CHAPTER II

THE CESARO-DENJOY-BOCHNER SCALE OF INTEGRATION

1. Introduction

Extension of special Denjoy integral to Cesaro-Perron integrals for real valued functions is due to J.C. Burkill [8, 9] who introduced a scale of Perron type integrals, the $C_nP$ integrals, such that the strength of the integrals is increased with $n$. Sargent [20, 21] also extended the special Denjoy integral to the Cesaro-Denjoy integrals for real valued function by introducing a scale of Denjoy type integrals, the $C_nD$ integrals. Sargent also proved that the $C_nP$ -integrals and the $C_nD$ -integrals are equivalent. We extend the Denjoy-Bochner integral introduced in Chapter I to Cesaro-Denjoy-Bochner integral for vector valued function by introducing a scale of integrals, the $C_nD_{XB}$ integrals, which are such that the strength of the integrals is increased with $n$.

2. Preliminaries

Definition 2.2.1. Let $F: [a, b] \to \chi$ and let $t_0 \in [a, b]$. Let $n$ be a positive integer. If there are constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ depending on $t_0$ but not on $h$ such that
\( F(t_0 + h) - F(t_0) = h \alpha_1 + \frac{h^2}{2!} \alpha_2 + \cdots + \frac{h^n}{n!} \alpha_n = O(h^n) \)

then \( \alpha_n \) is called the strong Peano derivative of \( F \) at \( t_0 \), of order \( n \) and is denoted by \( F^{(n)}(t_0) \). It is easily seen that if \( F^{(n)}(t_0) \) exists then \( F^{(k)}(t_0) \) (\( 1 \leq k \leq n \)) exists. In particular \( F^{(n)}(t_0) \) is the strong derivative of \( F \) at \( t_0 \). We shall write for convenience \( F^{(n)}(t_0) = F(t_0) \).

Let \( n \) be any non-negative integer. If \( F^{(n)}(t_0) \) exists, we write

\[
E_n(t_0, t) = E_n(F; t_0, t) = \frac{n!}{(t-t_0)^n} \left[ F(t) - \sum_{k=0}^{n} \frac{(t-t_0)^k}{k!} F^{(k)}(t_0) \right], \quad t \neq t_0
\]

Clearly then \( E_n(F; t_0, t) \) is strongly continuous. Let \( F^{(n)} \) exist in \([a, b]\). Let

\[
\Omega_n(c, d) = \max \left\{ \| a \| \| E_n(c, t) \|, \| a \| \| E_n(d, t) \| \right\}, \quad a \in \mathbb{R}, \quad c \leq t \leq d \leq a
\]

Then \( F \) is said to be strongly \( AC \) over a set \( E \subset [a, b] \) if \( F^{(n)} \) exists in \([a, b]\) and for arbitrary \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for all finite collection of non-overlapping intervals \((a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\) with end points in \( E \) and \( \sum_{r=1}^{m} (b_r - a_r) < \delta \) we have

\[
\sum_{r=1}^{m} \Omega_n(a_r, b_r) < \varepsilon.
\]
The function $F$ is said to be strongly $C_n G_*$ in $[a,b]$ if $F$ is strongly continuous on $[a,b]$ and if we can write $[a,b] = \bigcup E_i$, where $E_i$'s are non-overlapping closed sets and $F$ is strongly $C_n G_*$ over each $E_i$. (Note that the continuity of $F$ is necessary only for $n=0$. For $n>1$ it is a consequence of the existence of $F_{(n)}$).

The definition of $C_n G_*$ for real functions is as in [19, p.231] or [7] according as $n=0$ or $>0$.

**Lemma 2.2.2.** If $F: [a,b] \to \mathbb{R}$ is such that at a point $t_0 \in [a,b]$ $F_{(n)}(t_0)$ exists then for each $x^* \in X^*$, $(x^*F)_{(n)}(t_0)$ exists and $(x^*F)_{(n)}(t_0) = x^*(F_{(n)}(t_0))$. If $F$ is strongly $C_n G_*$ in $[a,b]$ then for all $x^* \in X^*$, $x^*F$ is a numerical valued $C_n G_*$ function in $[a,b]$.

Clearly $(x^*F)_{(1)}(t_0) = x^*(F_{(1)}(t_0))$. Supposing $(x^*F)_{(r)}(t_0)$ to be true it can be proved that $(x^*F)_{(r+1)}(t_0) = x^*(F_{(r+1)}(t_0))$ for each $x^* \in X^*$ and so the first part is clear. Also since for each $x^* \in X^*$

$$|\varepsilon_n(x^*F; t_0, t)| = |x^*\varepsilon_n(F; t_0, t)| \leq \|x^*\| \|\varepsilon_n(F; t_0, t)\|$$

we have

$$\omega_n(x^*F; c, d) \leq \|x^*\| \omega_n(F; c, d)$$

and hence the result follows.
Lemma 2.2.3. Let $F : [a, b] \rightarrow \chi$ be strongly continuous and let $F_{(n)}$ exist in $[a, b]$ and let

$$\phi(t) = \int_t^b F(f) df$$

for $a \leq t \leq b$;

then

i) $\phi_{(n+1)}(f) = F_{(n)}(f)$ for all $f \in [a, b]$

ii) If $F_{(n+1)}$ exists at a point $t_0$, then $\phi_{(n+2)}(t_0)$ exists and equals $F_{(n+1)}(t_0)$.

iii) If $F$ is $A_{C_n}$ over a set $E \subset [a, b]$ then $\phi$ is $A_{C_{n+1}}$ over $E$.

Proof. Let $t_0$ and $t_0 + h$ belong to $[a, b]$. Now

$$(2.2.1) \quad \phi(t_0 + h) - \phi(t_0) = \int_{t_0}^{t_0 + h} F(f) df$$

$$= \int_{t_0}^{t_0 + h} [F(t_0) + (f - t_0) F_{(1)}(t_0) + \cdots + \frac{(f - t_0)^n}{n!} F_{(n)}(t_0) + \frac{(f - t_0)^{n+1}}{(n+1)!} \varepsilon_{n}(F; t_0, f)] df$$

$$= h F(t_0) + \frac{h}{2!} F_{(1)}(t_0) + \cdots + \frac{h^{n+1}}{(n+1)!} F_{(n+1)}(t_0)$$

$$+ \int_{t_0}^{t_0 + h} \frac{(f - t_0)^n}{n!} \varepsilon_{n}(F; t_0, f) df.$$

Since $\|\varepsilon_{n}(F; t_0, t_0 + h)\| \to 0$ as $h \to 0$, we have

$$\lim_{h \to 0} \frac{(n+1)!}{h^{n+1}} \int_{t_0}^{t_0 + h} \frac{(f - t_0)^n}{n!} \varepsilon_{n}(F; t_0, f) = \Theta.$$ 

So, by (2.2.1) $\phi_{(n+1)}(t_0)$ exists and equal to $F_{(n)}(t_0)$.
Exactly as above we can show that $\phi_{(n+1)}$ exists and equal to $F_{(n+1)}$ at those points $t_o$ where $F_{(n+1)}$ exists.

Now by (2.2.1) we have

$$E_{n+1}(\phi; t_o, t_o+h) = \frac{(n+1)!}{h^{n+1}} \int_{t_o}^{t_o+h} \frac{(h-t)}{n!} E_n(F; t_o, f) df$$

from which (iii) follows.

**Lemma 2.2.4.** Let $n > 0$ and $F: [a, b] \rightarrow \mathbb{R}$ be such that $F_{(n)}$ exists in $[a, b]$ and $F$ be strongly $AC_{n+}$ in a closed set $E \subset [a, b]$. Then $F$ is strongly $AC_{n-}$ in $E$.

**Proof.** It can be shown as in [21, Lemma 1] that if $a < c < d < b$ then

$$\| F_{(n)}(d) - F_{(n)}(c) \| \leq A \omega_n(F; c, d)$$

where $A$ depends only on $n$. Hence if $c, d \in E$ then

$$\left| \| F_{(n)}(d) \| - \| F_{(n)}(c) \| \right| \leq A \omega_n(F; c, d)$$

and therefore the real valued function $\| F_{(n)} \|$ is absolutely continuous on $E$. The set $E$ being closed, $\| F_{(n)} \|$ is bounded on $E$. Let $M$ be the upper bound of $\| F_{(n)} \|$ on $E$. Then since

$$E_{n-1}(F; t_o, t_1) = \frac{t_1 - t_o}{n} \left[ F_{(n)}(t_o) + E_n(F; t_o, t_1) \right]$$

we have

$$\omega_{n-1}(F; c, d) \leq \frac{|d - c|}{n} \left[ M + \omega_n(F; c, d) \right]$$
whenever \( c, d \in E \). Thus \( F \) is strongly \( AC_{n-1} \) on \( E \).

**Corollary 2.2.5.** If \( F \) is strongly \( AC_{n} G_{+} \) on \([a,b]\) then \( F \) is strongly \( AC_{n-1} G_{+} \) on \([a,b]\).

**Lemma 2.2.6.** If a function \( \phi : [a,b] \to X \) is strongly \( AC_{n} G_{+} \) and \( \phi_{(n+1)} = \theta \) almost everywhere then \( \phi_{(n)} \) is constant.

**Proof.** Let \( x^{*} \in X^{*} \) be arbitrary. By Lemma 2.2.2 \( x^{*} \phi \) is numerical valued \( AC_{n} G_{+} \) function and moreover

\[
(x^{*} \phi)_{(n+1)} = x^{*} \phi_{(n+1)} = x^{*} \theta = 0
\]

almost everywhere. So, \( (x^{*} \phi)_{(n)} \) is constant [7, Corollary 17]. But \( \phi_{(n)} \) exists and \( (x^{*} \phi)_{(n)} = x^{*} \phi_{(n)} \).

Since \( x^{*} \) is arbitrary, the lemma is proved.

3. The \( AC_{n} D_{+} B \) –integral

**Definition 2.3.1.** A function \( f : [a,b] \to X \) is said to be \( AC_{n} D_{+} B \) (Cesaro-Denjoy-Bochner) integrable if there exists a function \( F : [a,b] \to X \) which is strongly \( AC_{n} G_{+} \) on \([a,b]\) and \( F_{(n+1)} = f \) almost everywhere in \([a,b]\). Then \( F_{(n)} \) is called an indefinite \( AC_{n} D_{+} B \) –integral of \( f \).
and \( \int_{c_n}^b - \int_{c_n}^a \) is called its definite \( C_n D_B \) integral in \([a,b]\) and is denoted by,

\[
(C_n D_B) \int_a^b f(t) \, dt.
\]

The definite integral of an integrable function is unique by Lemma 2.2.6. It can be verified that the class of all \( C_n D_B \) integrable functions in \([a,b]\) is a linear space and the \( C_n D_B \) integral is a linear operator from this linear space to \( X \), and this operator is additive on abutting intervals.

The \( D_B \) integral, as defined in Chapter I, is such that a function \( f: [a,b] \rightarrow X \) is \( D_B \) integrable in \([a,b]\) if there is a function \( F: [a,b] \rightarrow X \) such that \( F \) is strongly \( AC, G, \) in \([a,b]\) and \( AD F = f \) almost everywhere in \([a,b]\), where \( AD F \) denotes the strong approximate derivative of \( F \). This definition of the \( D_B \) integral is equivalent to that given in [24]. The definition of the \( C_B D_B \) integral here differs with that of \( D_B \) integral in using the strong derivative \( F^{(i)} \) instead of strong approximate derivative \( AD F \). Thus the \( C_B D_B \) integral is less comprehensive than the \( D_B \) integral in Chapter I. Nevertheless, we shall see that the \( C_B D_B \) integral is strictly more general than the Lebesgue-Bochner integral. We remark in passing that the \( C_B D_B \) integral will coincide with the \( D_B \) integral in Chapter I (and hence with the special
Denjoy-Bochner integral in [24]) if the indefinite $D_x B$-integral $F$ satisfies the additional condition that for each $t_0 \in [a, b]$ there is $M = M(t_0)$ such that

$$\| F(t) - F(t_0) \| \leq M |t - t_0|$$

for all $t$ in some neighbourhood of $t_0$, (see Lemma 3 of [2]). On the other hand $C_0 D_x B$-integral is analogous to the DB-integral defined in [22, p.45].

**Theorem 2.3.2.** If $f$ is $C_n D_x B$-integrable then $f$ is strongly measurable.

**Proof.** Let $x^* \in X^*$ be arbitrary. By hypothesis there is a function $F : [a, b] \rightarrow X$ which is strongly $AC_n G_x$ and $F_{(n+1)} = f$ almost everywhere. So, by Lemma 2.2.2 $x^* F$ is $AC_n G_x$ and $(x^* F)_{(n+1)} = x^* f$ almost everywhere. Thus $x^* f$ is $C_n D_x -$integrable (or, equivalently $C_n P -$integrable) scalar function. Hence $x^* f$ is measurable (see [4]). Since $x^*$ is arbitrary, $f$ is weakly measurable.

Now $F(1), F(2), \ldots, F(n)$ exist everywhere and $F_{(n+1)} = f$ almost everywhere. Since $F$ is continuous, the range of $F$ is separable. Let $X_0$ be the closure of the space spanned by the range of $F$. So, $X_0$ is separable. Since

$$\lim_{h \rightarrow 0} \frac{F(t+h) - F(t)}{h} = F_{(1)}(t),$$
so \( F_{(1)}(t) \in X_0 \) for all \( t \in [a, b] \). So, \( \frac{2!}{h^h} [F(t+h) - F(t)] \in X_0 \) for all \( h \neq 0 \). Hence its limit 

\[
F_{(1)}(t) = X_0 \text{ for all } t \in [a, b].
\]

After finite number of steps we can show that \( F_{(1)}(t), F_{(2)}(t), \ldots, F_{(n)}(t) \) belong to \( X_0 \) for all \( t \in [a, b] \). Therefore

\[
\frac{(n+1)!}{h^{n+1}} [F(t+h) - \sum_{k=0}^{n} \frac{h^k}{k!} F_{(k)}(t)] \in X_0
\]

for all \( h \neq 0 \) and for all \( t \in [a, b] \). Hence the limit \( F_{(n+1)}(t) \), if it exists, is in \( X_0 \). Since \( F_{(n+1)}(t) \) exists and equals \( f(t) \in X_0 \) for almost all \( t \in [a, b] \). Thus \( f \) is essentially separably valued.

Hence by [15, p.72, Theorem 3.5.3] \( f \) is strongly measurable.

**Theorem 2.3.3.** If \( f \) is Lebesgue-Bochner integrable in \( [a,b] \) then \( f \) is \( C_0 \)-integrable in \( [a,b] \) and the integrals are equal.

**Proof.** If \( f \) is Lebesgue-Bochner integrable in \( [a,b] \) then its indefinite integral \( F \) is strongly absolutely continuous in \( [a,b] \) [15, p.83, Theorem 3.7.11] and \( f \) is the strong derivative of \( F \) almost everywhere in \( [a,b] \) [15, p.68, Corollary 2]. Since a strongly absolutely continuous function is strongly \( AC_0 \), the result follows.

**Theorem 2.3.4.** If \( f \in C_{n-1} \)-integrable in \( [a,b] \) then it is \( C_n \)-integrable in \( [a,b] \) and the integrals are equal.
Proof. By the $C_{n-1} D^* B$ integrability of $f$ there is a function, say $F$, which is strongly $A C_{n-1} G^*$ and $F_{(n)} = f$ almost everywhere and $F_{(n-1)}$ is the $C_{n-1} D^* B$ indefinite integral of $f$. Since $F$ is strongly continuous, writing

$$\phi(t) = \int_a^t F'(f) \, df$$

we get by Lemma 2.2.3, that the function $\phi$ is strongly $A C_{n} G^*$ and $\phi_{(n)}(t) = F_{(n-1)}(t)$ everywhere and $\phi_{(n+1)}(t) = F_{(n)}(t) = f(t)$ almost everywhere in $[a,b]$. Thus $f$ is $C_{n} D^* B$ integrable in $[a,b]$. The rest is clear.

Theorem 2.3.5. If $f : [a,b] \to X$ is $C_{n} D^* B$ integrable in $[a,b]$ and

$$F(t) = (C_{n} D^* B) \int_a^t f(f) \, df , \, t \in [a,b]$$

then $F$ is $C_{n-1} D^* B$ integrable in $[a,b]$.

Proof. Let $\phi : [a,b] \to X$ be the function which is strongly $A C_{n} G^*$ and $\phi_{(n+1)} = f$ almost everywhere in $[a,b]$. By definition $\phi_{(n)}(t) - \phi_{(n)}(a) = F(t)$ for all $t \in [a,b]$. Also by Corollary 2.2.5, $\phi$ is strongly $A C_{n-1} G^*$ in $[a,b]$. Hence $F(t) + \phi_{(n)}(a)$ is $C_{n-1} D^* B$ integrable in $[a,b]$. Since $\phi_{(n)}(a)$ is a constant, $F$ is $C_{n-1} D^* B$ integrable in $[a,b]$. 
4. Integration by parts for \( C_\eta D_\nu B \)-integral

**Lemma 2.4.1.** If \( n \) is any positive integer then

\[
(2.4.1) \quad \sum_{\gamma=0}^{n} (-1)^{\gamma} \binom{n}{\gamma} \gamma! = 0 \quad \text{for} \quad i = 0, 1, \ldots, n-1
\]

\[
= n! \quad \text{for} \quad i = n
\]

\[
(2.4.2) \quad \sum_{\gamma=0}^{n-k} (-1)^{\gamma} \binom{n}{\gamma} \binom{n-\gamma}{k} = 0 \quad \text{for} \quad k = 0, 1, \ldots, n-1
\]

\[
= 1 \quad \text{for} \quad k = n
\]

**Proof.** The relation (2.4.1) is well known. To prove (2.4.2) we have, when \( 0 \leq k \leq n \)

\[
I = \sum_{\gamma=0}^{n-k} (-1)^{\gamma} \binom{n}{\gamma} \binom{n-\gamma}{k} = \frac{1}{k!} \sum_{\gamma=0}^{n-k} (-1)^{\gamma} \binom{n}{\gamma} \binom{n-\gamma}{k} (n-\gamma-1)(n-\gamma-2)\ldots(n-r-k+1)
\]

\[
= \frac{1}{k!} \sum_{\gamma=0}^{n} (-1)^{\gamma} \binom{n}{\gamma} (n-\gamma)(n-\gamma-1)\ldots(n-r-k+1)
\]

\[
= \frac{1}{k!} \sum_{\gamma=0}^{n} (-1)^{\gamma+k} \binom{n}{\gamma} \left[ \gamma^k + \sum_{j=1}^{K} p_j \gamma^{k-j} \right]
\]

where \( p_1, p_2, \ldots, p_k \) depend on \( n \) and \( k \) and not on \( r \). So, by (2.4.1) we have

\[
I = \frac{(-1)^{n-k}}{k!} \sum_{\gamma=0}^{n} (-1)^{\gamma} \binom{n}{\gamma} \left[ \gamma^k + \sum_{j=1}^{K} p_j \gamma^{k-j} \right]
\]

\[
= \frac{(-1)^{n-k}}{k!} \sum_{\gamma=0}^{n} (-1)^{\gamma} \binom{n}{\gamma} \gamma^k
\]

\[
= 0 \quad \text{if} \quad k = 0, 1, \ldots, n-1,
\]

\[
= 1 \quad \text{if} \quad k = n.
\]
Lemma 2.4.2. Let \( F: [a,b] \to X \) and \( G: [a,b] \to \mathbb{R} \) be two functions such that \( F_{(n)}(f) \), and \( G_{(n)}(f) \) exist. Then the function \( H = GF \) is such that \( H_{(n)}(f) \) exists and
\[
H_{(r)}(f) = \sum_{p=0}^{r} \binom{r}{p} F_{(p)}(f) G_{(r-p)}(f), \quad r = 0, 1, \ldots, n.
\]

Proof. We may suppose \( f = 0 \). Since \( F_{(n)}(0) \) and \( G_{(n)}(0) \) exist, we have
\[
F(h) = \sum_{r=0}^{n} \frac{h^r}{r!} F_{(r)}(0) + O(h^n)
\]
and
\[
G(h) = \sum_{r=0}^{n} \frac{h^r}{r!} G_{(r)}(0) + O(h^n)
\]
and hence
\[
H(h) = G(h) F(h)
\]
\[
= \sum_{r=0}^{n} \frac{h^r}{r!} \left[ \sum_{p=0}^{r} \frac{1}{p!(r-p)!} F_{(p)}(0) G_{(r-p)}(0) \right] + O(h^n)
\]
\[
= \sum_{r=0}^{n} \frac{h^r}{r!} \left[ \sum_{p=0}^{r} \binom{r}{p} F_{(p)}(0) G_{(r-p)}(0) \right] + O(h^n)
\]
and this proves the lemma.

Lemma 2.4.3. Let \( n \) be any positive integer and let \( F: [a,b] \to X \) be strongly \( A \subseteq n \mathcal{G} \) and \( G: [a,b] \to \mathbb{R} \) be such that \( G^{(n)} \) is bounded and \( G \) be \( A \subseteq n \mathcal{G} \) on \([a,b]\). Then \( GF \) is strongly \( A \subseteq n \mathcal{G} \) on \([a,b] \).
Proof. Since $F_{(n)}$ and $G^{(n)}$ exist, by Lemma 2.4.2 $(FG)_{(n)}$ exists. To prove the theorem, we first prove the following relation

$$
(2.4.3) \quad \Delta_n(FG; c, t) = F(t) \Delta_n(G; c, t) + \sum_{r=0}^{n} \left( \frac{n}{r} \right) G_{(r)}(c) \Delta_{n-r}(F; c, t)
$$

where

$$
\Delta_n(F; c, t) = F(t) - F(c).
$$

Applying Lemma 2.4.2 for $n=1$, we have

$$
\Delta_1(FG; c, t) = \frac{1}{t-c} \left[ F(t)G(t) - F(c)G(c) - F_{(1)}(c)G(c)(t-c) - F(c)G_{(1)}(c)(t-c) \right]
$$

Thus $(2.4.3)$ is true for $n=1$. Suppose that it is true for $n-1$. Now for any function $\phi : [a, b] \to \mathbb{R}$ having $n$-th strong Peano derivative we have from the definition of $\Delta_n$,

$$
(2.4.4) \quad \Delta_n(\phi; c, t) = \frac{n}{t-c} \Delta_{n-1}(\phi; c, t) - \phi'(c)
$$

where $n$ is any positive integer. So applying $(2.4.4)$, Lemma 2.4.2 and $(2.4.3)$ for $n-1$ we get

$$
(2.4.5) \quad \Delta_n(FG; c, t) = \frac{n}{t-c} \Delta_{n-1}(FG; c, t) - (FG)_{(n)}(c)
$$

$$
= \frac{n}{t-c} \left[ F(t) \Delta_{n-1}(G; c, t) + \sum_{r=1}^{n-1} \left( \frac{n-1}{r} \right) G_{(r)}(c) \Delta_{n-1-r}(F; c, t) \right]
$$

$$
- \frac{n}{t-c} \sum_{r=1}^{n-1} \left( \frac{n-1}{r} \right) G_{(r)}(c) \Delta_{n-1-r}(F; c, t)
$$

$$
= F(t) \Delta_{n-1}(G; c, t) + G(c) \Delta_{n-1}(F; c, t)
$$

$$
+ \frac{n}{t-c} \sum_{r=1}^{n-1} \left( \frac{n-1}{r} \right) G_{(r)}(c) \Delta_{n-1-r}(F; c, t)
$$

$$
- \sum_{r=1}^{n-1} \left( \frac{n}{r} \right) G_{(r)}(c) \Delta_{n-r}(F; c, t) + G_{(n)}(c) \Delta_n(F; c, t)
$$

where $n$ is any positive integer.
Now using \((2.4.4)\)
\[
\frac{n}{t-c} \frac{(n-1)}{\gamma} \frac{G_{(r)}(c)}{E_{n-r}(F; e, t)} - (\frac{n}{\gamma}) G_{(r)}(c) F_{(n-r)}(c)
\]
\[
= (\frac{n}{\gamma}) G_{(r)}(c) \left[ \frac{n-r}{t-c} E_{n-r}(F; e, t) - F_{(n-r)}(c) \right]
\]
\[
= (\frac{n}{\gamma}) G_{(r)}(c) E_{n-r}(F; e, t).
\]
Hence from \((2.4.5)\)
\[
E_n(F; G; e, t) = F(t) E_n(G; e, t) + \sum_{r=0}^{n} (\frac{n}{\gamma}) G_{(r)}(c) E_{n-r}(F; e, t)
\]
which shows that \((2.4.3)\) is true for \(n\). Thus by induction \((2.4.3)\) is true for all \(n\). Now, let
\[
\|F(t)\| = M
\]
\[
\|G_{(r)}(c)\| = M_r, \quad r = 0, 1, \ldots, n,
\]
Then using relation \((2.4.3)\) and the definition of \(\omega_n\) we get
\[
\omega_n(F; G; e, d) \leq M \omega_n(G; e, d) + \sum_{r=0}^{n} (\frac{n}{\gamma}) M_r \omega_{n-r}(F; e, d).
\]
Now since \(F\) and \(G\) are strongly \(A_{C_{n\gamma}}\), \([a, b] = \bigcup_{k=1}^{n} E_k\) such that \(F\) and \(G\) are \(A_{C_{n\gamma}}\) on each \(E_k\). By Lemma 2.2.4, \(F\) is strongly \(A_{C_{n-r\gamma}}\) on each \(E_k\) for \(r = 1, 2, \ldots, n\). Let \(\epsilon > 0\) be arbitrary and \(E_k\) be fixed. Then there are positive numbers \(\delta_0, \delta_1, \ldots, \delta_n\) and \(\eta > 0\) such that for every finite collection of non-overlapping intervals \([a_i, b_i]\) with end points in \(E_k\) we have
\[ \sum_{i} \omega_n (G; a_i, b_i) \leq \varepsilon \text{ whenever } \sum_{i} (b_i - a_i) < \varepsilon \]

\[ \sum_{i} \omega_n (F; a_i, b_i) \leq \varepsilon \text{ whenever } \sum_{i} (b_i - a_i) < \frac{\varepsilon}{r} \]

for \( r = 0, 1, \ldots, n \). Thus choosing \( \varepsilon = \min \left[ \delta_0, \delta_1, \ldots, \delta_n, \eta \right] \)

we get from (2.4.6)

\[ \sum_{i} \omega_n (FG; a_i, b_i) \leq \left[ M + \sum_{r=0}^{n} \binom{n}{r} M_r \right] \varepsilon \]

whenever \( \sum_{i} (b_i - a_i) < \varepsilon \). Hence \( FG \) is strongly \( AC^n \) on each \( E_k \). Since \( FG \) is strongly continuous, this completes the proof of the lemma.

**Theorem 2.4.4.** Let \( n \) be any positive integer and let \( F : [a, b] \to X \) be strongly \( AC^n G^+ \) and \( G : [a, b] \to R \) be such that \( G^{(n)} \) is \( AC^n G^+ \). Then the function \( \Psi \) defined by

\[ \Psi(\xi) = F(\xi) G(\xi) + \sum_{r=1}^{n} \binom{n}{r} \frac{1}{(r-1)!} \int_{a}^{\xi} (\xi - t)^{r-1} F(t) G^{(r)}(t) dt \]

is strongly \( AC^n G^+ \).

**Proof.** Since \( G^{(n)} \) is \( AC^n G^+ \), \( G^{(r)} \) is the \( n-r \) fold integral of \( G^{(n)} \). Hence by Lemma 2.2.3, \( G^{(r)} \) is \( AC^n G^+ \) for \( r = 0, 1, 2, \ldots, n \). Moreover, \( G^{(n)} \) is bounded and so \( (G^{(r)})^{(n-r)} \) are bounded on \([a, b]\) for \( r = 0, 1, 2, \ldots, n \). Now since \( F \) is \( AC^n G^+ \), by Lemma 2.2.4 it is \( AC^n G^+ \) for \( r = 1, 2, \ldots, n \). Thus by Lemma 2.4.3, \( FG^{(r)} \) is \( AC^n G^+ \) for \( r = 0, 1, 2, \ldots, n \). Now, since \( FG^{(r)} \) is strongly continuous, it is Lebesgue-Bochner
integrable and hence integrating by parts successively it is seen that the integral
\[ \int_a^f (t - \xi)^{-1} F(t) G^{(r)}(t) \, dt \]
is the \( r \)-fold integral of \( F G^{(r)} \), and hence by Lemma 2.2.3 it is strongly \( AC_n G_r \) for \( r = 1, 2, \ldots, n \). Thus every term of \( \psi \) is strongly \( AC_n G_r \) and so \( \psi \) is strongly \( AC_n G_r \) on \([a, b]\).

**Lemma 2.4.5.** Let \( n \) be any positive integer and let \( F: [a, b] \to \mathbb{X} \) and \( G: [a, b] \to \mathbb{R} \) be such that \( F^{(n)} \) exists in \([a, b]\) and \( G^{(n)} \) is continuous in \([a, b]\). Then the function
\[ \psi: [a, b] \to \mathbb{X} \]
defined by
\[ \psi(t) = F(t) G(t) + \sum_{r=1}^{n} (-1)^r \binom{n}{r} \int_a^f (t - \xi)^{r-1} F^{(r)}(t) G^{(r)}(t) \, dt \]
is such that
\[ \psi^{(n)} = F^{(n)} G \quad \text{in} \quad [a, b] . \]
Moreover, if \( F^{(n+1)}(t_0) \) and \( G^{(n+1)}(t_0) \) exist then \( \psi^{(n+1)}(t_0) \) exists and
\[ \psi^{(n+1)}(t_0) = F^{(n+1)}(t_0) G(t_0) + F^{(n)}(t_0) G^{(1)}(t_0) \]

**Proof.** By Lemma 2.4.2, for fixed \( r, 1 \leq r \leq n \) the function
\[ H = F G^{(r)} \]
is such that \( H^{(n-r)} \) exists in \([a, b]\). Let \( t_0 \in [a, b] \)
Then
\[ (2.4.7) \quad H(t) = \sum_{t=t_0}^{t} \binom{n-r}{i} \frac{(t-t_0)^i}{i!} H^{(i)}(t_0) + O((t-t_0)^{n-r}) \]
Now since \( H \) is continuous, integrating by parts successively
\[
(2.4.8) \quad \frac{1}{(\gamma-1)!} \int_{t_0}^{t} (\gamma-t)^{-1} H(t) \, dt = \int_{t_0}^{t} \frac{f \, dt}{t_0} \int_{t_0}^{t} \frac{f \, dt}{t_0} \cdots \int_{t_0}^{t} \frac{f \, dt}{t_0} \, \int_{t_0}^{t} H(t) \, dt
\]
and
\[
(2.4.9) \quad \int_{a}^{t_{0}} (\gamma-t)^{-1} H(t) \, dt = \int_{a}^{t_{0}} \left[ (\gamma-t_{0}) + (t_{0}-t) \right]^{-1} H(t) \, dt
\]
\[
= \sum_{i=0}^{\gamma-1} (\gamma_{i}^{-1}) (\gamma-t_{0}) \int_{a}^{t_{0}} (t_{0}-t)^{-i} H(t) \, dt.
\]
From (2.4.7), (2.4.8) and (2.4.9)
\[
\frac{1}{(\gamma-1)!} \int_{a}^{t} (\gamma-t)^{-1} H(t) \, dt = \frac{1}{(\gamma-1)!} \sum_{i=0}^{\gamma-1} (\gamma_{i}^{-1}) (\gamma-t_{0}) \int_{a}^{t_{0}} (t_{0}-t)^{-i} H(t) \, dt
\]
\[
+ \sum_{i=0}^{\gamma-1} \frac{(\gamma-t_{0})}{(1+i)!} \int_{a}^{t_{0}} (t_{0}-t)^{-i} H(t) \, dt + O((\gamma-t_{0})^{\gamma}).
\]
Thus since \( t_{0} \) is arbitrary
\[
\left[ \frac{1}{(\gamma-1)!} \int_{a}^{t} (\gamma-t)^{-1} H(t) \, dt \right]_{t_{0}} = H_{\gamma}(n-\gamma)
\]
everywhere in \([a,b]\). Therefore by Lemma 2.4.2 and (2.4.2)
\[
\Psi_{(n)}(\gamma) = \sum_{K=0}^{n} \binom{n}{K} F_{(K)}(\gamma) G^{(n-K)}(\gamma) + \sum_{r=1}^{n} (-1)^{r} \binom{n}{r} \sum_{K=0}^{n-r} F_{(K)}(\gamma) G^{(n-K)}(\gamma)
\]
\[
= \sum_{r=0}^{n} (-1)^{r} \binom{n}{r} \sum_{K=0}^{n-r} F_{(K)}(\gamma) G^{(n-K)}(\gamma)
\]
\[
= \sum_{K=0}^{n-\gamma} \left[ \sum_{r=0}^{\gamma} (-1)^{r} \binom{n}{r} \binom{n-r}{K} \right] F_{(K)}(\gamma) G^{(n-K)}(\gamma)
\]
\[
= F_{(n)}(\gamma) G(\gamma).
\]
Also at a point \( t_0 \), where \( F^{(n+1)} \) and \( G^{(n+1)} \) exist, for fixed \( r \), \( 1 \leq r \leq n \), the function \( H = F G^{(r)} \) is such that \( H^{(n-r+1)}(t_0) \) exists and so one gets as above

\[
\Psi^{(n+1)}(t_0) = \sum_{k=0}^{n+1} \binom{n+1}{k} F^{(k)}(t_0) G^{(n-k+1)}(t_0) + \sum_{\gamma=1}^{n} (-1)^{\gamma} \binom{n}{\gamma} \sum_{k=0}^{n-r+1} \binom{n-r+1}{k} F^{(k)}(t_0) G^{(n-k+1)}
\]

\[
= \sum_{\gamma=0}^{n} (-1)^{\gamma} \binom{n}{\gamma} \sum_{k=0}^{n-r+1} \binom{n-r+1}{k} F^{(k)}(t_0) G^{(n-k+1)}
\]

\[
= \sum_{k=1}^{n+1} \left[ \sum_{\gamma=0}^{n-r+1} (-1)^{\gamma} \binom{n-r+1}{\gamma} \right] F^{(k)}(t_0) G^{(n-k+1)} + \sum_{\gamma=0}^{n} (-1)^{\gamma} \binom{n}{\gamma} F^{(n+1)}(t_0) G^{(n+1)}
\]

\[
= \sum_{k=1}^{n+1} \left[ \sum_{\gamma=0}^{n-k+1} (-1)^{\gamma} \binom{n-k+1}{\gamma} \right] F^{(k)}(t_0) G^{(n-k+1)} + F^{(n+1)}(t_0) G^{(n+1)}
\]

Since

\[
\binom{n}{\gamma} = \binom{n+1}{\gamma} - \binom{n}{\gamma-1}
\]

we have

\[
\sum_{\gamma=0}^{n-k+1} (-1)^{\gamma} \binom{n}{\gamma} \binom{n-r+1}{k} = \binom{n+1}{k} + \sum_{\gamma=1}^{n-r+1} (-1)^{\gamma} \left[ \binom{n+1}{\gamma} - \binom{n}{\gamma-1} \right] \binom{n-r+1}{k}
\]

\[
\sum_{\gamma=0}^{n-k+1} (-1)^{\gamma} \binom{n+1}{\gamma} \binom{n-r+1}{k} = \sum_{\gamma=0}^{n-k+1} (-1)^{\gamma} \binom{n+1}{\gamma} \binom{n-r+1}{k} + \sum_{\gamma=0}^{n} (-1)^{\gamma} \binom{n}{\gamma} \binom{n-r+1}{k}
\]

Hence by (2.4.2)

\[
\Psi^{(n+1)}(t_0) = F^{(n+1)}(t_0) G(t_0) + F^{(n)}(t_0) G^{(1)}(t_0)
\]
Theorem 2.4.6. Let \( f : [a, b] \to X \) be \( C_n D_B \) -integrable on \([a, b] \) and
\[
\phi(t) = (C_n D_B) \int_a^t f(x) \, dx
\]
Let \( G : [a, b] \to R \) be such that \( G^{(n)} \) is absolutely continuous on \([a, b] \). Then \( fG \) is \( C_n D_B \) -integrable on \([a, b] \) and
\[
(C_n D_B) \int_a^b fG = [\phi G]_a^b - (C_{n-1} D_B) \int_a^b \phi G^{(1)}.
\]

Proof. We prove the theorem for \( n = 1 \). Let \( F : [a, b] \to X \) be the function such that \( F_{(1)} = \phi \) and \( F_{(2)} = f \) almost everywhere and \( F \) is strongly \( AC \). By Theorem 2.3.5 the function \( \phi \) is \( C_0 D_B \) -integrable. Since \( G^{(1)} \) is absolutely continuous, \( \phi G^{(1)} \) is \( C_0 D_B \) -integrable in \([a, b] \) (cf. Theorem 1.4.3). By Theorem 2.4.4 and Lemma 2.4.5 there exists a function \( \psi : [a, b] \to X \) which is strongly \( AC \) and
\[
\psi_{(1)} = F_{(1)} G = \phi G \quad \text{everywhere and} \quad \psi_{(2)} = F_{(1)} G^{(1)} + G F_{(2)},
\]
almost everywhere, that is \( \psi_{(2)} = \phi G^{(1)} + G f \) almost everywhere. Hence \( \phi G^{(1)} + G f \) is \( C_1 D_B \) -integrable and
\[
\psi(b) - \psi(a) = \int_a^b [\phi(t) G^{(1)}(t) + G(t) f(t)] \, dt,
\]
i.e.
\[
\phi(b) G(b) - \phi(a) G(a) = \int_a^b [\phi(t) G^{(1)}(t) + G(t) f(t)] \, dt.
\]
Now since \( \phi G^{(1)} \) is \( C_0 D_B \) -integrable, by Theorem 2.3.4
\[
[\Phi G]_a^b = (c_{m-D_B})_a^b \Phi G^{(t)} + (c_{m-D_B})_a^b f G
\]

proving the theorem for \( m = 1 \).

Now we suppose that the theorem is true for \( m = m - 1 \) and prove it for \( m = m - 1 \). The proof will then follow by induction. As above, let \( F \) be the function such that \( F_{(m)} = \Phi \) and \( F_{(m+1)} = f \) almost everywhere and \( F \) be strongly \( A C_{m+1} \). Since \( \Phi \) is \( C_{m-D_B} \) -integrable and \( (G^{(t)})^{(m-1)} = G^{(m)} \) is absolutely continuous, by induction hypothesis, \( \Phi G^{(t)} \) is \( C_{m-D_B} \) -integrable. Applying Theorem 2.4.4 and Lemma 2.4.5 there exists a function \( \Psi : [a,b] \to X \) which is \( A C_{m+1} \) and \( \Psi_{(m)} = F_{(m)} G = \Phi G \) everywhere and \( \Psi_{(m+1)} = F_{(m)} G^{(t)} + \int f(t) G G^{(t)} \) almost everywhere. So, \( \Phi G^{(t)} + f G \) is \( C_{m-D_B} \) -integrable and

\[
\Psi_{(m)}(b) - \Psi_{(m)}(a) = \int_a^b [\Phi(t) G^{(t)} + G(t) f(t)] dt.
\]

Now since \( \Phi G^{(t)} \) is \( C_{m-D_B} \) -integrable, by Theorem 2.3.4 it is \( C_{m-D_B} \) -integrable and hence

\[
(c_{m-D_B}) \int_a^b f G = [\Phi G]_a^b - (c_{m-D_B}) \int_a^b \Phi G^{(t)}
\]

proving the theorem for \( m = m \). This completes the proof.

6. Examples

Example 2.6.1. There exists a \( C_{m-D_B} \) -integrable function which is not Lebesgue-Bochner integrable.
Proof. Let $f$ be an everywhere finite real valued function on $[0, 1]$ which is $D_\infty$-integrable but not $L$-integrable. (For the existence of such a function see [19, p.187]). Hence there exists a function $F$ which is $ACG_\infty$ and $F^{(1)} = f$ almost everywhere in $[0, 1]$. Let $\{c_n\} \in l_2$ be fixed. Define $g: [0, 1] \to l_2$ by
\[
g(t) = \{c_n f(t)\}, \quad t \in [0, 1]
\]
and $G: [0, 1] \to l_2$ by
\[
G(t) = \{c_n F(t)\}, \quad t \in [0, 1]
\]
Then if $\sum c_n^2 = k^2$, for each $t \in [0, 1]$
\[
\|g(t)\| = |k| |f(t)|.
\]
Since $f$ is not $L$-integrable, $\|g(t)\|$ is not $L$-integrable in $[0, 1]$ and hence by [15, p.80, Theorem 3.7.4], the function $g$ is not Lebesgue-Bochner integrable on $[0, 1]$. Now
\[
\|G(t_1) - G(t_2)\| = \left[ \sum (c_n F(t_1) - c_n F(t_2))^2 \right]^{1/2}
\]
\[
= |k| |F(t_1) - F(t_2)|
\]
So, $G$ is strongly $ACG_\infty$. Also if at $t = t_0$, $F^{(1)}(t_0) = f(t_0)$ then at $t = t_0$,
\[
\|G(t_0 + h) - G(t_0) - g(t_0)\| = \left[ \sum c_n^2 \left( \frac{F(t_0 + h) - F(t_0)}{h} - f(t_0) \right)^2 \right]^{1/2}
\]
\[
= |k| \left| \frac{F(t_0 + h) - F(t_0)}{h} - f(t_0) \right| \to 0
\]
as $h \to 0$. Hence $G(t) = \emptyset$ almost everywhere in $[0, 1]$. Thus $g$ is $C_0 D_\# B$ -integrable in $[0, 1]$ and

$$(c_0 D_\# B) \int_0^1 g(t) \, dt = G(t)$$

completing the proof.

Now, if $f : [a, b] \to \mathbb{X}$ is LB-integrable then the function $F : [a, b] \to \mathbb{X}$ defined by

$$F(\xi) = (LB) \int_a^\xi f(t) \, dt, \quad \xi \in [a, b]$$

is strongly absolutely continuous [15, p.83, Theorem 3.7.11] and $F(\xi) = f$ almost everywhere in $[a, b]$. Hence $f$ is $C_0 D_\# B$ -integrable and

$$(c_0 D_\# B) \int_a^b f(t) \, dt = (LB) \int_a^b f(t) \, dt$$

Thus from the above example, the $C_0 D_\# B$ -integral is strictly more general than the LB-integral.

Example 2.6.2. For each $n > 0$ there exists a $C_n D_\# B$ integrable function which is not $C_{n-1} D_\# B$ -integrable.

Proof. Let $f$ be a real valued finite function in $[0, 1]$ which is $C_n P$ -integrable but not $C_{n-1} P$ -integrable. (For the definition of $C_r P$ -integral see [9].) For the existence of such a function see [8]. In fact, in [8] a function is given which is CP-integrable but not $D_\#$-integrable.
The same method may be applied to construct a function which is \( C_n \) -integrable but not \( C_{n-1} \) -integrable. So, there is a real valued function \( \phi \) in \([0, 1]\) such that \( \phi \) is \( A \) \( C_n G_* \) and \( \phi_{(n+1)} = f \) almost everywhere in \([0,1]\) [21]. We may suppose \( \phi_{(n)}(0) = 0 \). Let \( C = \{ c_r \} \in \ell_1 \) be fixed such that \( \sum c_r^* = 1 \). Define the function \( g \) and \( \psi \) on \([0, 1]\) with values in \( \ell_1 \) such that

\[
   g(t) = c f(t), \quad t \in [0, 1]
\]

\[
   \psi(t) = c \phi(t), \quad t \in [0, 1].
\]

Then \( \psi_{(i)} = c \phi_{(i)} \), for \( i = 1, 2, \ldots, n+1 \) and

\[
   E_n(\psi; t_0, t) = c E_n(\phi; t_0, t). \quad \text{So,} \quad \psi \text{ is strongly } A C_n G_* \text{ and } \psi_{(n+1)} = g \text{ almost everywhere in } [0, 1]. \text{ Hence } g \text{ is}
\]

\[
   C_n D_\# B \text{ -integrable in } [0, 1] \text{ and }
\]

\[
   \psi_{(n)}(t) = (C_n D_\# B) \int_0^t g(t) \, dt.
\]

If possible, let \( g \) be \( C_{n-1} D_\# B \) -integrable. Then there is \( q : [0, 1] \rightarrow \ell_1 \) such that \( q \) is strongly \( A C_{n-1} G_* \) and \( q_{(n)} = g \) almost everywhere in \([0, 1]\). By Lemma 2.2.3, the function

\[
   G(t) = \int_0^t q(\xi) \, d\xi
\]

is strongly \( A C_n G_* \) and \( G_{(n+1)} = q_{(n)} \) at each point where \( q_{(n)} \) exists. Thus \( \psi_{(n+1)} = G_{(n+1)} \) almost everywhere. Hence by Lemma 2.2.6, \( \psi_{(n)} - G_{(n)} \) is a constant and hence \( \psi_{(n+1)} - q \) is a polynomial of degree at most \( n-1 \). Hence

\[
   E_{n-1}(q; t_0, t) = E_{n-1}(\psi_{(n)}; t_0, t) = c E_{n-1}(\phi_{(n)}; t_0, t).
\]
Therefore, since \( q \) is strongly \( A \mathcal{C}_{n-1} \mathcal{G}_+ \), \( \phi_{(t)} \) is \( A \mathcal{C}_{n-1} \mathcal{G}_+ \). Also if \( q_{(n)}(t_o) \) exists and equals \( q(t_o) \) then since \( \psi_{(t)} \) and \( q \) differ by a polynomial of degree at most \( n-1 \),

\[
\| \frac{n}{t-t_o} \mathcal{E}_{n-1}(q ; t_o, t) - q(t_o) \| = \| \frac{n}{t-t_o} \mathcal{E}_{n-1}(\psi_{(t)}; t_o, t) - q(t_o) \|
\]

\[
= \| \frac{n}{t-t_o} \mathcal{E}_{n-1}(\phi_{(t)}; t_o, t) - f(t_o) \|
\]

which tends to 0 as \( t \to t_o \), showing that \( (\phi_{(t)})'_{(n)}(t_o) = f(t_o) \).

Hence \( (\phi_{(t)})_{(n)} = f \) almost everywhere. Thus \( f \) is \( \mathcal{C}_{n-1} \mathcal{P} \) - integrable in \([0, 1]\) (cf. [21]), which is a contradiction.