CHAPTER III

THE CESARO-DENJOY-PETTIS SCALE OF INTEGRATION

1. Introduction

The Cesaro-Denjoy-Bochner scale of integration being introduced in Chapter II, we now consider the analogous weak processes of integration, namely the Cesaro-Denjoy-Pettis scale of integration. In this chapter we extend the Denjoy-Pettis integral introduced in Chapter I to Cesaro-Denjoy-Pettis integral by introducing a scale of integrals, the $C_n \mathcal{D}_* \mathcal{P}$ -integrals, which are such that the strength of the integrals is increased with $n$, the $C_0 \mathcal{D}_* \mathcal{P}$-integral being the $\mathcal{D}_* \mathcal{P}$-integral introduced in Chapter I and that for each $n$, the $C_n \mathcal{D}_* \mathcal{P}$ -integral includes the $C_0 \mathcal{D}_* \mathcal{B}$ -integral.

2. Preliminaries

**Definition 3.2.1.** Let $F : [a,b] \to \mathbb{X}$ and let $\xi \in [a,b]$. Let $n$ be a positive integer. If there are constants $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{X}$ depending on $\xi$ such that

$$X^* \left[ F(t) - F(\xi) - (t-\xi)\xi_1 - \ldots - \frac{(t-\xi)^n}{n!} \xi_n \right] = O((t-\xi)^n) \quad (t \to \xi)$$
for all \( x^* \in X^* \) then \( \alpha_n \) is called the weak Peano derivative of \( F \) at \( x \) of order \( n \) and is denoted by
\[
F_{(n)}^\omega(x) = \frac{(t-x)^n}{n!} \frac{\epsilon}{t} - \frac{\epsilon}{x} F_{(n)}(x).
\]
It is easily seen that if \( F_{(n)}^\omega(x) \) exists then
\[
F_{(k)}^\omega(x) \quad (1 \leq k \leq n)
\]
exists. In particular \( F_{(1)}^\omega(x) \) is the weak derivative of \( F \) at \( x \). For convenience we shall write
\[
F_{(n)}^\omega(x) \quad \text{to mean } \frac{\epsilon}{x} F_{(n)}(x).
\]
It is clear that if the strong Peano derivative \( F_{(n)} \) exists at a point \( x \) then \( F_{(n)}^\omega(x) \) will also exist at \( x \) and
\[
F_{(n)}^\omega(x) = F_{(n)}(x).
\]

**Definition 3.2.2.** Let \( F: [a, b) \to X \) and let \( n \) be a positive integer. If there are functions \( F_i: [a, b) \to X \), \( i = 1, 2, \ldots, n \) such that
\[
F(t) = F(x) + (t-x) F_1(x) + \frac{(t-x)^2}{2!} F_2(x) + \cdots + \frac{(t-x)^n}{n!} F_n(x) + O((t-x)^n) \quad (t \neq x)
\]
for almost all \( x \in [a, b] \) for each \( x^* \in X^* \) (the exceptional set of measure zero may vary with \( x^* \)), then \( F \) is said to be the pseudo derivative of \( F \) on \([a, b]\) of order \( n \) and is denoted by \( D_p^n F \). It is easily seen that \( D_p^n F \) \((1 \leq k \leq n)\) exists if \( D_p^n F \) exists and that if \( F_{(k)} \) exists a.e. in \([a, b]\) then \( D_p^n F \) exists in \([a, b]\) and \( D_p^n F = F_{(k)} \).

**Definition 3.2.3.** Let \( \gamma \geq 0 \). A function \( F: [a, b) \to X \) is called weakly \( AC_{n \gamma} \) on \([a, b]\) if \( F_{(n)}^\omega \) exists in \([a, b]\) and if the real valued function \( x^*F \) is \( AC_{n \gamma} \) \([7]\) for each \( x^* \in X^* \).
Since \(|x^*F| \leq \|x^*\| \|F\|\) it is easy to verify that strong \(A \subset G\) defined in Chapter II implies weak \(A \subset G\).

**Theorem 3.2.4.** Let \(F\) be weakly \(A \subset G\) in \([a,b]\) and let \(D^{n+1}_p\) exist in \([a,b]\). If \(D^{n+1}_p F = 0\) in \([a,b]\) then \(F^{\omega}_{(n)}\) is constant.

**Proof.** Let \(x^* \in X^*\) be arbitrary. Then \(x^*F\) is real valued \(A \subset G\) function. Since \(x^* D^{n+1}_p F = 0\), \((x^*F)_{(n+1)} = 0\) a.e. So by [7, Theorem 16 coupled with Lemma 2] \((x^*F)_{(n)}\) is constant. But since \(F^{\omega}_{(n)}\) exists in \([a,b]\), \((x^*F)_{(n)} = x^*F^{\omega}_{(n)}\).

Hence \(x^*F^{\omega}_{(n)}\) is constant. Let \(\xi \in [a,b]\). Then \[x^*\left(F^{\omega}_{(n)}(\xi) - F^{\omega}_{(n)}(a)\right) = 0.\]

Since \(x^*\) is arbitrary, the theorem is proved.

**Theorem 3.2.5.** If \(F^{\omega}_{(n)}\), \(n \geq 1\) exists in \([a,b]\) then \(F^{\omega}_{(k)}\) are strongly measurable for \(k = 1, 2, \ldots, n\).

**Proof.** Since \(F\) is weakly continuous, it is strongly measurable by Theorem 1.2.3. Since \(F^{\omega}_{(n)}\) exists in \([a,b]\) for each \(t \in [a,b]\) and each \(x^* \in X^*\)

\[
\lim_{h \to 0} x^* \frac{1}{h} \left[ F(t+h) - F(t) \right] = x^* F^{\omega}_{(n)}(t).
\]
Taking any sequence \( \{ h_r \} \) which converges to 0 we get a sequence of strongly measurable functions \( \left\{ \frac{1}{h_r} \left[ F(t+h_r) - F(t) \right] \right\} \) which converges to \( F_\omega(t) \) weakly everywhere. So, by [15, p.74, Theorem 3.5.4], \( F_\omega \) is strongly measurable. Thus the theorem is true for \( n = 1 \). Suppose that it is true for \( n = m-1 \). Then, since \( F_\omega \) is strongly measurable for \( k = 0, 1, \ldots, m-1 \) and since by the existence of \( F_\omega \), we have

\[
\lim_{h \to 0} \mathcal{X}^* \left[ \frac{m!}{h^n} \left\{ F(t+h) - F(t) - h F_\omega(t) - \ldots - \frac{h^{m-1}}{(m-1)!} F_\omega^{(m-1)}(t) \right\} \right] = \mathcal{X}^* F_\omega(t)
\]

for each \( t \in [a,b] \) and each \( \mathcal{X}^* \), applying similar argument as above, \( F_\omega \) is strongly measurable. The proof is thus completed by induction.

3. The \( C_n D_* P \) -integral

**Definition 3.3.1.** A function \( f : [a,b] \to X \) is said to be weakly \( A C_n G_* \) function \( F : [a,b] \to X \) such that \( D_p^{n+1} F \) exists in \([a,b]\) and \( D_p^{n+1} F = f \) on \([a,b]\). Then \( F_\omega(t) \) is called an indefinite \( C_n D_* P \) integral of \( f \) and \( F_\omega(b) - F_\omega(a) \) is its definite \( C_n D_* P \) integral in \([a,b]\) and is denoted by

\[
(C_n D_* P) \int_a^b f(t) \, dt.
\]
The definite integral of an integrable function is unique by Theorem 3.2.4. Clearly by Theorem 3.2.5 an indefinite $C_nD_P$-integral is strongly measurable. It can be verified that the class of all $C_nD_P$-integrable functions in $[a,b]$ is a linear space and the $C_nD_P$-integral is a linear operator from this linear space to $X$, and this operator is additive on abutting intervals. In Chapter I we have defined the $D_P$-integral of a function $f : [a,b] \to X$ to be $F(b) - F(a)$ if there exists a weakly $ACG_+$ function $F : [a,b] \to X$ such that $D^*_PF = f$ on $[a,b]$. Clearly $f$ is $D_P$-integrable in $[a,b]$ if and only if $f$ is $C_nD_P$-integrable and the integrals are equal.

**Theorem 3.3.2.** The function $f$ is $C_nD_P$-integrable over $[a,b]$ if and only if there is a function $F : [a,b] \to X$ such that $F_\omega$ exists in $[a,b]$ and $x^*F_\omega$ is an indefinite $C_nD$-integral of $x^*f$ for each $x^* \in X^*$. Further, we have

$$x^*(F_{(n)}(b) - F_{(n)}(a)) = (C_nD) \int_a^b x^*f(t) \, dt.$$  

**Proof.** Let $f$ be $C_nD_P$-integrable. Then there is a weakly $ACG_+$ function $F : [a,b] \to X$ such that $D^{n+1}_PF = f$ in $[a,b]$. So, for each $x^* \in X^*$, $x^*F$ is $ACG_+$ and $(x^*F)_{(n+1)} = x^*f$ a.e. and since

$$(x^*F)_{(n)} = x^*F_\omega,$$  

so by the definition of $C_nD$-integral
we see that $x^* F_{(n)}^\omega$ is an indefinite $C_n D$ -integral of $x^* f$ and

$$x^* F_{(n)}^\omega(b) - x^* F_{(n)}^\omega(a) = (C_n D) \int_a^b x^* f(t) \, dt.$$  

Conversely, if $F_{(n)}^\omega$ exists and $x^* F_{(n)}^\omega$ is an indefinite $C_n D$ -integral of $x^* f$ for each $x^* \in X^*$ then since $(x^* F)_{(n)} = x^* F_{(n)}^\omega$, $x^* f$ is $A_n G_\infty$. Also since $(x^* F)_{(n)}$ is an indefinite $C_n D$ -integral of $x^* f$ we have $(x^* F)_{(n+1)} = x^* f$ a.e. and hence $D_{-1}^{n+1} f = f$. This completes the proof.

**Theorem 3.3.3.** If $f$ is $C_n D, P$ -integrable then $f$ is weakly measurable.

**Proof.** By Theorem 3.3.2, for arbitrary $x^* \in X^*$, the real valued function $x^* f$ is $C_n D$ -integrable and hence is $C_n P$ -integrable and so $x^* f$ is measurable [4]. Hence, $x^*$ being arbitrary, $f$ is weakly measurable.

**Theorem 3.3.4.** If $f$ is $L^P$-integrable then $f$ is $C_\infty D, P$ integrable and the integrals are equal.

**Proof.** Let $x^* \in X^*$ be arbitrary and $F(t) = \int_a^t f(f) \, df$. Then $x^* f$ is Lebesgue integrable with indefinite integral $x^* F$. So $x^* f$ is $C_\infty P$ -integrable with $C_\infty P$ -integral $x^* F$. Since $F_{(n)}^\omega = F$ the result follows by Theorem 3.3.2.
Theorem 3.3.5. A $C_{n-1} D_\ast P$ integrable function $f$ is $C_n D_\ast P$ integrable and the integrals are equal.

Proof. Let $f$ be $C_{n-1} D_\ast P$ integrable in $[a,b]$. Let $F$ be weakly $A C_{n-1} G_\ast$ and $D^n_p F = f$ in $[a,b]$. Since $F$ is weakly continuous, it is $LB$-integrable by Theorem 1.2.3 and hence $LP$-integrable. Let $G(t) = \int_a^t F$. Since $D^n_p F$ exists, for each $x^* \in X^*$ there is $E_{x^*} \subset [a,b]$ of measure zero such that for $\xi \notin E_{x^*}$

$$x^* \left[ F(t) - F(\xi) - \sum_{i=1}^n \frac{(t-\xi)^i}{i!} D^i_p F(\xi) \right] = o((t-\xi)^n) \quad (t \to \xi).$$

Hence since $F$ is $LP$-integrable,

$$x^* \left[ G(t) - G(\xi) - (t-\xi)F(\xi) - \sum_{i=1}^n \frac{(t-\xi)^i}{(i+1)!} D^i_p F(\xi) \right] = o((t-\xi)^{n+1}) \quad (t \to \xi).$$

Hence $D^{n+1}_p G = D^n_p F$. It can be shown that, since $F^{\omega}_{(n-1)}$ exists, $(x^* G)_{(n)} = (x^* F)_{(n-1)} = X^* F^{\omega}_{(n-1)}$ and hence $G^{\omega}_{(n)} = F^{\omega}_{(n-1)}$ and that, since $F$ is weakly $A C_{n-1} G_\ast$, $G$ is weakly $A C_n G_\ast$. Thus $f$ is $C_n D_\ast P$ integrable. Since $G^{\omega}_{(n)} = F^{\omega}_{(n-1)}$, the result follows.

Theorem 3.3.6. A $C_n D_\ast B$ integrable function is $C_n D_\ast P$ integrable.

This is obvious since strongly $A C_n G_\ast$ implies weakly $A C_n G_\ast$ and existence of strong Peano derivative implies the existence of weak Peano derivative.
Theorem 3.3.7. If $f$ is $C_n D_+ P$ -integrable and $F(t) = \int_a^t f$, then $F$ is $C_{n-1} D_+ P$ -integrable in $[a,b]$.

Proof. If $f$ is $C_n D_+ P$ -integrable then by Theorem 3.3.2 there is $\phi : [a,b] \to \mathcal{X}$ such that $\Phi^{\omega}_{(n)}$ exists in $[a,b]$ and $(x^* \phi)_{(n)}$ is an indefinite $C_n D_+ -integral of $x^* f$ for each $x^* \in \mathcal{X}^*$. Hence $\Phi^{\omega}_{(n-1)}$ exists in $[a,b]$ and

$(x^* \phi)_{(n-1)}$ is an indefinite $C_{n-1} D_+ -integral of $x^* F(t) = \int_a^t x^* f$ for each $x^* \in \mathcal{X}^*$. Hence $F$ is $C_{n-1} D_+ P$ -integrable by Theorem 3.3.2.

4. Integration by parts

Theorem 3.4.1. Let $f$ be $C_n D_+ P$ -integrable and let $F = \int_a^t f$. If $G : [a,b] \to \mathcal{R}$ is such that $G^{(n)}$ is absolutely continuous, then $fG$ is $C_n D_+ P$ -integrable in $[a,b]$ and

$$(C_n D_+ P) \int_a^b fG = [FG]_a^b - (C_{n-1} D_+ P) \int_a^b FG' .$$

Proof. We shall first prove the theorem for $n = 1$. Let $\phi : [a,b] \to \mathcal{X}$ be such that $\Phi^{\omega}_{(1)} = F$ and $D_p \phi = f$ and $\phi$ is weakly $\mathcal{A} C_1 G_\ast$. Since $\phi$ is weakly continuous and $G^{(1)}$ is continuous, so $\Phi G^{(1)}$ is weakly continuous.
and so by Theorem 1.2.3 it is LB-integrable and so LP-integrable. Now let
\[ \Psi(f) = \phi(f) g(f) - \int_a^b \phi(t) g''(t) \, dt \]
and \( x^* \in X^* \) be arbitrary. Then
\[ (x^* \Psi)_{(i)} = (x^* \phi)_{(i)} g + x^* \phi_{(i)} g_{(i)} - x^* \phi g_{(i)} \]
\[ = (x^* \phi)_{(i)} g \]
\[ = x^* (\phi_{(i)} g) \]
\[ = x^* F G \]

Therefore \( \Psi_{(i)} = F G \) and also (cf. [4])
\[ (x^* \Psi)_{(ii)} = x^* F G' + G x^* f \quad a.e. \]
\[ = x^* (F G' + f G) \quad a.e. \]

So, \( D^*_p \Psi = F G' + f G \).

Now since \( \phi \) is weakly \( AC_1 G_* \) and since
\[ x^* \Psi(f) = x^* \phi(f) g(f) - \int_a^b x^* \phi(t) g''(t) \, dt \]
by Theorem 2.4.4, \( \Psi \) is weakly \( AC_1 G_* \). Hence \( FG' + fG \)
is \( C, D_x F \) -integrable and
\[ [\Psi_{(ii)}] = (C, D_x F) \int_a^b (FG' + fG) \]

Now by Theorem 3.3.7, \( F \) is \( C_0 \, D_* \, P \) integrable and so \( F^c \) is \( C_0 \, D_* \, P \) integrable by Theorem 1.5.4 and this completes the proof for \( n = 1 \).

Now we assume the theorem for \( n = m - 1 \) and prove it for \( n = m \). The theorem will then follow by induction. Let 
\[
\phi : [a, b] \rightarrow X
\]
be such that \( \phi_{(m)} = F \) and \( D_{P}^{m+1} \phi = f \) and \( \phi \) is weakly \( A \, C_m \, G_\ast \). Since \( \phi \) is weakly continuous and \( G_{(m)} \) is continuous, so \( \phi \, G_{(r)} \) is \( \text{LB-integrable by Theorem 1.2.3} \) and so \( \text{LP-integrable, for } r = 1, 2, \ldots, m \). Setting 
\[
\psi(f) = \phi(f) 
\]
we get for arbitrary \( \chi^* \in X^* \)
\[
\chi^{*} \psi(f) = \chi^{*} \phi(f) = \sum_{r=1}^{m} \frac{(-1)^r}{(r-1)!} \int_{a}^{f} \, (\chi - t)^{r-1} \phi(t) \, G^{(r)}(t) \, dt,
\]
we get for arbitrary \( \chi^* \in X^* \)
\[
\chi^* \psi(f) = \chi^* \phi(f) = \sum_{r=1}^{m} \frac{(-1)^r}{(r-1)!} \int_{a}^{f} \, (\chi - t)^{r-1} \chi^* \phi(t) \, G^{(r)}(t) \, dt.
\]
By Lemma 2.4.5 \( \chi^* \psi_{(m)} = \chi^* (fG) \) for all \( \chi^* \) i.e. \( \psi_{(m)} = F \, G \) and \( (\chi^* \psi)_{(m+1)} = \chi^* (F \, G^{(1)} + fG) \) a.e. for all \( \chi^* \) i.e. \( D_{P}^{m+1} \psi = F \, G^{(1)} + fG \).

Also by Theorem 2.4.4, \( \psi \) is weakly \( A \, C_m \, G_\ast \). Hence \( F \, G^{(1)} + fG \) is \( C_m \, D_* \, P \) integrable and
\[
(F \, G)(t) = (C_m \, D_* \, P) \int_{a}^{t} (F \, G^{(1)} + fG).
\]
Now by Theorem 3.3.7, \( F \) is \( C_{m-1} \, D_* \, P \) integrable and since \( G^{(m-1)} = G^{(m)} \) is absolutely continuous, \( F \, G^{(1)} \) is \( C_{m-1} \, D_* \, P \) integrable and hence by Theorem 3.3.5, it is \( C_m \, D_* \, P \) integrable. So we have
\[
[ F \, G ]_{a}^{b} = (C_{m-1} \, D_* \, P) \int_{a}^{b} F(t) \, G^{(1)}(t) \, dt + (C_m \, D_* \, P) \int_{a}^{b} f(t) \, G(t) \, dt,
\]
completing the proof.
Example 3.5.1. There exists a \( C_0 \, D_\star \, P \) integrable function which is not LP-integrable.

Proof. Let \( f \) be an everywhere finite real valued function on \([0,1]\) which is \( D_\star \)-integrable but not \( L \)-integrable and let \( F(t) \) be its indefinite integral with \( F(0) = 0 \). Let \( \{ c_n \} \subseteq L_2 \) be fixed. Define \( g : [0,1] \to L_2 \) by

\[ g(t) = \{ c_n f(t) \}, \quad t \in [0,1] \]

and \( G : [0,1] \to L_2 \) by

\[ G(t) = \{ c_n F(t) \}, \quad t \in [0,1] \]

Now, if \( x^* \in L_2^+ = L_2 \) then there exists a sequence \( \{ d_n \} \subseteq L_2 \) such that

\[ x^* g(t) = f(t) \sum c_n d_n = A f(t) \]

where \( \sum c_n d_n = A \). Since \( f(t) \) is not \( L \)-integrable so \( x^* g(t) \) is not \( L \)-integrable. So \( g \) is not LP-integrable. On the other hand \( f(t) \) being \( D_\star \)-integrable \( x^* g(t) \) is \( D_\star \) integrable and

\[
\int_0^f x^* g(t) \, dt = \int_0^f A f(t) \, dt = A \int_0^f f(t) \, dt = A F(f) = F(f) \sum c_n d_n = x^* G(f).
\]

Since \( G(\omega) = G \), by Theorem 3.3.2, \( g(t) \) is \( C_0 \, D_\star \, P \) integrable on \([0,1]\).

Example 3.5.2. For each \( n > 0 \) there exists a \( C_n \, D_\star \, P \) integrable function which is not \( C_{n-1} \, D_\star \, P \) integrable.
Proof. Let \( f \) be a real valued finite function in \([0,1]\) which is \( C_n P \) integrable but not \( C_{n-1} P \) integrable. Then there is a real valued function \( \phi \) in \([0,1]\) such that \( \phi \) is \( A C_n G \) and \( \phi_{(n)}(t) \) is the indefinite integral of \( f(t) \) with \( \phi_{(n)}(c) = 0 \). Let \( \{c_r\} \in l_2 \). Define the function \( g \) and \( \psi \) on \([0,1]\) with values in \( l_2 \) such that
\[
\begin{align*}
g(t) &= \{ c_r f(t) \}, t \in [0,1] \\
\psi(t) &= \{ c_r \phi(t) \}, t \in [0,1].
\end{align*}
\]
Then the strong Peano derivative \( \psi_{(i)} \) exists at each point where \( \phi_{(i)} \) exists and
\[
\psi_{(i)} = \{ c_r \phi_{(i)} \} \quad \text{for} \quad i = 1, 2, \ldots, n+1.
\]
Let \( x^* \in l^*_1 = l_2 \). Then there is a sequence \( \{ d_r \} \in l_2 \) such that
\[
x^* g(t) = f(t) \sum c_r d_r = A f(t).
\]
Since \( f(t) \) is not \( C_{n-1} P \) integrable, so \( x^* g(t) \) is not \( C_{n-1} P \) integrable. Hence by Theorem 3.3.2, \( g(t) \) is not \( C_{n-1} D_p P \) integrable.

Again since \( f(t) \) is \( C_n P \) integrable, we see \( x^* g(t) \) is \( C_n P \) integrable on \([0,1]\) and
\[
\begin{align*}
\int_0^1 x^* g(t) dt &= \int_0^1 A f(t) dt = A \int_0^1 f(t) dt = A \phi_{(n)}(f) \\
&= \sum c_r \phi_{(n)}(f) d_r = x^* \psi_{(n)}(f).
\end{align*}
\]
So, by Theorem 3.3.2, \( g(t) \) is \( C_n D_p P \) integrable on \([0,1]\).