedge, \{ \mu(u_i) \land \sigma(v_i) \} \right]
\]

Therefore, \( t - \varepsilon < \left[ \bigwedge, \{ \mu(u_i) \land \sigma(v_i) \} \right] \)

Moreover,

\[
[f(\mu)f(\sigma)](y) = \vee_y \left[ \bigwedge, \{ f(\mu)(p_i) \land f(\sigma)(q_i) \} \right], \text{ where } p_i, q_i \in K, y = \sum_{i=1}^{n} p_i q_i. \quad \text{This gives us,}
\]

\[
[f(\mu)f(\sigma)](y) \geq \bigwedge, \{ f^{-1}(f(\mu)(u_i)) \land f^{-1}(f(\sigma)(v_i)) \}
\]

\[
\geq \bigwedge, \{ \mu(u_i) \land \sigma(v_i) \}
\]

\[
> t - \varepsilon
\]

Therefore, \( [f(\mu)f(\sigma)](y) \geq [f(\mu \sigma)](y) \).

Conversely, let \( [f(\mu)f(\sigma)](y) = t. \)

Since \( [f(\mu)f(\sigma)](y) = \vee_y \left[ \bigwedge, \{ f(\mu)(p_i) \land f(\sigma)(q_i) \} \right] \) where \( p_i, q_i \in K, y = \sum_{i=1}^{n} p_i q_i, \)

\[
= \vee_y \left[ \bigwedge, \{ \vee_{f(z_i) = p_i} \mu(z_i) \land \vee_{f(w_i) = q_i} \sigma(w_i) \} \right]
\]

for any \( \varepsilon > 0 \) we have,

\[
t - \varepsilon < \left[ \bigwedge, \{ \mu(n_i) \land \sigma(m_i) \} \right] \text{ for some } f(n_i) = p_i, f(m_i) = q_i.
\]

\[
\leq (\mu \sigma)(x) \text{ where } n_i, m_i \in N, x = \sum_{i=1}^{n} n_i m_i
\]
\[ \leq \vee_{f(x,y)}(\mu \sigma)(z) \quad \text{as} \quad y = \sum_{i=1}^{n} p_i q_i = f(x) \]

\[ = [f(\mu \sigma)](y) \]

Therefore \([f(\mu \sigma)f(\sigma)](y) \leq [f(\mu \sigma)](y)\) and this completes the proof. □

### 2.3 Fuzzy N-subgroup and fuzzy ideal of an N-group \(E\).

#### 2.3.1. Definition:[63] Let \(\mu\) be a fuzzy subset of an N-group \(E\). Then \(\mu\) is said to be a fuzzy N-subgroup of \(E\) if for all \(n \in \mathbb{N}\) and \(x, y \in E\),

(i) \(\mu(x + y) \geq \mu(x) \wedge \mu(y)\)

(ii) \(\mu(x) = \mu(-x)\)

(iii) \(\mu(nx) \geq \mu(x)\)

#### 2.3.2. Remarks: If \(\mu\) is a fuzzy N-subgroup of \(N\)-group \(E\), then for all \(x \in E\), the following are equivalent:

(i) \(\mu(-x) \geq \mu(x)\)

(ii) \(\mu(-x) \leq \mu(x)\)

(iii) \(\mu(x) = \mu(-x)\)

#### 2.3.3. Example: We consider the Dihedral group \(Q = \{0, a, 2a, 3a, b, a + b, 2a + b, 3a + b\}\) over the zero near-ring \(N\). Then \(Q\) is an \(N\)-group. If a fuzzy subset \(\mu\) of \(Q\) is defined such that \(\mu(0) > \mu(2a) > \mu(b) = \mu(2a + b) > \mu(a) = \mu(a + b) = \mu(3a + b)\), then it is seen that \(\mu\) is a fuzzy N-subgroup of \(Q\).

#### 2.3.4. Lemma: If \(\mu\) is a fuzzy N-subgroup of an N-group \(E\), then \(\mu^*\) is also an N-subgroup of \(E\).
Proof: It is obvious that \( \mu^* \) is a subgroup of \((E, +)\). Let \( n \in \mathbb{N} \) and \( x \in \mu^* \).

Then \( \mu(x) > 0 \) and \( \mu \) being a fuzzy N-subgroup of \( E \), we have \( \mu(nx) > \mu(x) > 0 \). Consequently, \( nx \in \mu^* \) and hence \( \mu^* \) is an N-subgroup of \( E \). □

2.3.5. Theorem: [[63]] A nonempty subset \( S \) of \( E \) is an N-subgroup of \( E \) if and only if the characteristic function \( \chi_S \) of \( S \) is a fuzzy N-subgroup of \( E \).

Proof: Let \( S \) be an N-subgroup of \( E \). Thus \( S \) is a subgroup of \((E, +)\) and hence \( \chi_S \) is a fuzzy subgroup of \((E, +)\). Also let \( n \in \mathbb{N} \) and \( x \in E \).

If \( x \in S \) then clearly \( nx \in S \) which gives \( \chi_S(nx) \geq \chi_S(x) \). Also if \( x \notin S \) then \( \chi_S(x) = 0 \) and thus the result follows.

Conversely, let \( \chi_S \) be a fuzzy N-subgroup of \( E \). Then clearly \( S \) is a subgroup of \((E, +)\). Let \( x \in NS \). Then there exists \( n \in \mathbb{N} \), \( y \in E \) such that \( x = ny \). Hence \( \chi_S(y) = 1 \). Also \( \chi_S(x) = \chi_S(ny) \geq \chi_S(y) = 1 \) gives \( x \in S \). Thus \( S \) is an N-subgroup of \( E \). □

2.3.6. Theorem: [[63]] A fuzzy subset \( \mu \) of \( E \) is a fuzzy N-subgroup of \( E \) if and only if \( \mu_t \), for all \( t \in \text{Im} \mu \), is an N-subgroup of \( E \).

Proof: Let \( \mu \) be a fuzzy N-subgroup of \( E \). Thus \( \mu_t \), for all \( t \in \text{Im} \mu \) is a subgroup of \((E, +)\).

Also let \( n \in \mathbb{N} \) and \( x \in \mu_t \). Then \( \mu(nx) \geq \mu(x) \geq t \). Hence \( nx \in \mu_t \). Thus \( \mu_t \), for all \( t \in \text{Im} \mu \), is an N-subgroup of \( E \).

Conversely, let \( n \in \mathbb{N} \) and \( x \in E \). Let \( \mu(x) = s \). Then \( x \in \mu_s \) and \( \mu_s \) being an N-subgroup of \( E \), we have \( nx \in \mu_s \) which shows \( \mu(nx) \geq \mu(x) \). This proves the result. □

2.3.7. Theorem: A fuzzy subset \( \mu \) of \( E \) is a fuzzy N-subgroup of \( E \) if and only if

\[
(i) \quad \mu(0) \geq \mu(x)
\]
(ii) $\mu( mx + ny ) \geq \mu( x ) \wedge \mu( y )$ for all $m, n \in \mathbb{N}$ and $x, y \in E$.

**Proof:** Let $\mu$ be a fuzzy $N$-subgroup of $E$.

Clearly $\mu(0) = \mu( -x ) \geq \mu( x ) \wedge \mu( -x ) = \mu( x )$, for all $x \in E$.

Also,

$$\mu( mx + ny ) \geq \mu( mx ) \wedge \mu( ny )$$

$$\geq \mu( x ) \wedge \mu( y )$$

for all $m, n \in \mathbb{N}$ and $x, y \in E$.

Conversely, we assume conditions (i) and (ii).

Now $\mu( x + y ) = \mu( 1x + 1y ) \geq \mu( x ) \wedge \mu( y )$ and

$$\mu( -x ) = \mu( -1x + 0.0 )$$

$$\geq \mu( x ) \wedge \mu( 0 )$$

$$= \mu( x ) .$$

Thus by remark 2.3.2, $\mu( x ) = \mu( -x )$, for all $x \in E$.

Also

$$\mu( nx ) = \mu( nx + m0 )$$

$$\geq \mu( x ) \wedge \mu( 0 )$$

$$= \mu( x )$$

where $n \in \mathbb{N}$ and $x \in E$.

Thus $\mu$ is a fuzzy $N$-subgroup of $E$. \[\square\]

The proof of the following theorem follows from 1.4.6.(vi).

**2.3.8. Theorem:** A fuzzy subset $\mu$ of $E$ is a fuzzy $N$-subgroup of $E$ if and only if

(i) $\mu( x - y ) \geq \mu( x ) \wedge \mu( y )$, for all $x, y \in E$.

and (ii) $n\mu \subseteq \mu$, for all $n \in \mathbb{N}$.
2.3.9. Theorem: [[63]] Let $E$ and $F$ be two $N$-groups and $f: E \to F$ be an $N$-epimorphism. Let $\mu$ be a fuzzy $N$-subgroup of $E$. Then $f(\mu)$ is a fuzzy $N$-subgroup of $F$.

Proof: By 1.2.14(i), $f(\mu)$ is a fuzzy subgroup of $(F, +)$. Now let $y \in F$ and $n \in N$. Then there exists $z \in E$ such that $f(z) = y$ and hence $f(nz) = ny$.

Now,

$$[f(\mu)](ny) = \{\bigvee_{f(z)=y} \mu(x) : x \in E\} \supseteq \{\bigvee_{f(z)=y} \mu(nz) : nz \in E\} = \{\bigvee_{f(z)=y} \mu(nz) : nz \in E\} \supseteq \{\bigvee_{f(z)=y} \mu(z) : z \in E\} = f(\mu)(y)$$

Thus $f(\mu)$ is a fuzzy $N$-subgroup of $F$. $lacksquare$

2.3.10. Theorem: [[63]] Let $E$ and $F$ be two $N$-groups and $f: E \to F$ be an $N$-homomorphism. Let $\mu$ be a fuzzy $N$-subgroup of $F$. Then $f^{-1}(\mu)$ is a fuzzy $N$-subgroup of $E$.

Proof: By 1.2.14(ii), $f^{-1}(\mu)$ is a fuzzy subgroup of $(E, +)$. Let $x \in E$ and $n \in N$. Then $f^{-1}(\mu)(nx) = \mu(f(nx)) = \mu(nx) f(x) \supseteq \mu f(x) = f^{-1}(\mu)(x)$.

Thus $f^{-1}(\mu)$ is a fuzzy $N$-subgroup of $E$. $lacksquare$

We omit the proof of the following theorem, as it is routine matter of verification.

2.3.11. Theorem: The intersection of a non-empty family of fuzzy $N$-subgroups of an $N$-group $E$ is again a fuzzy $N$-subgroup of $E$.

2.3.12. Definition: [[63]] Let $\mu$ be a fuzzy subset of an $N$-group $E$. Then $\mu$ is said to be a fuzzy ideal of $E$ if,

(i) $\mu$ is a fuzzy normal subgroup of the additive group $E$. 
(ii) $\mu[n(a + x) - na] \geq \mu(x)$, for all $n \in \mathbb{N}$ and $a, x \in E$.

2.3.13. Example: We consider the near-ring $S = \{0, a, b, c\}$ under the addition and multiplication defined as follows:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>.</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

If a fuzzy subset $\mu: S \to [0,1]$ is defined such that $\mu(0) > \mu(a) > \mu(b) = \mu(c)$, then it can be seen that $\mu$ is a fuzzy ideal of the near-ring group $S$.

2.3.14. Theorem: A fuzzy normal subgroup $\mu$ of $(E, +)$ is a fuzzy ideal of $E$ if and only if one of the following holds:

(i) $\mu[-na + n(a + x)] \geq \mu(x)$, for all $n \in \mathbb{N}$ and $a, x \in E$.

(ii) $\mu[-na + n(x + a)] \geq \mu(x)$, for all $n \in \mathbb{N}$ and $a, x \in E$.

2.3.15. Lemma: If $\mu$ and $\theta$ are fuzzy ideals of $E$, then the following hold:

(i) $\mu(0) \geq \mu(x)$, for all $x \in E$.

(ii) $\mu(x) = \mu(-x)$ for all $x \in E$.

(iii) $\mu(x - y) = \mu(0)$ imply $\mu(x) = \mu(y)$ where $x, y \in E$.

(iv) $\mu_0 \cap \theta_0 \subseteq \left(\mu \cap \theta\right)_0$ and $\mu_0 \cap \theta_0 = \left(\mu \cap \theta\right)_0$ if $\mu(0) = \theta(0)$

(v) $\mu^*$ is an ideal of $E$. 
2.3.16. Theorem: [[63]] Let \( S \) be a nonempty subset of \( E \). Then \( S \) is an ideal of \( E \) if and only if \( \chi_S \) is a fuzzy ideal of \( E \).

**Proof:** Let \( S \) be an ideal of \( E \). Hence \( \chi_S \) is a fuzzy normal subgroup of \((E, +)\), by 1.2.21. Let \( n \in \mathbb{N} \) and \( a, x \in E \). Then if \( x \in S \), then \( n(a + x) - na \in S \) which gives

\[
\chi_S[n(a + x) - na] = \chi_S(x).
\]

Also if \( x \notin S \), then \( \chi_S(x) = 0 \) and hence \( \chi_S[n(a + x) - na] \geq \chi_S(x) \).

Thus \( \chi_S \) is a fuzzy ideal of \( E \).

Conversely if \( \chi_S \) is fuzzy ideal of \( E \), then by 1.2.21, \( S \) is a normal subgroup of \((E, +)\).

Let \( n \in \mathbb{N} \), \( x \in S \) and \( a \in E \). Then \( \chi_S(x) = 1 \) and since \( \chi_S \) is fuzzy ideal of \( E \), \( \chi_S[n(a + x) - na] \geq \chi_S(x) = 1 \) which gives \( n(a + x) - na \in S \).

Consequently, \( S \) is an ideal of \( E \). □

2.3.17. Theorem: [[63]] A fuzzy subset \( \mu \) of \( E \) is a fuzzy ideal of \( E \) if and only if \( \mu_t \), for all \( t \in \text{Im} \mu \), is an ideal of \( E \).

**Proof:** Let \( \mu \) be a fuzzy ideals of \( E \). Then \( \mu_t \), for all \( t \in \text{Im} \mu \) is a normal subgroup of \((E, +)\).

Let \( n \in \mathbb{N} \), \( x \in \mu_t \) and \( a \in E \) where \( t \in [0, \mu(0)] \). Then \( \mu(x) \geq t \).

But \( \mu[n(a + x) - na] \geq \mu(x) \geq t \). Hence \( n(a + x) - na \in \mu_t \), and therefore we get the desired result.

The proof of the converse part is a routine matter of verification. □

The following theorem plays an important role in the determination of fuzzy prime ideals, fuzzy maximal ideals. The proof is a routine matter of verification and we omit it.
2.3.18. **Theorem:** Let $A$ be an ideal of $E$. Then the fuzzy subset $\mu$ of $E$ defined as follows is a fuzzy ideal of $E$:

$$
\mu(x) = \begin{cases} 
t, & \text{if } x \in A, \\
s, & \text{otherwise},
\end{cases}
$$

where $t, s \in [0, 1]$ such that $s \leq t$ ($\neq 0$).

2.3.19. **Theorem:** Let $E$ and $F$ be two $N$-groups. Let $f: E \to F$ be an $N$-homomorphism. Let $\mu$ and $\theta$ be fuzzy ideals of $E$ and $F$ respectively. Then the following are satisfied:

(i) $[f(\mu)](0') = \mu(0)$, where $0$ and $0'$ are zero element of $E$ and $F$ respectively.

(ii) $f^{-1}(\theta)(0) = \theta(0')$

(iii) $f(\mu_0) \subseteq [f(\mu)]_0$ and the equality holds if $\mu$ has sup. Property.

(iv) $[f(\mu)](f(x)) = \mu(x), \forall x \in E$ if $\mu$ is $f$–invariant.

(v) $f(\mu)$ is a fuzzy ideal of $F$, if $f$ is onto.

(vi) $f^{-1}(\theta_0) = [f^{-1}(\theta)]_0$

(vii) $f^{-1}(\theta)$ is a fuzzy ideal of $E$.

(viii) $f^{-1}(\theta)$ is $f$–invariant.

(ix) $(f^{-1} \circ f)(\mu) = \mu$, if $\mu$ is $f$–invariant.

(x) $(f \circ f^{-1})(\theta) = \theta$, if $f$ is onto.

**Proof:** We prove (v) and (vii) and the rest are easy to prove.

(v) Let $x, y \in F$. By 1.2.23(i), we have,

$$
f(\mu)(x - y) \geq f(\mu)(x) \wedge f(\mu)(y),$$

and

$$
f(\mu)(x + y) = f(\mu)(y + x).$$
Also let \( n \in \mathbb{N} \) and \( b, y \in F \). Since \( f \) is onto, there exists \( a, x \in E \) such that \( f(a) = b \) and \( f(x) = y \).

Now,

\[
f(\mu)[n(b + y) - nb] = \vee_{f(u) = n(b + y) - nb} \mu(u)
\]

\[
\geq \mu(n(a + x) - na), \text{ as } n(b + y) - nb = f[n(a + x) - na].
\]

Thus,

\[
f(\mu)[n(b + y) - nb] \geq \mu(x), \text{ whenever } f(x) = y.
\]

Therefore, \( f(\mu)[n(b + y) - nb] \geq \vee_{f(x) = y} \mu(x) \).

\[
= f(\mu)(y)
\]

Thus \( f(\mu) \) is a fuzzy ideal of \( F \).

(vii) By 1.2.23(ii), \( f^{-1}(\theta) \) is normal fuzzy subgroup of \( (E,+) \). Let \( n \in \mathbb{N} \) and \( a, x \in E \).

Then,

\[
f^{-1}(\theta)[n(a + x) - na] = \theta[f\{n(a + x) - na\}]
\]

\[
= \theta[nf(a + x) - nf(a)]
\]

\[
= \theta[nf(a) + f(x) - nf(a)]
\]

\[
\geq \theta(f(x))
\]

\[
= f^{-1}(\theta)(x)
\]

Thus \( f^{-1}(\theta) \) is a fuzzy ideal of \( E \).  

The following two theorems can be obtained as in 2.2.36. and 2.2.37.

2.3.20. Theorem: If \( \mu \) is a fuzzy ideal of \( E \) and \( \theta \) is a fuzzy \( N \)-subgroup of \( E \), then \( \mu + \theta \) is a fuzzy \( N \)-subgroup of \( E \).

2.3.21. Theorem: If \( \mu \) and \( \theta \) are fuzzy ideals of \( E \), then \( \mu + \theta \) is a fuzzy ideal of \( E \).
2.3.22. Theorem: The intersection of a non-empty family of fuzzy ideals of an N-group is again a fuzzy ideal of E.

2.4. Fuzzy factor N-group

2.4.1. Definition:[15] Let $\mu$ be a fuzzy ideal of E and $a \in E$. Then the fuzzy subset $a + \mu$ defined by $(a + \mu)(x) = \mu(x - a)$, for all $x \in E$, is called a fuzzy coset of $\mu$.

2.4.2 Lemma:[[63]] If $\mu$ is a fuzzy ideal of E, then $(a + \mu)(nx) \geq \mu(x) \land \mu(a)$, for all $n \in \mathbb{N}$ and $a, x \in E$.

2.4.3. Lemma:[[63]] Let $\mu$ be a fuzzy ideal of E and $a, b \in E$. Then

$$a + \mu = b + \mu \text{ if and only if } \mu(b - a) = \mu(0).$$

Proof: Let $a + \mu = b + \mu$, where $a, b \in E$.

Then $\mu(b - a) = (a + \mu)(b) = (b + \mu)(b) = \mu(0)$.

Conversely, let $\mu(b - a) = \mu(0)$.

Then for $x \in E$, $(a + \mu)(x) = \mu(x - a) = \mu(x - b + b - a)$

$$\geq \mu(x - b) \land \mu(b - a)$$

$$= \mu(x - b), \quad \text{by 2.3.15(i)}$$

$$= (b + \mu)(x)$$

Thus $a + \mu \geq b + \mu$.

Similarly we get $b + \mu \geq a + \mu$ and the result follows. □

2.4.4. Lemma:[[63]] Let $\mu$ be a fuzzy ideal of E and $a, b \in E$.

Then $a + \mu = b + \mu$ if and only if $-a + \mu = -b + \mu$. 
Proof: Let \( a, b \in E \).

Then \( a + \mu = b + \mu \)

\[ \Leftrightarrow \mu(b - a) = \mu(0) \]

\[ \Leftrightarrow \mu(-a + b) = \mu(0) \]

\[ \Leftrightarrow \mu((-a) - (-b)) = \mu(0) \]

\[ \Leftrightarrow -b + \mu = -a + \mu. \]

2.4.5 Lemma: Let \( \mu \) be a fuzzy ideal of \( E \). Then for any \( x, y \in E \),

\( x + \mu = y + \mu \) if and only if \( x + \mu_0 = y + \mu_0 \).

Proof: Let \( x + \mu = y + \mu \), where \( x, y \in E \). Then \( \mu(x - y) = \mu(0) \). This gives

\( x - y \in \mu_0 \) and hence \( x + \mu_0 = y + \mu_0 \). Converse part is straightforward. \( \square \)

The theorem 2.3.15.(iii) is restated below and an alternative proof is given using the notion of fuzzy cosets.

2.4.6 Lemma: Let \( \mu \) be a fuzzy ideal of \( E \). Then for any \( x, y \in E \),

\[ \mu(x - y) = \mu(0) \Rightarrow \mu(x) = \mu(y). \]

Proof: Let \( x, y \in E \). Then, \( \mu(x - y) = \mu(0) \)

\[ \Rightarrow x + \mu = y + \mu \]

\[ \Rightarrow (x + \mu)(0) = (y + \mu)(0) \]

\[ \Rightarrow \mu(0 - x) = \mu(0 - y) \]

\[ \Rightarrow \mu(-x) = \mu(-y) \]

\[ \Rightarrow \mu(x) = \mu(y). \] \( \square \)
2.4.7. Theorem: Let \( \mu \) be a fuzzy ideal of \( E \). Let \( E/\mu \) be the set of all fuzzy cosets of the fuzzy ideal \( \mu \). Then \( E/\mu \) is an \( N \)-group under addition and scalar multiplication defined as follows: For \( n \in \mathbb{N} \) and \( a, b \in E \),
\[
(a + \mu) + (b + \mu) = (a + b) + \mu
\]
and \( n(a + \mu) = na + \mu \).

Proof: Let us first prove that the compositions are well defined.

Let \( a, b, u, v \in E \) such that \( a + \mu = u + \mu \) and \( b + \mu = v + \mu \).

Then \( \mu(u-a) = \mu(0) \) and \( \mu(v-b) = \mu(0) \).

Now, \( \mu((u + v) - (a + b)) = \mu(u + v - b - a) \)
\[
= \mu((u + v - b) - a)
\]
\[
= \mu(-a + (u + v - b))
\]
\[
= \mu((-a + u) + (v - b))
\]
\[
\geq \mu(-a + u) \land \mu(v - b)
\]
\[
= \mu(u - a) \land \mu(v - b)
\]
\[
= \mu(0)
\]
Thus \( (a + b) + \mu = (u + v) + \mu \).

Also let \( n \in \mathbb{N} \) and \( a + \mu = b + \mu \), where \( a, b \in E \).

Then \( \mu(b - a) = \mu(0) \).

Now, \( \mu(nb - na) = \mu\{n(a + (-b + a)) - na\} \)
\[
\geq \mu(-(-b + a))
\]
\[
= \mu(-b + a)
\]
\[
= \mu(a - b).
\]
Thus \[ na + \mu = nb + \mu. \]

Hence the compositions are well defined.

It is straightforward to verify that \( E/\mu \) is a group under addition defined as above.

Next let \( m, n \in \mathbb{N} \) and \( a \in E.\)

Then, (i) \[(m + n)(a + \mu) = (m + n)a + \mu = (m + n)a + \mu = (ma + na) + \mu = (ma + \mu) + (na + \mu) = m(a + \mu) + n(a + \mu).\]

(ii) \[(mn)(a + \mu) = (mn)a + \mu = m(na) + \mu = m(na + \mu) = m(n(a + \mu)).\]

(iii) It is obvious that \( 1(a + \mu) = a + \mu. \)

Thus \( E/\mu \) is an N-group. \( \square \)

2.4.8. Theorem:\,[63] Let \( \mu \) be a fuzzy ideal of \( E. \) Then for any \( a, b \in E, \)

\[ \{ x \in E : x + \mu = a + \mu \} + \{ x \in E : x + \mu = b + \mu \} = \{ x \in E : x + \mu = (a + b) + \mu \}. \]

Proof: If \( x = y + z \) such that \( y + \mu = a + \mu \) and \( z + \mu = b + \mu, \) then we have,

\[ x + \mu = (y + z) + \mu. \]

\[ = (y + \mu) + (z + \mu) \]

\[ = (a + \mu) + (b + \mu) \]

\[ = (a + b) + \mu. \]
Also let, \( x + \mu = (a + b) + \mu \).

Now, \( \mu (a + b - x) = (x + \mu)(a + b) \)

\[ = ((a + b) + \mu)(a + b) \]

\[ = \mu(0). \]

If \( y \in E \), then \( ((x - b) + \mu)(y) = \mu(y - (x - b)) \)

\[ = \mu(y + b - x) \]

\[ = (x + \mu)(y + b) \]

\[ = (a + b + \mu)(y + b) \]

\[ = \mu((y + b) - (a + b)) \]

\[ = \mu(y - a) \]

\[ = (a + \mu)(y). \]

Thus \( (x - b) + \mu = a + \mu \), and since \( x = (x - b) + b \), so the result follows. ⚫

2.4.9. Theorem:([63]) Let \( \sigma \) be a fuzzy N-subgroup of \( E \). A fuzzy subset \( \sigma/\mu \) of the N-group \( E/\mu \) is defined as follows:

\[ \sigma/\mu(a + \mu) = \vee_{x \in (a + \mu)} \sigma(x) \text{ where } a, x \in E. \]

Then \( \sigma/\mu \) is a fuzzy N-subgroup of \( E/\mu \).

Proof: Let \( a + \mu, b + \mu \in E/\mu \).

Then, \( \sigma/\mu((a + \mu) + (b + \mu)) = \sigma/\mu(a + b + \mu) \)

\[ = \vee_{x + \mu = (a + b) + \mu} \sigma(x) \]

\[ = \vee_{x + \mu = (a + \mu) + (b + \mu)} \sigma(x) \]

\[ = \vee_{y + \mu = a + \mu, z + \mu = b + \mu} \sigma(y + z) \]
\[\geq \bigvee_{y + \mu = a + \mu} \bigvee_{z + \mu = b + \mu} \sigma(y) \land \sigma(z)\]

\[= \left(\bigvee_{y + \mu = a + \mu} \sigma(y)\right) \land \left(\bigvee_{z + \mu = b + \mu} \sigma(z)\right)\]

\[= \sigma/\mu(a + \mu) \land \sigma/\mu(b + \mu)\]

Also since \(a + \mu \in E/\mu\), \(-(a + \mu) = -a + \mu\). So we have,

\[
\sigma/\mu(-(a + \mu)) = \sigma/\mu(-a + \mu)
\]

\[= \bigvee_{x + \mu = -a + \mu} \sigma(-x)\]

\[= \bigvee_{x + \mu = a + \mu} \sigma(-x)\]

\[= \bigvee_{x + \mu = a + \mu} \sigma(x)\]

\[= \sigma/\mu(a + \mu)\]

Also let \(x + \mu \in E/\mu\), \(n \in \mathbb{N}\). Then

\[
\sigma/\mu(n(x + \mu)) = \sigma/\mu(nx + \mu)
\]

\[= \bigvee_{y + \mu = nx + \mu} \sigma(y)\]

\[\geq \bigvee_{z + \mu = nx + \mu} \sigma(nz)\]

\[= \bigvee_{n(x + \mu) = n(z + \mu)} \sigma(nz)\]

\[\geq \bigvee_{(x + \mu) = (z + \mu)} \sigma(nz)\]

\[\geq \bigvee_{(x + \mu) = (z + \mu)} \sigma(z)\]

\[= \sigma/\mu(x + \mu)\]

Hence \(\sigma/\mu\) is a fuzzy N-subgroup of \(E/\mu\).

2.4.10. Definition:([63]) The fuzzy N-subgroup \(\sigma/\mu\) of the N-group \(E/\mu\) defined as in 2.4.9.

is called fuzzy factor N-group of \(\sigma\) with respect to \(\mu\).
2.4.11. Theorem: Let \( \sigma \) be a fuzzy N-subgroup and \( \mu \) be a fuzzy ideal of \( E \). Then the mapping \( f: E \rightarrow E/\mu \) given by \( f(x) = x + \mu \), \( x \in E \) is an N-epimorphism whose kernel is \( \mu_0 \) and \( f(\sigma) = \sigma/\mu \).

Proof: Let \( x, y \in E \), \( n \in \mathbb{N} \).

Then \( f(x + y) = (x + y) + \mu \)

\[
= (x + \mu) + (y + \mu)
\]

\[
= f(x) + f(y)
\]

This shows \( f \) is an N-homomorphism.

Also whenever \( x + \mu \in E/\mu \), we have \( x \in E \) such that \( f(x) = x + \mu \) and thus \( f \) is onto.

Let \( x \in \text{Ker} f \Leftrightarrow f(x) = \mu \Leftrightarrow x + \mu = \mu \Leftrightarrow \mu(x) = \mu(0) \Leftrightarrow x \in \mu_0 \).

This proves that \( \text{Ker} f = \mu_0 \).

Next, we have

\[
f(\sigma)(x + \mu) = \vee_{y \in f^{-1}(x+\mu)} \sigma(y)
\]

\[
= \vee f(y)_{y + \mu = x + \mu} \sigma(y)
\]

\[
= \vee_{y + \mu = x + \mu} \sigma(y)
\]

\[
= \sigma/\mu(x + \mu)
\]

Hence \( f(\sigma) = \sigma/\mu \).
2.4.12. Theorem: Let $E$ and $F$ be two $N$-groups and $f: E \to F$ be an $N$-epimorphism. Let $\mu$ and $\sigma$ be fuzzy ideals of $E$ such that $\mu, = \{x : \mu(x) = \mu(0) = i\} \subseteq \ker f$. If $\varphi : E/\mu \to F$ is defined such that $\varphi(x + \mu) = f(x)$ then $\varphi$ is an $N$-epimorphism with $\varphi(\sigma/\mu) = f(\sigma)$.

Proof: First we show that $\varphi$ is well defined.

Let $x + \mu, y + \mu \in E/\mu$ such that $x + \mu = y + \mu$.

Then $\mu(y - x) = \mu(0)$ and hence $y - x \in \mu \subseteq \ker f$.

$$\Rightarrow f(y - x) = 0.$$

Also $\varphi(y + \mu) = f(y) = f(y - x + x) = f(y - x) + f(x) = \varphi(x + \mu)$.

It is easy to verify that $\varphi$ is an $N$-epimorphism.

Let $y \in F$.

Then, $\varphi(\sigma/\mu)(y) = \vee_{x + \mu \in \sigma/\mu}(\sigma/\mu)(x + \mu)$

$$= \vee_{x \in f^{-1}(y)}(\sigma/\mu)(x + \mu)$$

$$= \vee_{x \in f^{-1}(y)}[\vee_{x + \mu \in \sigma}(\sigma(z))$$

$$= \vee_{x \in f^{-1}(y)} \sigma(z)$$

$$= f(\sigma)(y).$$

Thus we get the required result. $\blacksquare$