CHAPTER I

SPECIAL DENJOY-BOCHNER AND DENJOY-PETTIS INTEGRALS

1. Introduction

The Denjoy-Bochner integral and the Denjoy-Pettis integral which are generalisation of Lebesgue-Bochner and Lebesgue-Pettis integral (see [14, pp. 41-48], [15, pp. 76-85]) are discussed in [1, 22, 24] and in [1, 22] respectively. Alexiewicz [1] studied the general Denjoy-Bochner integral (DB-integral) and the general Denjoy-Pettis integral (DP-integral) which are the Banach space version of the general Denjoy-integral or the D-integral (also called the Denjoy-Khintchine integral). We consider the special Denjoy-Bochner integral (D*B-integral) and the special Denjoy-Pettis integral (D*P-integral) which are the Banach space version of the special Denjoy-integral or the D*-integral (also called the Denjoy-Perron integral). In [24] the author introduced an abstract concept of generalised oscillation for Banach valued functions of intervals and using this he defined an abstract totalisation process for Banach valued functions to obtain Cauchy and Harnack type extension of an integral. By giving a suitable meaning to this
generalised oscillation the definition of the Denjoy-Bochner integral is obtained. Our definition of the $D^*B$-integral and the $D^*P$-integral are analogous to those for the $DB$-integral and $DP$-integral respectively as considered in [1] and agree with the $D^*$-integral [19, p.241] when the Banach space is, in particular, the space of reals. Moreover our $D^*B$-integral agrees with that suggested in [24] (see the remark in p.713 and Theorem 5.4 of [24]).

2. The strongly and weakly $ACG_*$ function

Let $I = [a,b]$ be a fixed interval in $\mathbb{R}$ and $F : I \to X$. Then $F$ is said to be strongly continuous at $\xi \in I$ if

$$\| F(\xi + h) - F(\xi) \| \to 0 \quad \text{as} \quad h \to 0,$$

and $F$ is said to be strongly differentiable at $\xi$ if there is $x(\xi) \in X$ such that

$$\| h^{-1} [ F(\xi + h) - F(\xi) ] - x(\xi) \| \to 0 \quad \text{as} \quad h \to 0,$$

being called the strong derivative of $F$ at $\xi$, which will be denoted by $D_sF(\xi)$. The function $F$ will be said to be strongly approximately differentiable at $\xi$ if there exists $x(\xi) \in X$ such that

$$\lim_{h \to 0} a \| h^{-1} [ F(\xi + h) - F(\xi) ] - x(\xi) \| = 0$$

and $x(\xi)$, in this case, is called the strong approximate derivative of $F$ at $\xi$ and will be denoted by $AD_sF(\xi)$. For $[c,d] \subset I$ let
$\omega^0(F; c, d) = \sup_{c \leq t \leq d} || F(t) - F(c) ||$ \\
$\omega^I(F; c, d) = \sup_{c \leq t \leq d} || F(d) - F(t) ||$ \\
$\omega(C; c, d) = \max \left[ \omega^0(F; c, d), \omega^I(F; c, d) \right]$.

Clearly $F$ is strongly continuous at $\xi$ if and only if

$$\lim_{h \to 0} \omega^0(F; \xi, \xi + h) = 0 = \lim_{h \to 0} \omega^I(F; \xi - h, \xi).$$

The function $F$ is said to be strongly $AC_*$ on a set $E \subset I$ if for arbitrary $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sum_k \omega(C; c_k, d_k) < \varepsilon,$$

for every finite sequence $\{c_k, d_k\}$ of pairwise disjoint intervals with end points in $E$ for which

$$\sum_k (d_k - c_k) < \delta.$$

The function $F$ is said to be strongly $ACG_*$ in $I$ if $F$ is strongly continuous in $I$ and if there is a sequence $\{E_n\}$ of closed sets $E_n \subset I$ such that $I = \bigcup_n E_n$ and $F$ is strongly $AC_*$ on every $E_n$.

The function $F$ is called weakly continuous if the numerical valued function $x^*F$ is continuous for each $x^* \in X^*$.

The function $F$ is said to be pseudo-differentiable on $I$ if there is a function $f : I \to X$ such that

$$\lim_{h \to 0} \frac{x^*F(f + h) - x^*F(f)}{h} = x^*f(f).$$
for almost all \( f \in I \) and for each \( x^* \in X^* \) (the exceptional set of measure zero may vary with \( x^* \)); the function \( f \) is called the pseudo-derivative of \( F \) on \( I \) and will be denoted by \( D_pF \).

The function \( F \) is said to be weakly \( AC^{**} \) on \( I \) if the numerical valued function \( x^*F \) is \( AC^{**} \) \([19, \text{p.231}]\) on \( I \) for each \( x^* \in X^* \).

**Theorem 1.2.1.** If \( F \) is weakly \( AC^{**} \) and \( D_pF = \theta \) on \( I \) then \( F \) is constant.

**Proof.** Since \( F \) is weakly \( AC^{**} \), the numerical valued function \( x^*F \) is \( AC^{**} \) on \( I \) for each \( x^* \in X^* \). Also since \( D_pF = \theta \) on \( I \), for each \( x^* \), \( \frac{d}{df} [x^*F(f)] = 0 \) for almost all \( f \in I \). Hence \( x^*F \) is constant on \( I \) \([19, \text{p.225, Theorem 6.2}]\) for each \( x^* \in X^* \). Since \( x^* \) is arbitrary, \( F \) is constant.

**Corollary 1.2.2.** If \( F \) is strongly \( AC^{**} \) on \( I \) and \( AD_sF = \theta \) almost everywhere in \( I \) then \( F \) is constant.

**Proof.** Since \( F \) is strongly \( AC^{**} \), it is weakly \( AC^{**} \) on \( I \). Also since \( AD_sF = \theta \) almost everywhere on \( I \), for each \( x^* \in X^* \), the approximate derivative \( AD_\left[ x^*F \right] \) of \( x^*F \) vanishes almost everywhere. Since \( x^*F \) is \( AC^{**} \), the ordinary \( \frac{d}{df} [x^*F(f)] \) exists almost everywhere
[19, p.230, Theorem 7.2] and so \( \frac{d}{df} [ x^* f(\xi) ] = 0 \) almost everywhere. Hence \( D_x F = \emptyset \) on \( I \). So, the result follows from Theorem 1.2.1.

**Theorem 1.2.3.** A weakly continuous function \( f : I \to X \) is strongly measurable and bounded on \( I \) and hence \( f \) is LB-integrable.

**Proof.** That a weakly continuous function is strongly measurable has been discussed in [15, p.73]. In fact, if \( \Lambda = \{ f(\xi) : \xi \in I \} \) and if \( \Lambda_1 \), is the set of all convex combination of the members of \( \Lambda \) with rational coefficients then \( \Lambda \) and \( \Lambda_1 \), are countable. If \( \Lambda_2 \) be the set of all convex combinations of the members of \( \Lambda \) with real coefficients, then \( \Lambda_1 \subset \Lambda_2 \) and \( \Lambda_1 \) is dense in \( \Lambda_2 \). Thus \( \Lambda_2 \) is separable. Hence the closure of \( \Lambda_2 \) is also separable. Let \( f(t_0) \in f(I) \). Then there is a sequence of rationals \( \{ \xi_n \} \subset I \) such that \( \xi_n \to t_0 \). Since \( f \) is weakly continuous \( f(\xi_n) \to f(t_0) \) weakly. Hence by [25, p.120, Theorem 2] \( f(t_0) \) is in the closure of \( \Lambda_2 \). Hence \( f(I) \) is contained in the closure of \( \Lambda_2 \) and so \( f(I) \) is separable. Since \( f \) is weakly continuous, it is weakly measurable and so by Pettis measurability theorem [14, p.42] \( f \) is strongly measurable.

Since \( x^*f \) is continuous on \( I \) for all \( x^* \in X^* \), \( x^*f \) is bounded on \( I \) for all \( x^* \). Therefore by uniform boundedness
principle [23, p.202, Theorem 4.4-A] the function $f$ is bounded on $I$.

Theorem 1.2.4. If $F : I \rightarrow \mathbb{X}$ and $G : I \rightarrow \mathbb{R}$ are such that $AD_s F$ and $G'$ exists at $\xi \in I$ then $AD_s [FG]$ exists at $\xi$ and

$$AD_s [FG](\xi) = AD_s F(\xi) G(\xi) + F(\xi) G'(\xi).$$

The proof is usual and hence omitted.

3. The Riemann-Stieltje's integrals

Let $F : I \rightarrow \mathbb{X}$ and $G : I \rightarrow \mathbb{R}$ be given. Let $\Pi : a = t_0 < t_1 < \cdots < t_n = b$ be a partition of the interval $I = [a, b]$ and let $|\Pi| = \max_{1 \leq i \leq n} |t_i - t_{i-1}|$.

Let $T_i \subseteq [t_{i-1}, t_i]$. If the sum

$$S_\Pi (F, G) = \sum_{i=1}^{n} F(T_i) \left[ G(t_i) - G(t_{i-1}) \right]$$

tends to a limit (in the norm topology of $\mathbb{X}$) as $|\Pi| \rightarrow 0$ for all choices of the points $T_i$; then $F$ is said to be strongly Stieltjes-integrable relative to $G$ and the limit is called the Stieltjes integral of $F$ relative to $G$ and is denoted by

$$(s) \int_{a}^{b} F \, dG.$$
The function $F$ is said to be weakly Stieltje's integrable relative to $G$ if there is an element $x_0 \in X$ such that for every $x^* \in X^*$, the function $x^*F$ is Stieltje's integrable relative to $G$ and
\[(s) \int_a^b x^*F \, dG = x^*(x_0).\]

The element $x_0$ is called the weak Stieltjes integral of $F$ relative to $G$ and is denoted by
\[\int_a^b F \, dG.\]

**Theorem 1.3.1.** (Krein) If $F$ is weakly continuous on $[a, b]$ then $F$ is (ws)-integrable relative to every function of bounded variation on $[a, b]$.

For a proof see § 10.4 of [1, p.124].

**Theorem 1.3.2.** If $F : [a, b] \to X$ is weakly continuous and $G : [a, b] \to R$ is absolutely continuous then
\[(ws) \int_a^b F \, dG = (LB) \int_a^b F(t)G'(t) \, dt.\]

**Proof.** By Theorem 1.3.1 the left hand side exists. By Theorem 1.2.3 the function $F$ is bounded and strongly measurable. By [15, p.74, Theorem 3.5.4] $FG'$ is strongly measurable and since
\[\int_a^b \| F(t)G'(t) \| \, dt \leq M \int_a^b |G'(t)| \, dt < \infty\]
where \( M = \sup_{a \leq t \leq b} \| F(t) \| \), by [15, p.80, Theorem 3.7.4]

\( F'G' \) is \( \text{LB-integrable} \) and so the right hand side exists. We show that they are equal. Let \( x^* \in X^* \) be arbitrary. Since \( x^*F \) is continuous, by [17, p.265, Theorem 3]
\[
(s) \int_a^b x^*F \, dG = (L) \int_a^b x^*F(t) \, G'(t) \, dt.
\]

Since \( F \) is weakly Stieltje's integrable relative to \( G \),
\[
(s) \int_a^b x^*F \, dG = x^*(\omega s)\int_a^b F \, dG.
\]

Also since \( FG' \) is \( \text{LB-integrable} \), it is \( \text{LP-integrable} \) and
\[
(L) \int_a^b x^*F(t) \, G'(t) \, dt = x^*(LB)\int_a^b F(t) \, G'(t) \, dt,
\]
(cf. [15, p.80]). Hence
\[
x^*(\omega s)\int_a^b F \, dG = x^*(LB)\int_a^b F(t) \, G'(t) \, dt.
\]

Since \( x^* \) is arbitrary
\[
(\omega s)\int_a^b F \, dG = (LB)\int_a^b F(t) \, G'(t) \, dt.
\]

**Corollary 1.3.3.** If \( F: [a,b] \to X \) is strongly continuous and \( G: [a,b] \to R \) is absolutely continuous then
\[
(s)\int_a^b F \, dG = (LB)\int_a^b F(t) \, G'(t) \, dt.
\]

**Proof.** By [15, p.63, Theorem 3.3.2] the strong Stieltje's integral in the left hand side exists and hence it is equal to \( (\omega s)\int_a^b F \, dG \). Theorem 1.3.2 completes the proof.
If \( E \subseteq X \), we denote by \( \text{Conv} E \), the set of all linear combinations \( \sum_{i=1}^{k} c_i x_i \), of the members \( x_i \) of \( E \) where \( c_i \geq 0 \) for all \( i \), \( 1 \leq i \leq k \), and \( \sum_{i=1}^{k} c_i = 1 \) and by \( \overline{\text{Conv}} E \) the closure of \( \text{Conv} E \) with respect to the norm topology in \( X \).

**Lemma 1.3.4.** Let \( F: [a,b] \to X \) be strongly continuous and let \( G: [a,b] \to R \) be non-decreasing. Then there is \( x_0 \in \overline{\text{Conv}} F([a,b]) \), where \( F([a,b]) \) is the image of \([a,b]\) under \( F \) such that

\[
(s) \int_{a}^{b} F \, d\alpha = [G(b) - G(a)] x_0.
\]

**Lemma 1.3.5.** Let \( F: [a,b] \to X \) be strongly continuous and let \( G: [a,b] \to R \) be non-decreasing. If \( \phi(t) = (s) \int_{a}^{t} F \, d\alpha \) then \( D_s \phi(t) = F(t) G'(t) \) at all points \( t \) at which \( G'(t) \) exists finitely.

Lemma 1.3.4 and Lemma 1.3.5 are proved by Alexiewicz (see \\S\\S 5.2, 5.3, pp.109-110 of [1]).

### 4. The \( D*B \)-integral

**Definition 1.4.1.** A function \( f: I \to X \) is said to be special Denjoy-Bochner integrable or \( D*B \)-integrable on \( I \) if there exists a function \( F: I \to X \) such that \( F \) is strongly
ACG* on I and \( \mathcal{F} \mathcal{D}_s F = f \) almost everywhere in I. The function \( F \) is then called an indefinite \( \mathcal{D}_s B \)-integral of \( f \) on I and \( F(b) - F(a) \) is called its definite integral on I and is denoted by

\[
(\mathcal{D}_s B) \int_a^b f(t) \, dt.
\]

It follows from Corollary 1.2.2 that the indefinite integrals of a function differ by a constant while the definite integral is unique. Clearly if a function \( f \) is LB-integrable then \( f \) is \( \mathcal{D}_s B \)-integrable and the two integrals are equal. For, if \( f : [a, b] \to \mathbb{X} \) is LB-integrable then by [15, p.83, Theorem 3.7.11] the function

\[
F(\xi) = (\mathcal{L} B) \int_a^\xi f(t) \, dt, \quad \xi \in [a, b]
\]

is strongly absolutely continuous and by [15, p.88, Corollary 2] the strong derivative \( \mathcal{D}_s F = f \) almost everywhere in \([a, b]\).

We need the following lemmas.

**Lemma 1.4.2.** Let \( F : [a, b] \to \mathbb{X} \) be strongly ACG* and let \( G : [a, b] \to \mathbb{R} \) be of bounded variation. Then the function \( \phi : [a, b] \to \mathbb{X} \) where

\[
\phi(t) = F(t) G(t) - (s) \int_a^t F \, dG, \quad t \in [a, b]
\]

is strongly ACG*.
Proof. We may suppose that $G$ is non-decreasing. Since $F$ is strongly $\text{AC}^*_c$ on $[a,b]$, there is a sequence of closed sets \{ $E_n$ \} such that $[a,b] = \bigcup E_n$ and $F$ is strongly $\text{AC}^*_c$ on each $E_n$. We show that $\phi$ is strongly $\text{AC}^*_c$ on each $E_n$. Let $\varepsilon > 0$ be arbitrary. Since $F$ is strongly continuous on $[a,b]$ there is $\delta_1 > 0$ such that

\begin{equation}
\| F(\xi) - F(\eta) \| < \frac{\varepsilon}{2[G(b) - G(a)] + 1}
\end{equation}

whenever $|\xi - \eta| < \delta_1$. Also since $F$ is strongly $\text{AC}^*_c$ on $E_n$ there is $\delta_2 > 0$ such that

\begin{equation}
\sum_K O(F; c_K, d_K) < \frac{\varepsilon}{2M+1}
\end{equation}

for every finite sequence \{ $(c_K, d_K)$ \} of pairwise disjoint intervals with end points on $E_n$ for which $\sum_K (d_K - c_K) < \delta_2$, where $M = \max_{t \in \{c_K\}} |G(t)|$. Let $\delta = \min(\delta_1, \delta_2)$. Let \{ $(c_i, d_i)$ \} be any finite sequence of pairwise disjoint intervals with end points on $E_n$ for which $\sum_i (d_i - c_i) < \delta$. Then for any $t \in [c_i, d_i]$ we have by Lemma 1.3.4

\[
\phi(t) - \phi(c_i) = F(t) G(t) - F(c_i) G(c_i) - (s) \int_{c_i}^t F \, dG
\]

\[
= [F(t) - F(c_i)] G(t) + F(c_i) [G(t) - G(c_i)] - (s) \int_{c_i}^t F \, dG
\]

\[
= [F(t) - F(c_i)] G(t) + [G(t) - G(c_i)][F(c_i) - x_0]
\]
where \( x_0 \in \text{Conv} F([c_i, t]) \). Now we claim that

\[
\| F'(c_i) - x_0 \| \leq \sup_{\beta} \| F(\beta) - F(\eta) \|,
\]

\( \forall \beta, \eta \in [c_i, t] \)

For in the contrary case, there are \( x_1, x_2, \ldots, x_n \in F([c_i, t]) \)
and real numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that \( \sum_{j=1}^{n} \alpha_j = 1 \),
\( \alpha_j > 0 \) and

\[
L = \sup_{\beta} \| F(\beta) - F(\eta) \| < \| F(c_i) - \sum_{j=1}^{n} \alpha_j x_j \|,
\]

\[
= \| \sum_{j=1}^{n} \alpha_j \left[ F(c_i) - x_j \right] \|.
\]

\[
\leq \sum_{j=1}^{n} \alpha_j \| F(c_i) - x_j \|.
\]

\[
\leq \sum_{j=1}^{n} \alpha_j L = L,
\]

which is a contradiction. Hence

\[
(1.4.3) \quad \| \phi(t) - \phi(c_i) \| \leq M \| F(t) - F(c_i) \| + \sum_{\eta \in [c_i, t]} \| F(\eta) - F(c_i) \|.
\]

Hence from (1.4.1)

\[
\omega_r(\phi; c_i, d_i) \leq M \omega_r(F; c_i, d_i) + \frac{E \left[ G(d_i) - G(c_i) \right]}{2 \left[ G(b) - G(a) \right] + 1}.
\]

Similarly

\[
\omega_l(\phi; c_i, d_i) \leq M \omega_l(F; c_i, d_i) + \frac{E \left[ G(d_i) - G(c_i) \right]}{2 \left[ G(b) - G(a) \right] + 1}.
\]

Hence

\[
\phi(\phi; c_i, d_i) \leq M \phi(F; c_i, d_i) + \frac{E \left[ G(d_i) - G(c_i) \right]}{2 \left[ G(b) - G(a) \right] + 1}.
\]
So from (1.4.2)
\[ \sum_{i} O(\phi; c_i, d_i) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \]

Hence \( \phi \) is strongly \( AC_{g} \) on each \( E_n \). To show that \( \phi \) is strongly continuous on \([a, b]\), let \( c \in [a, b] \). If \( t \in [a, b] \) then as in (1.4.3)

\[ \| \phi(t) - \phi(c) \| \leq M \| F(t) - F(c) \| + \| G(t) - G(c) \| \sup_{\| f \|} \| F(f) - F(\xi) \| \]

where the 'sup' is taken over all \( f, \xi \) lying in the interval with end points \( c \) and \( t \). Letting \( t \to c \), since \( F \) is strongly continuous at \( c \), we have \( \phi(t) \to \phi(c) \). Thus \( \phi \) is strongly continuous on \([a, b]\) and so \( \phi \) is strongly \( AC_{g} \) on \([a, b]\).

**Theorem 1.4.3.** Let \( f: [a, b] \to X \) be \( D_{*}B \)-integrable and let \( F(f) = (D_{*}B) \int_{a}^{b} f(t) \, dt \). Let \( G: [a, b] \to R \) be of bounded variation. Then \( fG \) is \( D_{*}B \)-integrable on \([a, b]\) and

\[ (D_{*}B) \int_{a}^{b} f(t) G(t) \, dt = F(b)G(b) - F(a)G(a) - (s) \int_{a}^{b} F \, dG. \]

**Proof.** By Lemma 1.4.2 the function

\[ \phi(t) = F(t)G(t) - (s) \int_{a}^{t} F \, dG \]

is strongly \( AC_{g} \). Also since \( \text{AD}_{g} F = f \) almost everywhere, applying Theorem 1.2.4 and Lemma 1.3.5,
A D_{s} \phi = fG + FG' - FG' = fG \quad \text{almost everywhere.}

Hence \( fG \) is \( D_{s}B \) integrable and

\[
(D_{s}B) \int_{a}^{b} f(t) \,G(t) \, dt = \phi(b) - \phi(a) = F(b)G(b) - F(a)G(a) - (5) \int_{a}^{b} F \, dG.
\]

**Corollary 1.4.4.** Under the hypothesis of Theorem 1.4.3 if moreover \( G \) is absolutely continuous then

\[
(D_{s}B) \int_{a}^{b} f(t) \,G(t) \, dt = F(b)G(b) - F(a)G(a) - (5) \int_{a}^{b} F \, dG.
\]

The proof follows from Corollary 1.3.3 and Theorem 1.4.3.

5. The \( D_{s}P \)-integral

**Definition 1.5.1.** A function \( f : I \to X \) is called special Denjoy-Pettis integrable or \( D_{s}P \)-integrable on \( I \), if there exists a function \( F : I \to X \) such that \( F \) is weakly \( A \subset A_{x} \) and \( D_{s}P F = f \) on \( I \). The function \( F \) is called an indefinite \( D_{s}P \)-integral of \( f \) and \( F(b) - F(a) \) is called its definite integral on \( I \) and is denoted by

\[
(D_{s}P) \int_{a}^{b} f(t) \, dt.
\]

From Theorem 1.2.1 it follows that the indefinite \( D_{s}P \)-integrals of a function differ by a constant and the definite
integral is unique. If \( f \) be \( D_*P \)-integrable and \( F \) is its indefinite \( D_*P \)-integral then \( F \) is weakly \( \text{ACG}_* \) and \( D_{p}F = f \) on \( I \). So, for each \( x^* \in X^* \), \( x^*F \) is \( \text{ACG}_* \) and \( x^*f \) is almost everywhere the derivative of \( x^*F \). Hence \( x^*F \) is an indefinite \( D_* \)-integral of \( x^*f \) [19, p.241] and

\[
x^*F(b) - x^*F(a) = (D_*) \int_a^b x^*f(t) \, dt.
\]

Conversely, if for each \( x^* \in X^* \), \( x^*F \) is an indefinite \( D_* \)-integral of \( x^*f \) then \( F \) is weakly \( \text{ACG}_* \) and \( D_{p}F = f \) on \( I \). Hence we get the following characterisation of the \( D_*P \)-integral.

**Theorem 1.5.2.** Let \( f : I \to X \) and \( F : I \to X \). Then \( f \) is \( D_*P \)-integrable with \( F \) its indefinite \( D_*P \)-integral if and only if for each \( x^* \in X^* \), \( x^*f \) is \( D_* \)-integrable with \( x^*F \) as its indefinite \( D_* \)-integral. Further

\[
x^*(F(b) - F(a)) = (D_*) \int_a^b x^*f(t) \, dt.
\]

**Theorem 1.5.3.** If \( f : [a, b] \to X \) is \( D_*B \)-integrable then it is \( D_*P \)-integrable and the integrals are equal.

**Proof.** Let \( f \) be \( D_*B \)-integrable and let \( F \) be an indefinite \( D_*B \)-integral. Then \( F \) is strongly \( \text{ACG}_* \) and hence it is weakly \( \text{ACG}_* \) on \( I \). Let \( x^* \in X^* \) be arbitrary. Then since \( AD_{s}F = f \) almost everywhere \( AD_{s}[x^*F] = x^*f \) almost everywhere in \( I \).
Since $x^*F$ is $A^CG_*$, we conclude $D[x^*F] = x^f$ almost everywhere in $I$ [19, p.230, Theorem 7.2]. So, $x^*f$ is $D_*$-integrable with $x^*F$ its indefinite $D_*$-integral. By Theorem 1.5.2 the result follows.

**Theorem 1.5.4.** Let $f: [a, b] \to X$ be $D_*P$-integrable and let $F(f) = (D_*P) \int_a^b f(t) \, dt$. Let $G: [a, b] \to \mathbb{R}$ be of bounded variation. Then $f G$ is $D_*P$-integrable on $[a, b]$ and

$$
(D_*P) \int_a^b f(t) G(t) \, dt = F(b) G(b) - F(a) G(a) - (ws) \int_a^b F \, dG
$$

**Proof.** Let $x^* \in X^*$ be arbitrary. Since $f$ is $D_*P$-integrable and $F$ is an indefinite $D_*P$-integral, by Theorem 1.5.2, $x^*f$ is $D_*$-integrable and $x^*F$ is an indefinite $D_*$-integral of $x^*f$. So by [19, p.246, Theorem 2.5], $x^*f G$ is $D_*$-integrable. Since $x^*F$ is ACG* and $G$ is of bounded variation in $[a, b]$, the function

$$(1.5.1) \quad x^*F(t) G(t) - (s) \int_a^t x^*F \, dG$$

is ACG* on $[a, b]$ (cf. [19, p.245, Lemma 2.2]). Since $F$ is weakly continuous by Theorem 1.3.1, $F$ is $(ws)$-integrable relative to $G$ and

$$(1.5.2) \quad (s) \int_a^t x^*F \, dG = x^* \left[ (ws) \int_a^t F \, dG \right].$$

Writing

$$\phi(t) = F(t) G(t) - (ws) \int_a^t F \, dG, \quad t \in [a, b]$$
We have from (1.5.1) and (1.5.2)

\[ x^* \phi(t) = x^* F(t) G(t) - (s) \int_a^t x^* F \, d\alpha. \]

So, \( x^\phi \) is ACG and applying [19, p.244, Theorem 2.1(ii)]

\[ (x^\phi)' = (x^F)' G + (x^F)G' - (x^F)G' = x^f \, G \] almost everywhere in \([a,b]\). Hence \( x^\phi \) is an indefinite \( D^- \)-integral of \( x^f \, G \). Since \( x^\phi \) is arbitrary, by Theorem 1.5.2 \( f \, G \) is \( D^-\, P \)-integrable and \( \phi \) is an indefinite \( D^-\, P \)-integral of \( f \, G \). So

\[ (D^-\, P) \int_a^b f(t) G(t) \, dt = \phi(b) - \phi(a) = \int_a^b F \, G(b) - F(a) \, G(a) \]

Corollary 1.5.5. Under the hypothesis of Theorem 1.5.4 if moreover \( G \) is absolutely continuous then

\[ (D^-\, P) \int_a^b f(t) G(t) \, dt = F(b) G(b) - F(a) G(a) - (\omega s) \int_a^b F \, d\alpha. \]

The proof follows from Theorem 1.3.2 and Theorem 1.5.4.