Chapter V

Section 5.1: Our aim in this chapter is to extend the theory of fixed points of mapping in a quasi-proximity space. It is a difficult task to carry notion of a contraction mapping from a metric space to a more generalised space. This is evident from the fact that theory of fixed points in a uniform space has not developed as fast as in a metric space. We have found that J.Dugundji in [26] has proved certain fixed point theorems for mappings in a metric space by using positive definite function over it. K.K.Tan and C.S.Wong in [27] and C.S.Wong in [28] have extended the idea of positive definiteness of a function in a uniform space, and have proved some fixed point theorems therein.

In this chain, A.Choudhury and A.C.Babu in [29] further extended this idea in the setting of a quasi-uniform space. In this chapter we have advanced this concept of positive definiteness in a quasi-proximity space, and with its aid have proved a fixed point theorem.

Section 5.2: We first describe some important results as can be found in J.Dugundji's work [26]:

Let \( P \) be a non-negative real-valued mapping on a metric space \((X, d)\). Let \( A \) be a subset of \( X \). Then \( P \)
is said to be positive definite mod $A$ if

$$P_A(r) = \inf \{ P(x) : d(x, A) \geq r \}$$

is positive for all $r > 0$. \[ \inf \emptyset = \infty. \]

J. Dugundji has got following result in [26]. A mapping

$$P : (X \times X, D) \rightarrow [0, \infty)$$

is positive definite mod $\Delta (X)$ iff for any $r > 0$, there exists $\vartheta (r) > 0$ such that

$$P : (x, y) < \vartheta (r) \implies d(x, y) < r,$$

where the metric $D$ on $X \times X$ is defined by

$$D((x, y), (x', y')) = d(x, x') + d(y, y'); x, y, x', y' \in X.$$ 

An important part of J. Dugundji's work [26] in the characterisation of a complete metric space in terms of notion of positive definiteness of mapping as described in Theorem 5.2.1.

**Theorem 5.2.1:** Let $(X, d)$ be a metric space. Then the following conditions are equivalent.

(i) $X$ is complete.

(ii) For every l.s.c. mapping $\nabla$ of $X$ into $[0, \infty)$ such that $\inf \nabla (X) = 0$

and $P_\nabla : X \times X \rightarrow [0, \infty)$ defined by

$$P_\nabla (x, y) = \nabla (x) + \nabla (y), (x, y) \in X \times X$$

is positive definite mod $\Delta (X)$, then $\nabla (x) = 0$ for some $x \in X$. 


Moreover, if (ii) is satisfied and if \( \{ x_n \} \) is a sequence in \( X \) such that \( \{ V(x_n) \} \) converges to 0, then \( \{ x_n \} \) converges to \( x \). In order to obtain a fixed point for a mapping \( T \) of \( X \) into \( X \), it is only natural to take \( V \) in Theorem 5.2.1 as a function on \( X \) such that
\[
V(x) = d(x, T(x))
\]
for all \( x \) in \( X \). Thus J.Dugundji [26] fixed point theorem runs as follows:

**Theorem 5.2.2:** Let \((X, d)\) be a complete metric space. Also let \( T : X \rightarrow X \) be a mapping. Suppose that the mapping \( V \) defined by \( V(x) = d(x, T(x)) \), \( x \in X \) is l.s.c. with \( \inf V(X) = 0 \), and \( P_V : X \times X \rightarrow [0, \infty) \) defined by
\[
P_V(x, y) = V(x) + V(y); (x, y) \in X \times X
\]
is positive definite mod \( \Delta(X) \). Then \( T \) has a unique fixed point.

For definition of a separated quasi-proximity space see Definition 3.4.2. We now prove Theorem 5.2.3, the result of which will be used in proving Theorem 5.3.1.

**Theorem 5.2.3:** Let \((X, \partial)\) be a separated quasi-proximity space. Then \( A \not\subseteq B \) implies that there exists a quasi-proximal mapping \( f : (X, \partial) \rightarrow (I, \mathcal{P}) \) where \( I = [0, 1] \) and \( \mathcal{P} \) is the quasi-proximity to induce the upper topology \( \mathcal{T}_\mathcal{P} \) in \( I \) such that \( f(A) = 0 \) and \( f(B) = 1 \).

**Proof:** From Theorem 4.4.2, suppose that \((X, \partial, \partial^*\) is the compactification of \((X, \partial, \partial^*)\) where
\[ A \nsubseteq B \text{ iff } \overline{\partial^* \text{ cl}(A)} \cap \overline{\partial \text{ cl}(B)} = \emptyset; \]
\[ A, B \subseteq X. \]

Then as shown in Theorem 3.2.3, \((X, \partial, \partial^*)\) is pairwise completely regular; and hence it is pairwise compact. Thus \((X, \partial, \partial^*)\) becomes pairwise normal by Corollary 2.3.1, since every pairwise completely regular space is also pairwise regular. As \(\overline{\partial^* \text{ cl}(A)} \cap \overline{\partial \text{ cl}(B)} = \emptyset\), by Theorem 2.2.1, there is a mapping \(g : X \rightarrow I\) satisfying \(g(A) = 0\) and \(g(B) = 1\);

i.e. \(g : (X, \tau, \tau^*) \rightarrow (I, \tau_0, \tau_{0^*})\)
is continuous. So on applying Theorem 4.3.4 we have \(g\) as a quasi-proximal mapping. Put \(f \equiv \text{Restriction } g \text{ on } X\), and \(f\) is the required mapping.

**Corollary 5.2.1** (Theorem 7.13 in [31]): In a separated proximity space \((X, \partial), A \nsubseteq B\) implies that there exists a proximity mapping \(f : X \rightarrow [0,1]\) such that \(f(A) = 0\) and \(f(B) = 1\).

**Section 5.3:** To define a complete quasi-proximity space we have found that the key concept is that of 'small sets' in the space. Motivation however owes to the work of S.Leader who in [30] has employed pseudo-metrics to define 'gauges' in the
space. Now we define a 'gauge' in a quasi-proximity space with aid of quasi-pseudo-metrics.

**Definition 5.3.1**: Let \((X, \partial)\) be a quasi-proximity space. A quasi-pseudo-metric \(d\) on \(X\) is called a gauge if it satisfies the following conditions:

Given \(A \subseteq B\) and \(\epsilon > 0\) there exist \(a \in A\) and \(b \in B\) such that \(d(a, b) < \epsilon\).

Let \(\partial^*\) be the quasi-proximity conjugate to \(\partial\). Since \(A \subseteq B \iff B \subseteq A\) and \(d(a, b) = d^*(b, a)\), it is clear that \(d\) is a gauge on \((X, \partial)\) iff \(d^*\) is a gauge on \((X, \partial^*)\).

**Theorem 5.3.1** below demonstrates how a quasi-proximity \(\partial\) on \(X\) is generated by the family of all gauges on it.

**Theorem 5.3.1**: \(A \nsubseteq B\) in a separated quasi-proximity space \((X, \partial)\) iff \(d(A, B) = 0\) for every gauge \(d\) on \(X\) \[d(A, B) = \text{d-dist}(A, B)\].

**Proof**: Necessity is trivial. For sufficiency suppose \(d(A, B) = 0\) for every gauge \(d\) on \(X\); but \(A \nsubseteq B\). By Theorem 5.2.3 above, there is a quasi-proximal mapping \(f : (X, \partial) \to [I, P]\) satisfying \(f(A) = 0\) and \(f(B) = 1\). Define \(d(x, y) = \max\{0, f(y) - f(x)\}\),
then $d$ is a gauge on $X$ and $d(A, B) > 0$, a contradiction.

**Definition 5.3.2**: A family of subsets of a quasi-proximity space $(X, \mathfrak{d})$ is said to have small members if for every gauge $d$ on $X$ and for every $\varepsilon > 0$, there is a member of the family with $d$-diameter $< \varepsilon$.

**Definition 5.3.3** (see [30]): A funnel $\mathcal{F}$ is a family of non-empty subsets of $X$ directed by set inclusion i.e., given $A$ and $B$ in $\mathcal{F}$ there is a member $C \in \mathcal{F}$ with $C \subset A \cap B$.

That every filter [11, p.78] is a funnel now follows from Definitions.

**Definition 5.3.4**: A funnel in a quasi-proximity space $(X, \mathfrak{d})$ is said to be local if it contains small members.

**Definition 5.3.5**: A quasi-proximity space is called complete if every local funnel of closed subsets of $X$ has a non-empty intersection.

The following theorem on the product of a family 
\[ \left\{ (X_\alpha, \mathfrak{d}_\alpha) : \alpha \in \Delta \right\} \] of quasi-proximity spaces is needed for achieving further results.
Theorem 5.3.2: Let \( \{ (X_{\alpha}, \partial_{\alpha}) : \alpha \in \Delta \} \) be a family of quasi-proximity spaces. Also let \( X = \prod X_{\alpha} \) denote the cartesian product of these spaces. Then a binary relation \( \partial \) defined on \( \mathcal{P}(X) \) by
\[
A \partial B \iff \text{for each pair of finite covers } \{A_i : i = 1, \ldots, m\} \text{ and } \{B_j : j = 1, \ldots, n\}\text{ of } A \text{ and } B \text{ respectively, there exists an } A_i \text{ and a } B_j \text{ such that }
\]
\[
P_{\alpha}[A_i] \partial_{\alpha} P_{\alpha}[B_j] \text{ for each } \alpha \in \Delta,
\]
where \( A, B \subseteq X \), and \( P_{\alpha} \) denotes the projection of \( X \) onto \( X_{\alpha} \), is a quasi-proximity on \( X \).

For proof we refer to Theorem 4.13 of [31].

Lemma 5.3.1 and Lemma 5.3.2 show how the family of gauges on \((X, \partial)\) determines another family of gauges on \((\prod X, \partial \times \partial^*)\) which generates the product quasi-proximity \( \partial \times \partial^* \).

Lemma 5.3.1: If \( d \) is a gauge on \((X, \partial)\) then \( D \) defined by \( D((x, y), (x', y')) = d(x, x') + d^*(y, y') \) is a gauge on \((\prod X, \partial \times \partial^*)\).

Proof: Clearly \( D \) is a quasi-pseudo-metric. Now let
\[
A \partial \times \partial^* B \text{ for two subsets } A \text{ and } B \text{ of } \prod X.
\]
Then by Theorem 5.3.2 we have
Now $D(A, B) = [\text{inf}\left\{ d((x, y), (x', y')) : (x, y) \in A \text{ and } (x', y') \in B \right\}]
= [\text{inf}\left\{ d(x, x') + d^*(y, y') : (x, y) \in A \text{ and } (x', y') \in B \right\}]
= [d(P_1(A), P_1(B)) + d^*(P_2(A), P_2(B))]

[using (5.3.1), and that $d_\cdot$ is a gauge]

$= 0$. $\left[ d(A, B) = d_{-\text{dist}}(A, B) \right]$. 

Lemma 5.3.2: If $A \nabla x \nabla^* B$, then $D(A, B) > 0$
for some $D$ defined as in Lemma 5.3.1.

Proof: $A \nabla x \nabla^* B \Rightarrow P_1(A) \nabla P_1(B),$
in case $P_2(A) \nabla^* P_2(B)$ or vice-versa. Therefore $d(P_1(A), P_1(B)) > 0$ for some gauge $d$ on $X$.

Let $D$ be defined on $X \times X \times X$ as $D((x, y), (x', y')) = d(x, x') + d^*(y, y')$.

It is now routine work to see that $D(A, B) > 0$.

Theorem 5.3.3: For any gauge $d$ on $X$ and $\epsilon > 0$,
let $B_{d, \epsilon} = \{(x, y) \in X \times X : d^*(x, y) < \epsilon \}$.

Then $\{B_{d, \epsilon}\}$ is a $\mathcal{T}_{\Theta \times \Theta^*}$-neighbourhood base for $\Delta(X)$ in $(X \times X, \mathcal{T}_{\Theta \times \Theta^*})$. 
Proof: We first show that each $B_{d, \epsilon}$ is an $\tau_{\Theta x \Theta^*}$-open neighbourhood of $\Delta(X)$. Clearly for $(u, v) \in (X \times X \setminus B_{d, \epsilon})$,

$$D((x, x), (u, v)) = d(x, u) + d^*(x, v) = d^*(u, x) + d^*(x, v) \geq d^*(u, v) \geq \epsilon.$$ 

Hence $D(\Delta(X), (X \times X \setminus B_{d, \epsilon})) \geq \epsilon$, and, therefore $\Delta(X) \not\subseteq \Theta^*(X \times X \setminus B_{d, \epsilon})$, showing thereby that $B_{d, \epsilon}$ is a $\tau_{\Theta x \Theta^*}$-open neighbourhood of $\Delta(X)$.

We now show that each $\tau_{\Theta x \Theta^*}$-neighbourhood of $\Delta(X)$ contains a set of the form $\{(x, y) : d^*(x, y) < r : r > 0\}$. Let $N$ be a $\tau_{\Theta x \Theta^*}$-neighbourhood of $\Delta(X)$. So $\Delta(X) \not\subseteq \Theta^*(X \times X \setminus N)$. Then for some gauge $D$ on $X \times X$ we have

$$D(\Delta(X), (X \times X \setminus N)) > 0,$$

say, $> r (> 0)$.

Thus $D((x, x), (x, y)) \geq r$ for all $(x, y) \in N$ i.e., we have $d(x, x) + d^*(x, y) \geq r$ \Rightarrow $d^*(x, y) \geq r$.

If $L = \{(x, y) : d^*(x, y) < r\}$. Then $(X \times X \setminus N) \subset (X \times X \setminus L)$ implies $L \subset N$. 


Lemma 5.3.3 : For any gauge, \( d \) on quasi-proximity space \((X, \mathcal{D})\),
\[
d(A, B) \leq d(A, C) + d(C) + d(C, B)
\]
\[
[d(C) = d - \text{Diam } C] \quad \text{for all subsets } A, B, C \text{ of } X \text{ with } C \neq \emptyset.
\]

Proof : Since \( d(x, y) \leq d(x, z) + d(z, z') + d(z', y) \)
\[
it follows, taking lower bound for \( z \) in \( C \), that, given \( \epsilon > 0 \),
\[
d(x, y) \leq d(x, C) + d(l, z') + d(z', y) + \frac{\epsilon}{2}
\]
for some \( l \in C \).

Again taking lower bound for \( z' \) in \( C \), we obtain
\[
d(x, y) \leq d(x, C) + d(l, l') + d(C, y) + \frac{\epsilon}{2} + \frac{\epsilon}{2}
\]
for some \( l' \in C \).

\[
\leq d(x, C) + d(C) + d(C, y),
\]
since \( \epsilon > 0 \) is arbitrary.

Now taking lower bounds for \( x \) in \( A \) and \( y \) in \( B \)
successively we obtain
\[
d(A, B) \leq d(A, C) + d(C) + d(C, B).
\]

Section 5.4 : Here we introduce concept of positive definiteness in a quasi-proximity space, and study some consequences of same inorder to obtain a fixed point Theorem.
Definition 5.4.1: Let $(X, \partial)$ be a quasi-proximity space. A mapping $P : X \to [0, \infty)$ is called positive definite mod $A$ if

$$P_A(B) = \inf \{ P(x) : x \in (X \setminus B)^p \}$$

is positive for every $B$ such that $A \subseteq^* B$ (equivalently $A \not\subseteq^* (X \setminus B)$).

Theorem 5.4.1: If $P$ is positive definite mod $A$ in $(X, \partial)$, then $P^{-1}(\{0\}) \subset \partial \text{-} cl(A)$.

**Proof:** $x \in P^{-1}(\{0\}) \implies P(x) = 0 \implies x \notin (X \setminus B)$ for all $B$ such that $A \subseteq^* B$ implies $x \in \bigcap_{A \subseteq^* B} A = \partial \text{-} cl(A)$ by Theorem 3.4.4.

Theorem 5.4.2: Let $(X, \partial)$ be a quasi-proximity space, and $X \times X \times X \times \partial$ be endowed with product quasi-proximity $\partial \times \partial^*$. Also let $P$ be a mapping on $X \times X \times X$ to $[0, \infty)$. Then following conditions are equivalent.

(a) $P$ is positive definite mod $\Delta(X)$.

(b) For any gauge $d$ and $\varepsilon > 0$ there exists $K = K(d, \varepsilon)$ such that $P(x, y) < K$ implies $d(x, y) < \varepsilon$. 
Proof: \( (a) \Rightarrow (b) \)

Suppose \( P \) is positive definite mod \( \Delta(X) \) in \((X \times X, \Theta \times \Theta^*)\) but (b) is false i.e., there are \( d \) and \( \epsilon (\epsilon > 0) \) such for any \( \rho (\rho > 0) \) we have
\[
(X, \rho) \in X \times X \quad \text{for which} \quad d(x, y) \geq \epsilon,
\]
and
\[
P(x, y) < \rho. \quad \text{Let}
\]
\[
(X \times X \setminus B) = \{ (x, y) : d(x, y) \geq \epsilon \}.
\]
Since
\[
D^*(X, x, u, v), (x, y) \in X \times X \setminus B
\]
\[
= d^*(x, u) + d(x, v)
\]
\[
= d(u, x) + d(x, v) \geq d(u, v) \geq \epsilon,
\]
it follows that
\[
D^*(\Delta(X), (X \times X \setminus B)) \geq \epsilon.
\]
So
\[
\Delta(X) \not\ni \Theta^* \Theta (X \times X \setminus B)
\]
Thus
\[
(x, y) \in (X \times X \setminus B)
\]
and therefore
\[
P_{\Delta(X)}(B) = 0 \quad \text{a contradiction,}
\]
\( (b) \Rightarrow (a) \)

Suppose the contrary i.e., \( P \) is not positive definite mod \( \Delta(X) \) in \((X \times X, \Theta \times \Theta^*)\) i.e.,
\[
P_{\Delta(X)}(B) = 0 \quad \text{for some} \ B \quad \text{satisfying}
\]
\[
\Delta(X) \not\ni \Theta^* \Theta (X \times X \setminus B).
\]
By Theorem 5.3.3, there is no loss of generality if we take
\[
B = \{ (x, y) : d(x, y) < \epsilon \}, \quad \epsilon > 0.
\]
There exists a sequence \( \{(x_n, y_n)\} \) in \( X \times X \)
such that \( \{ P(x_n, y_n) \} \) converges to zero, where \( (x_n, y_n) \in (X \times X \setminus B) \) for all \( n \). So for any \( \rho > 0 \), \( \{ x_n, y_n \} \) is eventually in \( P^{-1}(\left[ 0, \rho \right)) \) i.e., \( P(x_n, y_n) < \rho \) eventually; but \( d(x_n, y_n) \geq \varepsilon \) leads to a contradiction. The proof is now complete.

**Theorem 5.4.3**: Let \( A \) be compact in \( (X, T_{\theta}) \) where \( \theta \) is a quasi-proximity on \( X \). Then for every mapping \( f: X \to [0, \infty) \) which is l.s.c. in \( (X, T_{\theta}) \) with \( \inf f(x) = 0 \), and is positive definite mod \( A \) in \( (X, \theta^*) \), \( f(a) = 0 \) for some \( a \in A \).

**Proof**: If possible, let \( \inf \{ f(x) : x \in A \} = \beta > 0 \). Then \( G = \{ x : f(x) > \frac{1}{2} \beta \} \) is a \( T_{\theta} \)-open set containing \( A \). Let \( B = \{ x : f(x) > \frac{3}{4} \beta \} \); clearly \( B \) is \( T_{\theta} \)-open and \( B \subset G \). Also \( A \subset B \).

Now \( x \in A \Rightarrow \{ x \} \subset B \Rightarrow \{ x \} \subset H_x \subset B \)

for some \( H_x \subset X \) by (Q.5). The family

\[ \{ H_x : x \in A \} \]

is an open cover for \( A \). By compactness of \( A \) we find a finite sub-family \( H_{x_1}, H_{x_2}, \ldots, H_{x_n} \) such that \( H_{x_i} \subset B (i = 1, \ldots, n) \) and \( A \subset \bigcup_{i=1}^{n} H_{x_i} \).

So \( A \subset \bigcup_{i=1}^{n} H_{x_i} \subset B \subset G \).

Since \( \inf f(x) = 0 \), we have \( \inf \{ f(x) : x \in (X \setminus G) \} = 0 \),
and hence \( \inf \{ f(x) : x \in (X \setminus B) \} = 0 \).

This contradicts that \( f \) is positive definite mod \( A \) in \((X, \mathcal{A}^*)\). Therefore \( \inf \{ f(x) : x \in A \} = 0 \).

Since \( A \) is compact and restriction \( f \) over \( A \) is l.s.c., there is an element \( a \in A \) such that
\[
f(a) = \inf \{ f(x) : x \in A \} = 0.
\]

**Lemma 5.4.1**: \( \mathcal{T}_{\mathcal{A} \times \mathcal{A}^*} \mathcal{C} \mathcal{L} (\Delta (X)) = \Delta (X) \)

if \((X, \mathcal{A})\) is separated.

**Proof**: By Theorem 3.4.4 we have,
\[
\mathcal{T}_{\mathcal{A} \times \mathcal{A}^*} \mathcal{C} \mathcal{L} (\Delta (X)) = \cap \{ B : \Delta (X) \not\subseteq \mathcal{A} (X \times X \setminus B) \}
\]

If possible, let \( \mathcal{T}_{\mathcal{A} \times \mathcal{A}^*} \mathcal{C} \mathcal{L} (\Delta (X)) \neq \Delta (X) \).

Let \((x, y) \in (\mathcal{T}_{\mathcal{A} \times \mathcal{A}^*} \mathcal{C} \mathcal{L} (\Delta (X)) \setminus \Delta (X))\).

Since \((X, \mathcal{A})\) is separated we have a gauge \( d \) such that \( d(x, y) > 0 \). From Theorem 5.3.3, it now follows that \((x, y) \notin \mathcal{T}_{\mathcal{A} \times \mathcal{A}^*} \mathcal{C} \mathcal{L} (\Delta (X))\), a contradiction.

We now prove the main theorem of this chapter.

**Theorem 5.4.4**: Let \((X, \mathcal{A})\) be a complete and separated quasi-proximity space and \( \mathcal{V} : X \to [0, \infty) \) such that

(i) \( \mathcal{V} \) is l.s.c. on \((X, \mathcal{T}_{\mathcal{A}})\)

(ii) \( \inf \{ \mathcal{V}(x) : x \in X \} = 0 \)
(iii) \( P_v : X \times X \to [0, \infty) \) defined by
\[ P_v(x, y) = V(x) + V(y) : (x, y) \in X \times X \]

is positive definite mod \( \Delta(X) \) where \( X \times X \) is endowed with topology \( \mathcal{T}_{\mathcal{A} \times \mathcal{A}^*} \). Then \( V(p) = 0 \) for some \( p \in X \). The point \( p \) is unique. Moreover, if \( \{ x_n \} \) is a sequence in \( X \) such that \( \{ V(x_n) \} \) converges to zero, then \( \{ x_n \} \) converges to \( p \).

Proof: Let \( F_n = \{ x \in X : V(x) \leq \frac{1}{n} \} \); \( n = 1, 2, \ldots, \)

Since \( V \) is l.s.c., and \( \inf V(X) = 0 \) we have each \( F_n \) as a non-empty closed set in \( (X, \mathcal{T}_\mathcal{A}) \). The family \( \{ F_n \} \) is a funnel of \( \mathcal{T}_\mathcal{A} \)-closed subsets of \( X \). We now show that \( \{ F_n \} \) contains small sets. Since \( P_v \) is positive definite mod \( \Delta(X) \) in \( (X \times X, \mathcal{T}_{\mathcal{A} \times \mathcal{A}^*}) \), by Theorem 5.4.2, given \( d \) and \( \epsilon > 0 \), there exists \( k = k(d, \epsilon) \) such that \( d(x, y) < \epsilon \) whenever \( P_v(x, y) < k \).

Choose \( N > 2/k \). Let \( x, y \in F_n \), then
\[ P_v(x, y) = V(x) + V(y) \leq 2/N < k \implies d(x, y) < \epsilon. \]

Thus the family \( \{ F_n \} \) contains small sets. Since \( (X, \mathcal{A}) \) is complete, and \( \{ F_n \} \) is a local funnel \( \bigcap_{n=1}^\infty F_n \neq \emptyset \).

Let \( x \in \bigcap_{n=1}^\infty F_n \). Then \( V(x) = 0 \). Now suppose
\[ V(x) = V(y) = 0. \] Then \( P_v(x, y) = 0 \).

Since \( P_v \) is positive definite mod \( \Delta(X) \), by Theorem 5.4.1,
\( (x, y) \in \mathcal{T}_{\mathcal{A} \times \mathcal{A}^*} \cdot \text{cl} (\Delta(X)) \).
which is equal to \( \Delta(X) \) by application of Lemma 5.4.1, because \((X, \partial)\) is separated. Hence \( x = y \). Now assume that \( \{x_n\} \) is a sequence such that \( \{V(x_n)\} \) converges to zero. Without loss of generality we assume that \( \{V(x_n)\} \) monotonically tends to zero. For each \( i \) choose \( n_i \) so that \( V(x_n) < 1/i \) for \( n > n_i \).

Set \( E_{n_i} = \{x_n : n > n_i\} : i = 1, 2, \ldots \).

Clearly \( \{E_{n_i}\} \) is decreasing, and the family \( \{\tau_\partial \cdot \text{cl}(E_{n_i})\} \) is a local funnel with \( \tau_\partial \cdot \text{cl}(E_{n_i}) \subseteq F_i : i = 1, 2, \ldots \).

Let \( x' \in \tau_\partial \cdot \text{cl}(E_{n_i}) \), and \( G \) be a \( \tau_\partial \)-open neighbourhood of \( x' \). There exists a gauge \( d \) such that \( d(x', (X \setminus G)) > 0 \). Since \( \{\tau_\partial \cdot \text{cl}(E_{n_i})\} \) is a local funnel we have \( d(\tau_\partial \cdot \text{cl}(E_{n_i})), d(x', (X \setminus G)) \)
eventually. So by Lemma 5.3.3, we have,

\[
d(x', (X \setminus G)) \leq d(x', \tau_\partial \cdot \text{cl}(E_{n_i}))+d(\tau_\partial \cdot \text{cl}(E_{n_i})), (X \setminus G)) + d(\tau_\partial \cdot \text{cl}(E_{n_i})), (X \setminus G))
\]

\[
= d(\tau_\partial \cdot \text{cl}(E_{n_i})), (X \setminus G)) + d(\tau_\partial \cdot \text{cl}(E_{n_i})), (X \setminus G)),
\]

and this implies \( d(\tau_\partial \cdot \text{cl}(E_{n_i})), (X \setminus G)) > 0 \)
eventually. So \( \tau_\partial \cdot \text{cl}(E_{n_i}) \notin (X \setminus G) \), eventually.

So \( \{x_n\} \) converges to \( x' \), and \( V(x') \leq 1/n \) for all \( n \). Thus \( V(x') = 0 \). By uniqueness \( x' = p \).
Section 5.5: In this section we arrive at a fixed point theorem, and we need Lemma 5.5.1 and Lemma 5.5.2 in this connection.

**Lemma 5.5.1:** Let $(X, d)$ be a quasi-pseudo-metric space and $T: (X, \mathcal{T}_d) \rightarrow (X, \mathcal{T}_{d^*})$ be a continuous mapping. Then $d(X, T(x))$ is a $\mathcal{T}_d$-l.s.c. function of $x$.

**Proof:** For any $k > 0$ it suffices to show that $A = \{ x \in X : d(x, T(x)) \leq k \}$ is $\mathcal{T}_d$-closed. Let $\{y_n\}$ be a net in $(A \setminus \{y\})$ converging to $y$. Take an $E > 0$. We can find an $N$ such that $d(y, y_N) < \frac{1}{2}E$ and $d^*(T(y), T(y_N)) < \frac{1}{2}E$. So

$$d(y, T(y)) \leq d(y, y_N) + d(y_N, T(y_N)) + d(T(y_N), T(y)) \leq \frac{1}{2}E + k + \frac{1}{2}E = k + E.$$

This shows that $d(y, T(y)) \leq k$. Hence $y \in A$.

**Lemma 5.5.2:** If $(X, \mathcal{D})$ is a quasi-proximity space with $\mathcal{D}$ as the family of all gauges $d$ on $X$ and $T: (X, \mathcal{D}) \rightarrow (X, \mathcal{D}^*)$ is quasi-proximally continuous, then for each member $d \in \mathcal{D}$, $T: (X, \mathcal{T}_d) \rightarrow (X, \mathcal{T}_{d^*})$ is continuous.
Proof: By quasi-proximal continuity of $T$ we have

$$A \ni B \implies T(A) \ni T(B)$$

i.e.,

$$d(A, B) = 0 \implies d^*(T(A), T(B)) = 0$$

for each $d \in \mathcal{D}$. i.e., $T : (X, \mathcal{E}_d) \to (X, \mathcal{E}_{d^*})$ is quasi-proximally continuous [ $\mathcal{E}_d$ is the quasi-proximity induced by $d$. ]

So $T : (X, \mathcal{T}_d) \to (X, \mathcal{T}_{d^*})$ is continuous.

Theorem 5.5.1 (Fixed point Theorem): Let $(X, \mathcal{E})$ be a separated quasi-proximity space which is also complete; and let $\mathcal{D}$ be the family of all gauges $d$ on $X$ and $T : (X, \mathcal{E}) \to (X, \mathcal{E}^*)$ be quasi-proximally continuous. Let $V_d(x) = d(x, T(x))$ for $x \in X$.

Suppose (a) there is a sequence $\{x_n\}$ in $X$ such that

$$\sup \{ V_d(x_n) : d \in \mathcal{D} \}$$

converges to 0,

and (b) for any gauge $d$ and $\varepsilon > 0$ there exists

$$K = K(d, \varepsilon) > 0$$

such that for any

$$x, y \in X, \quad V_d(x) + V_d(y) < K$$

implies

$$d(x, y) < \varepsilon.$$ Then $T$ has a unique fixed point.

Proof: By Lemma 5.5.1 and Lemma 5.5.2, $V_d(x)$ is a $\mathcal{T}_\mathcal{E}$-l.s.c. mapping of $x$ in $X$. Define $V$ on $X$ by

$$V(x) = \sup \{ \min [V_d(x), 1] : d \in \mathcal{D} \}.$$ Clearly, $V$ is $\mathcal{T}_\mathcal{E}$-l.s.c. from $X$ to $[0, \infty)$. 

such that \( \inf \forall (x) = 0 \). For \( x, y \in X \), let \( P_V(x, y) = \forall(x) + \forall(y) \). Then \( P_V \) is positive definite mod \( \Delta(X) \) by Theorem 5.4.2. So by Theorem 5.4.4, \( \forall(x) = 0 \) for some \( x \in X \). Thus

\[
\forall_d(x) = 0 \quad \text{for all } d \quad \text{i.e., } \{x\} \in \{T(x)\}.
\]

Since \((X, \mathcal{D})\) is separated we have \( x = T(x) \). Next let \( x, y \) be fixed points of \( T \). Then \( \forall(x) = \forall(y) = 0 \).

By (b) we have \( d(x, y) < \varepsilon \) for every gauge \( d \) and \( \varepsilon > 0 \). Thus \( \{x\} \in \{y\} \), and since \((X, \mathcal{D})\) is separated we have \( x = y \).

As a special case we have the following fixed point Theorem in a quasi-metric space.

**Theorem 5.5.2**: Let \((X, d)\) be a quasi-metric space which is also complete [same as Definition 5.3.5] and

\[
T : (X, d) \longrightarrow (X, d^*)
\]

be continuous. Define \( \forall \) on \( X \) by \( \forall(x) = d(x, T(x)) \) for all \( x \in X \).

Suppose that

(i) \( \inf \forall(x) = 0 \)

(ii) for every \( \varepsilon > 0 \) there exists \( \mathcal{D}(\varepsilon) > 0 \) such that

\[
\forall(x) + \forall(y) < \mathcal{D}(\varepsilon) \implies d(x, y) < \varepsilon.
\]

Then \( T \) has a unique fixed point.

**Proof**: By Lemma 5.5.1, \( \forall \) is l.s.c. The rest of the proof is same as that of Theorem 5.5.1.