Chapter - III

Integration of vector-valued functions with respect to an operator-valued measure

As mentioned in the introduction this chapter extends the theory of vector integration to the case of vector-valued functions and operator-valued measures. The study of integration of vector-valued functions with respect to an operator-valued measure was made by several authors in [25], [26], [48] and [67]. In all of these papers, either the integrands or the integrals or both have their values in Banach spaces. Debieve [19] has generalized this theory to locally convex spaces. He has considered the normed space valued integrands and the locally convex space valued integrals. An integration theory of similar type has been developed in this chapter. However, our idea of integrability is more general than that of [19]. Also our integration theory includes all the cases cited above.

1. Notations and preliminaries. Throughout this chapter, unless otherwise stated, \( \gamma \) is a \( \delta \)-ring of subsets of a non-empty set \( T \), that is, \( \gamma \) is a collection of subsets of \( T \) closed under relative complement, finite union and countable intersection. \( C(\gamma) \) is the \( \sigma \)-algebra of sets locally in \( \gamma \). Let \( X \) be a normed linear space (n.l.s.) and \( Y \) a locally convex Hausdorff linear topological space (l.c.s.) generated by the family \( \{ q_\beta \} _{\beta \in \mathbb{C}} \) of continuous semi-norms. The scalar field of \( X \) and \( Y \) may be either the real or the complex numbers and is denoted by \( \mathbb{C} \). Let \( X' \) and \( Y \) be the topological duals of \( X \) and \( Y \) respectively, and \( L(X, Y) \) the
space of all continuous linear operators from $X$ to $Y$, equipped with the topology of bounded convergence. The family of semi-norms

$$u \mapsto \| u \|_\beta = \text{Sup} \left\{ q_\beta (u(x)) : \| x \| \leq 1 \right\}$$

generates the topology of bounded convergence on $L(X, Y)$ and under this topology $L(X, Y)$ becomes a locally convex space [64].

**Definition 1.1** An operator-valued measure $\mu : \mathcal{C} \to L(X, Y)$ is an additive set function with

$$\mu \left( \bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu (E_n)$$

for all mutually disjoint sequences $\{ E_n \} \subset \mathcal{C}$ with $\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$, the series being unconditionally convergent with respect to the topology of simple convergence.

**Theorem 1.2** If $\mu : \mathcal{C} \to L(X,Y)$ is an operator-valued measure, then for each $x \in X$, the set function $\mu_x : \mathcal{C} \to Y$, defined by $\mu_x (E) = \mu(E)x$ is a vector measure and conversely, if for each $x \in X$, $\mu(\cdot)x$ is a vector measure, then $\mu : \mathcal{C} \to L(X, Y)$ is countably additive with respect to the topology of simple convergence in $L(X, Y)$.

**Proof** :- Let $\{ E_n \}$ be a sequence of mutually disjoint sets in $\mathcal{C}$ with $\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$. Then for each $\beta \in J$,

$$q_\beta \left( \mu_x \left( \bigcup_{n=1}^{\infty} E_n \right) - \sum_{n=1}^{k} \mu_x (E_n) \right)$$

$$= q_\beta \left( \mu \left( \bigcup_{n=1}^{\infty} E_n \right) x - \sum_{n=1}^{k} \mu (E_n) x \right)$$
Thus we have  

\[
q_\beta \left( \sum_{n=1}^{k} \mu \left( E_n \right) \right) \leq \left\| \mu \left( \bigcup_{n=1}^{\infty} E_n \right) \right\| - \sum_{n=1}^{k} \mu \left( E_n \right) - \sum_{n=1}^{\infty} \mu \left( E_n \right) \right\|_\beta \leq \left\| x \right\|
\]

Therefore 

\[
q_\beta \left( \sum_{n=1}^{k} \mu_x \left( E_n \right) \right) \leq \lim_{k \to \infty} q_\beta \left( \sum_{n=1}^{k} \mu_x \left( E_n \right) \right) = \lim_{k \to \infty} \left\| \mu \left( \bigcup_{n=1}^{k} E_n \right) \right\| - \sum_{n=1}^{k} \mu \left( E_n \right) \right\|_\beta \leq \left\| x \right\|
\]

Hence \( \mu_x : \mathcal{C} \to Y \) is a vector-valued measure.

Conversely, suppose that for each \( x \in X \), \( \mu \left( \cdot \right) x \) is a vector measure. We shall show that \( \mu : \mathcal{C} \to L(X, Y) \) is an operator-valued measure.

Now for each \( \beta \in J \) and a sequence \( \{ E_n \} \) of pairwise disjoint sets in \( \mathcal{C} \) with \( \bigcup_{n=1}^{\infty} E_n \in \mathcal{C} \), we have
\[ \| \mu \left( \bigcup_{n=1}^{\infty} E_n \right) - \sum_{n=1}^{\infty} \mu(E_n) \| = \lim_{\beta} \| \mu \left( \bigcup_{n=1}^{\infty} E_n \right) - \sum_{n=1}^{\infty} \mu(E_n) \| \]

\[ = \lim_{\beta} \sup_{\| x \| \leq 1} q_{\beta} \left( (\mu \left( \bigcup_{n=1}^{\infty} E_n \right) - \sum_{n=1}^{k} \mu(E_n) \right) x \right) \]

\[ = \sup_{\| x \| \leq 1} \lim_{\beta} q_{\beta} \left( (\mu \left( \bigcup_{n=1}^{\infty} E_n \right) - \sum_{n=1}^{\infty} \mu(E_n) \right) x \right) \]

\[ = \sup_{\| x \| \leq 1} q_{\beta} \left( (\mu \left( \bigcup_{n=1}^{\infty} E_n \right) x - \sum_{n=1}^{\infty} \mu(E_n)x) \right) \]

\[ = 0. \]

So \( \mu : \tau \to L(X, Y) \) is an operator-valued measure.

With the help of the above theorem, it can be easily proved that for each \( y' \in Y' \), the set function \( y' \mu : \tau \to X' \) defined by \( (y'\mu)(E)x = y'(\mu(E)x) \) for each \( E \in \tau \) is an \( X' \)-valued measure.

**Definition** 1.3 For each \( \beta \in J \), we define the \( \beta \)-variation of \( \mu \), which is a non-negative, not necessarily finite, countably additive set function on \( C(\tau) \), as

\[ v_{\beta}(\mu, E) = \sup_{1=1}^{n} \| \mu(\bigcap_{1}^{E_n}) \| \beta, \ E \in C(\tau) \]

where the supremum is taken over all finite pairwise disjoint collections \( \{E_i\} \subset \tau \).
For each $y' \in Y'$, we write $v (y', \mu, E)$, the variation of $y' \mu$, as

$$v (y', \mu, E) = \sup_{i=1}^{n} \| y' (E \cap E_i) \|$$

**Definition 1.4** We define the $\beta$-semi-variation of $\mu$ as

$$\hat{\mu}_\beta (E) = \sup_{y' \leq q_\beta \leq Y} v (y', \mu, E), \text{ } E \in C, (\mathcal{C}),$$

which is non-negative and not necessarily finite ($y' \leq q_\beta$ means $|y'(y)| \leq q_\beta (y)$ for all $y \in Y$).

Note that $\hat{\mu}_\beta (E) < \infty$ whenever $v (y', \mu, E) < \infty$ for each $y' \in Y'$ (cf. Lemma 1 of [27]).

2. Integration with respect to an operator-valued measure. In this section we have introduced the theory of integration of normed space valued functions with respect to an operator-valued measure having values in a locally convex space in the sense of Pettis. With the help of this theory Lebesgue dominated convergence theorem is proved here under certain condition. In the remainder of this work, $\mu$ is a fixed $L (X, Y)$-valued measure defined on $\mathcal{C}$ with $v (y', \mu, E) < \infty$ for each $E \in \mathcal{C}$ and $y' \in Y'$. Also the integrands are assumed to be measurable.

**Definition 2.1** If $E \in \mathcal{T}$, then $X_E$ will always denote its characteristic function on $T$. By a $\mathcal{C}$-simple function $f$ on $T$ with values in $X$, we mean a function of the form
\[ f = \sum_{i=1}^{n} x_i X_{E_i}, \]

where \( x_i \in X, \ E_i \in \mathcal{C} \) and \( E_i \cap E_j = \emptyset \) for \( i \neq j, \ 1, j=1,2,\ldots,n. \)

**Definition 2.2** A function \( f : T \rightarrow X \) is said to be \( \mu \)-integrable if

(i) \( f \) is \( y' \mu \)-integrable (in the sense of [25]), and

(ii) for each \( E \in \mathcal{C} (\mathcal{C}), \) there is an element \( y_E \in Y \) such that

\[ y' (y_E) = \int_{E} f(t) y' \mu (dt) \text{ for each } y' \in Y'. \]

If \( f \) is \( \mu \)-integrable, we denote \( y_E \) by \( \int_{E} f(t) \mu (dt). \) It follows from Definition 2.2 that every simple function defined as in 2.1, is \( \mu \)-integrable and the integral of such a function is given by

\[ \int_{E} f(t) \mu (dt) = \sum_{i=1}^{n} \mu (E \cap E_i) x_i. \]

**Lemma 2.3** If \( f : T \rightarrow X \) is \( y' \mu \)-integrable, then \( f \) is \( v(y'\mu,.) \)-integrable.

**Proof:** Since \( f \) is \( y' \mu \)-integrable in the sense of [25], there is a sequence \( \{f_n\} \) of simple functions which converges to \( f \) \( v(y'\mu,.)\)-a.e. Then \( \|f_n\| \) converges \( v(y'\mu,.)\)-a.e. to \( \|f\| \), which implies the result.
Lemma 2.4 If $f : T \to X$ is $y^\mu$-integrable, then

$$\left| \int_E f(t) y^\mu (dt) \right| \leq \int_E \| f(t) \| v (y^\mu, dt)$$

for each $E \in C (\tau)$.

Proof: If $f$ is $y^\mu$-integrable then by Lemma 2.3, $\| f \|$ is $v(y^\mu, \cdot)$-integrable. So, if $\{ f_n \}$ is a defining sequence of simple functions for $y^\mu$-integrability of $f$, then $\| f_n \|$ is a defining sequence corresponding to the function $\| f \|$ and

$$\left| \int_E f_n(t) y^\mu (dt) \right| \leq \int_E \| f_n(t) \| v (y^\mu, dt), E \in C(\tau)$$

which yields the required inequality.

Theorem 2.5. If $f : T \to X$ is a bounded $\mu$-integrable function, then for each $\beta \in J$ and $E \in C(\tau)$,

$$q_\beta \left( \int_E f(t) \mu (dt) \right) \leq \| f \|_T \beta_\mu (E), \text{ where } \| f \|_T = \sup_{t \in T} \| f(t) \|.$$  

Proof: By lemma 2.4, we have

$$q_\beta \left( \int_E f(t) \mu (dt) \right) = \sup_{y \leq q_\beta} \left| \int_E f(t) y^\mu (dt) \right| \leq \sup_{y \leq q_\beta} \int_E \| f(t) \| v (y^\mu, dt) \leq \sup_{t \in T} \| f(t) \| \sup_{y \leq q_\beta} v(y^\mu, E) = \| f \|_T \beta_\mu (E).$$
Theorem 2.6  If $f : T \to X$ is $\mu$-integrable, then the set function defined by

$$\lambda(E) = \int_E f(t) \mu(dt)$$

is a measure on $C(\tau)$.

Proof: If $\{E_n\}$ is a sequence of disjoint members of $C(\tau)$, then for each $y' \in Y'$ we can write as in [25]

$$y' \left( \int_\infty f(t) \mu(dt) \right) = \int_\infty f(t) y' \mu(dt) \bigg|_{E_n} $$

$$= \sum_{n=1}^\infty f(t) y' \mu(dt) \bigg|_{E_n} $$

$$= \sum_{n=1}^\infty y' \left( \int f(t) \mu(dt) \right) \bigg|_{E_n} $$

Thus $\lambda$ is weakly countably additive. Since the same argument applied to any subsequence of $\{E_n\}$ remains valid, it is clear from [54] that $\lambda$ is countably additive.

Theorem 2.7  Let $g : T \to X$ be a $\mu$-integrable function with

$$\lim_{n \to \infty} \int_{E_n} \|g(t)\| \nu(y', \mu, dt) = 0 \text{ uniformly for } y' \leq \nu \beta \text{ for each } \beta \in J$$

and $E_n \not\subseteq \phi$. Let $\{f_n\}$ be a sequence of $\mu$-integrable functions which converges pointwise to $f$ on $T$ and $\|f_n(t)\| \leq \|g(t)\|$ for each $n$. Then $f$ is $\mu$-integrable whenever $Y$ is sequentially complete.
In this case

\[
\int f(t) \mu(dt) = \lim_{n \to \infty} \int f_n(t) \mu(dt),
\]

uniformly for each \( E \in C(\gamma) \).

**Proof:** By applying dominated convergence theorem [25], we see that \( f \) is \( y' \mu \)-integrable and

\[
\int f(t) y' \mu(dt) = \lim_{n \to \infty} \int f_n(t) y' \mu(dt)
\]

for each \( E \in C(\gamma) \).

For fixed \( \varepsilon > 0 \), let \( F_n = \{ t \in T : \|f(t) - f_n(t)\| > \varepsilon \|g(t)\| \} \)

and \( E_n = \bigcup_{k=n}^{\infty} F_k \). Then \( \{E_n\} \) is a decreasing sequence of sets with \( E_n \neq \emptyset \).

Now for each \( \beta \in J \),

\[
q_\beta \left( \int_{E_n} f_n(t) \mu(dt) - \int f(t) \mu(dt) \right) \]

\[
\leq \sup_{y' \leq q_\beta} \int_{E \sim E_n} (f - f_n)(t)y' \mu(dt) + \sup_{y' \leq q_\beta} \int_{E \cap E_n} (f - f_n)(t)y' \mu(dt) \]

\[
+ \sup_{y' \leq q_\beta} \int_{E \sim E_m} (f - f_m)(t)y' \mu(dt) + \sup_{y' \leq q_\beta} \int_{E \cap E_m} (f - f_m)(t)y' \mu(dt) \]
\[ \leq \varepsilon \sup_{y' \leq q_{\beta}} \int_{E \sim E_n} \| g(t) \| v(y', \mu, dt) + 2 \sup_{y' \leq q_{\beta}} \int_{E \cap E_n} \| g(t) \| v(y', \mu, dt) \]
\[ + \varepsilon \sup_{y' \leq q_{\beta}} \int_{E \sim E_m} \| g(t) \| v(y', \mu, dt) + 2 \sup_{y' \leq q_{\beta}} \int_{E \cap E_m} \| g(t) \| v(y', \mu, dt) \]

for all \( n, m \) and \( E \in C(\gamma) \).

So \( \{ \int_{E} f_n(t) \mu(dt) \} \) is Cauchy uniformly with respect to \( E \in C(\gamma) \). Since \( Y \) is sequentially complete, there is an element \( y_E \) in \( Y \) such that
\[ y'(y_E) = y' \left( \lim_{n} \int_{E} f_n(t) \mu(dt) \right) = \int_{E} f(t) y'(\mu, dt). \]

Hence \( f \) is \( \mu \)-integrable and \( \int_{E} f(t) \mu(dt) = \lim_{n} \int_{E} f_n(t) \mu(dt) \).

**Theorem 2.8** If \( Y \) is sequentially complete and \( \hat{\mu}_\beta(\cdot) \) is continuous at \( \emptyset \) on \( C(\gamma) \) for each \( \beta \in J \), then every bounded measurable function \( f : T \rightarrow X \) is \( \mu \)-integrable.

**Proof** :- Since \( f \) is a bounded measurable function, there is a sequence \( \{ f_n \} \) of simple functions such that \( \{ f_n \} \) converges pointwise to \( f \) on \( T \) and \( \| f_n \|_T \leq \| f \|_T \) for \( n = 1, 2, \ldots \ldots \).

Let \( \varepsilon > 0 \) be fixed, \( F_n = \{ t \in T : \| f(t) - f_n(t) \| > \varepsilon \} \) and \( E_n = \bigcup_{k=n}^{\infty} F_k \). Since \( E_n \neq \emptyset \) for each \( y' \in Y \) there exists a positive integer \( n_0 \) such that \( v(y', \mu, E_n) < \varepsilon \) for all \( n \geq n_0 \). Now if we write \( \| f \|_T = M \), then
\[
\int_E \|f(t) - f_n(t)\| v(y', \mu, dt) \\
\leq \int E \|f(t) - f_n(t)\| v(y', \mu, dt) + \int E \|f_n(t) - f_m(t)\| v(y', \mu, dt)
\]
\[
\leq \varepsilon v(y', \mu, E \sim E_n) + 2 M v(y', \mu, E \cap E_n)
\]
\[
\leq \varepsilon (v(y', \mu, E \sim E_n) + 2 M) \text{ for } n \geq n_0.
\]

So f is \(y', \mu\)-integrable and \(\int f(t) y' \mu(dt) = \lim_{n \to \infty} \int f_n(t) y' \mu(dt)\)
for each \(y' \in Y'\).

Since \(\hat{\mu}_\beta(\cdot)\) is continuous at \(0\) on \(C(Y)\), there is a positive integer \(N\) such that \(\hat{\mu}_\beta(\cdot) < \varepsilon\) for \(n \geq N\) and therefore
\[
q_\beta \left( \int E f_n(t) \mu(dt) - \int E f_m(t) \mu(dt) \right)
\]
\[
= \sup_{y' \leq q_\beta} \left| \int E f_n(t) y' \mu(dt) - \int E f_m(t) y' \mu(dt) \right|
\]
\[
\leq \sup_{y' \leq q_\beta} \int E (f_n(t) - f(t)) y' \mu(dt) + \int E (f(t) - f_m(t)) y' \mu(dt)
\]
\[
\leq \sup_{y' \leq q_\beta} \int E \|f(t) - f_n(t)\| v(y' \mu, dt) + \sup_{y' \leq q_\beta} \int E \|f(t) - f_m(t)\| v(y' \mu, dt)
\]
\[
\leq \varepsilon (\hat{\mu}_\beta(E \sim E_n) + 2 M) + \varepsilon (\hat{\mu}_\beta(E \sim E_m) + 2 M)
\]
for all \(n, m \geq N\) and \(E \in C(Y)\).

This inequality establishes that \(f\) is \(\mu\)-integrable.

**Corollary 2.9** Let \(f : T \to X\) be a \(\mu\)-integrable function such that
\[
\lim_{n \to \infty} \int E (f(t)) v(y' \mu, dt) = 0 \text{ uniformly for } y' \leq q_\beta, E_n \setminus \emptyset.
\]
If \(Y\) is
sequentially complete, then $\phi.f$ is $\mu$-integrable for every bounded scalar measurable function $\phi$.

Proof: Without loss of generality we may suppose that $|\phi(t)| \leq 1$ for each $t \in T$. Since $\phi$ is scalar measurable, we can choose a sequence of scalar $\gamma$-simple functions $\{\phi_n\}$ which converges to $\phi$ on $T$, for which $|\phi_n(t)| \leq 1$ for all $t \in T$.

For each $n$, let $E_n = \{ t \in T : \|f(t)\| \leq n \}$. If $f_n = f \chi_{E_n}$ then $\{f_n\}$ is a sequence of bounded integrable functions converging pointwise to $f$. So $\{\phi_n . f_n\}$ is a sequence of integrable functions which converges pointwise to $\phi . f$.

Moreover,

$$\| (\phi_n . f_n) (t) \| = |\phi_n(t)|\|f_n(t)\| \leq \|f(t)\|.$$ 

Hence by applying Theorem 2.7 we see that $\phi . f$ is $\mu$-integrable.

Theorem 2.10 Let $Y$ be sequentially complete and let $f : T \to X$ be $\gamma'$ $\mu$-integrable and such that $\lim\sup_{E_n} \|f(t)\| \vee (\gamma' \mu, dt) = 0$ uniformly for each $\gamma' \leq q_\beta$ and $E_n \setminus \phi$. Then following statements are equivalent:

(i) $f$ is $\mu$-integrable.

(ii) There is a sequence $\{f_n\}$ of bounded measurable functions which converges pointwise to $f$ and for which $\left\{ \int_E f_n(t) \mu(dt) \right\}$ is Cauchy uniformly with respect to $E \in C(\gamma)$.
(iii) There is a sequence \( \{ f_n \} \) of simple functions which converges pointwise to \( f \) and for which the integrals \( \int_{E} f_n(t) \mu(dt) \) form a Cauchy sequence uniformly with respect to \( E \in \mathcal{C}(\tau) \).

(iv) There is a sequence \( \{ f_n \} \) of simple functions which converges pointwise to \( f \) and for which the integrals \( \int_{E} f_n(t) \mu(dt), n = 1, 2, \ldots \) are uniformly countably additive on \( \mathcal{C}(\tau) \).

**Proof:** \((i) \Rightarrow (ii)\). If \( f \) is \( \mu \)-integrable then the sequence \( \{f_n\} \) considered in corollary 2.9 is a sequence of bounded measurable functions which satisfies the conditions given in Theorem 2.7 and (ii) follows.

\((ii) \Rightarrow (iii)\). Since \( \{ f_n \} \) is a sequence of bounded measurable functions, for each \( n \) there is a sequence of simple functions \( \{ f_{n,k} \} \) such that \( \{ f_{n,k} \} \) converges pointwise to \( f_n \) and \( \| f_{n,k} \|_T \leq \| f_n \|_T \) for \( k = 1, 2, \ldots \). If \( g_n = f_{n,n} \) then the sequence \( \{ g_n \} \) of simple functions converges pointwise to \( f \) on \( T \) and \( \| g_n \|_T \leq \| f_n \|_T \). Since \( \{ \int_{E} f_n(t) \mu(dt) \} \) is Cauchy uniformly with respect to \( E \in \mathcal{C}(\tau) \), it is clear that \( \{ \int_{E} g_n(t) \mu(dt) \} \) form a Cauchy sequence uniformly with respect to \( E \in \mathcal{C}(\tau) \).

\((iii) \Rightarrow (iv)\). Let \( \{ f_n \} \) be a sequence of simple functions converges pointwise to \( f \) and for which \( \{ \int_{E} f_n(t) \mu(dt) \} \) is Cauchy uniformly with respect to \( E \in \mathcal{C}(\tau) \). If \( \lambda_n(E) = \int_{E} f_n(t) \mu(dt) \)
then \( \{\lambda_n\} \) is a sequence of \( Y \)-valued measures for which \( \lim_{n} \lambda_n(E) \) exists for each \( E \in C(\gamma) \) and which are \( \mu \)-continuous. So by the Vitali-Hahn-Saks theorem [39], \( \{\lambda_n\} \) is uniformly \( \mu \)-continuous.

Let \( E = \bigcup_{i=1}^{\infty} E_i \), \( E_i \cap E_j = \emptyset \) for \( i \neq j \). If \( F_j = \bigcup_{i=1}^{j} E_i \) then \( \{F_j\} \) is an increasing sequence of sets with \( \lim_j (E \sim F_j) = \emptyset \).

So \( q_\beta (\lambda_n(E \sim F_j)) \to 0 \) uniformly (over \( n \)) as \( j \to \infty \). This shows that \( q_\beta (\lambda_n(E) - \sum_{i=1}^{j} \lambda_n(E_i)) \to 0 \) as \( j \to \infty \), uniformly on \( C(\gamma) \) for \( n = 1,2, \ldots \) and (iii) \( \Rightarrow \) (iv) is proved.

(iv) \( \Rightarrow \) (i). For each \( n \), let us define \( \lambda_n(E) = \int_{E} f_n(t) \mu(dt) \).

If \( \lambda_n \) are uniformly countably additive then by 3.9 of [39], \( \lambda_n \) are uniformly \( \mu \)-continuous.

Let \( E_n \) be defined as in Theorem 2.8. For each \( \varepsilon > 0 \) and \( \beta \in J \),

\[ q_\beta (\int_{E} f_n(t) \mu(dt) - \int_{E} f_m(t) \mu(dt)) \leq \sup_{y' \leq q_\beta E \sim E_n} \|f(t) - f_n(t)\| v(y', \mu, dt) + \sup_{y' \leq q_\beta E \sim E_n} \|f(t)\| v(y', \mu, dt) + q_\beta (\int_{E_n} f_n(t) \mu(dt)) + \sup_{y' \leq q_\beta E \sim E_m} \|f(t) - f_m(t)\| v(y', \mu, dt) + q_\beta (\int_{E_m} f_m(t) \mu(dt)), \]

\[ + \sup_{y' \leq q_\beta E \sim E_m} \|f(t)\| v(y', \mu, dt) + q_\beta (\int_{E_m} f_m(t) \mu(dt)), \]

\[ + \sup_{y' \leq q_\beta E \sim E_m} \|f(t)\| v(y', \mu, dt) + q_\beta (\int_{E_m} f_m(t) \mu(dt)), \]
which shows that \( \{ f_n(t) \mu(dt) \} \) is Cauchy for each \( E \in C(\gamma) \) and since \( Y \) is sequentially complete, \( f \) is \( \mu \)-integrable.

**Remark.** For (i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii), the sequential completeness is superfluous. The implication (iii) \( \Rightarrow \) (i) shows that if \( f \) is integrable in the sense of Debieve [19] then it is also integrable in our sense. We give an example below to show that the converse is not true. Also we have established a relationship between our integrability and Dobrakov integrability [26]. This type of relationship has been studied by Swartz [67]. He has shown that the class of all integrable functions in the sense of [67] coincides with that of Dobrakov [26]. Our class of integrable functions forms a subclass of [67]. This is due to the fact that Swartz assumed the integrability of \( f \) with respect to \( y' \mu \) in the sense of [26], whereas we assume the integrability of \( f \) with respect to \( y' \mu \) in the sense of [25], which is a weaker condition. However, in [67], the domain of \( y' \mu \) is a \( \sigma \)-algebra whereas in our case this is a \( \delta \)-ring.

The following example enables us to get more integrable functions than Debieve.

**Example.** Let \( T \) be the set of all natural numbers, \( \gamma \) the \( \sigma \)-algebra of all subsets of \( T \) and \( X = R \), the space of all real numbers, and \( Y = C_0 \). Let the set function \( \mu \) be defined on \( \gamma \) with values in \( L(X,Y) \) by \( \mu(\{k\}) = x \epsilon_k \), where \( x \epsilon R \), \( k \epsilon T \) and \( e_k = (0,0,\ldots,0,1,0,\ldots) \epsilon C_0 \) and let \( \mu(E) = \sum_{k \epsilon E} \mu(\{k\}) \) for \( E \epsilon \gamma \). Then \( \mu \) is an operator-valued measure countably additive in the topology of simple convergence.
Let us define the function \( f : T \to X \) by \( f(k) = \frac{1}{k} \) and the functions \( f_n \) by \( f_n(k) = \left( \frac{1}{k} \right) x_{E_n}(k) \), where \( E_n = 1, 2, \ldots, n \) for all \( k \in T \). Then \( \{f_n\} \) is a sequence of \( y'\mu \)-integrable functions converging pointwise to \( f \). So by Theorem 3 of [25] (p.136), \( f \) is \( y'\mu \)-integrable and \( \int_E f(t) y'\mu(\text{d}t) = \lim \int f_n(t) y'\mu(\text{d}t) \) for each \( E \in \mathcal{E} \). Now

\[
\int f_n(t) \mu(\text{d}t) = \sum_{k=1}^{\infty} f_n(t) \mu(\text{d}t) = \sum_{k=1}^{\infty} \frac{1}{k} x_{E_n}(t) \mu(\text{d}t) = \sum_{k=1}^{n} \mu(\{k\}) \frac{1}{k} = \frac{e_k}{k}.
\]

This shows that \( \lim \int f_n(t) \mu(\text{d}t) \) exists for each \( E \in \mathcal{E} \).

Since \( \lim \int f_n(t) \mu(\text{d}t) \) exists, it is clear that \( f \) is \( \mu \)-integrable.

But \( f \) is not \( \mu \)-integrable in the sense of Debieve, since \( \hat{\mu}(\cdot) \) is not continuous at \( \emptyset \), as \( \hat{\mu}(E) = 1 \) for all \( E \in \mathcal{E} \) (cf. Prop. 6, [19]).

3. Relation between \( \mu \)-integrability and \( v_B(\mu, \cdot) \)-integrability. In this section the relationship between the \( \mu \)-integrability and the integrability with respect to \( v_B(\mu, \cdot) \) is investigated.
**Definition 3.1** A function $f : T \to X$ is said to be integrable with respect to a non-negative measure $\nu$ if there is a sequence of $\tau$-simple functions $\{f_n\}$ converging to $f$ $\nu$-a.e. such that

$$
\lim_{m, n \to \infty} \int \|f_n(t) - f_m(t)\| \nu(dt) = 0
$$

for each $E \in C(\tau)$.

**Theorem 3.2.** If $f$ is $\nu_\beta(\mu, \cdot)$-integrable for each $\beta \in J$, then $f$ is $\mu$-integrable whenever $Y$ is sequentially complete.

Moreover,

$$
\nu_\beta \left( \int f(t) \mu(dt), E \right) \leq \int_E \|f(t)\| \nu_\beta(\mu, dt)
$$

for each $E \in C(\tau)$ ($\nu_\beta(\cdot)$ denotes the total variation of the integral on the left hand side).

**Proof:** We shall first show that $f$ is $y' \mu$-integrable for each $y' \in Y'$. Now for each $y' \in Y'$ there exists $M > 0$ and a semi-norm $\beta [45]$ such that $\nu(y'\mu, E) \leq M \nu_\beta(\mu, E)$ for all $E \in C(\tau)$, for,

$$
\nu(y'\mu, E) = \sup \sum_{i=1}^{n} \|y'\mu(E \cap E_i)\|$

$$
= \sup \sum_{i=1}^{n} \sup \|y'\mu(E \cap E_i)\|_{\|x\| \leq 1}$

$$
\leq \sup \sum_{i=1}^{n} \sup M q_\beta(\mu(E \cap E_i)\|x\| \leq 1$

$$
= \sup \sum_{i=1}^{n} \|\mu(E \cap E_i)\| \beta$

$$
= M \nu_\beta(\mu, E)$.$
So it is clear that $f$ is $v(y'y,\mu)$-integrable for each $y' \in Y'$ and therefore, there exists a Cauchy sequence $\{f_n\}$ of $v(y'y,\mu)$-integrable simple functions converging to $f$ $v(y'y,\mu)$-a.e. for which

$$\lim_{n, m \to \infty} \|f_n(t) - f_m(t)\| v(y'y, dt) = 0.$$ 

Thus $f$ is $y'\mu$-integrable and $\int_E f(t)y'y(\mu(\mathbb{R})) = \lim_{n \to \infty} \int_E f_n(t)y'y(\mu(\mathbb{R}))$.

Thus, $f$ is $y'\mu$-integrable and $\int_E f(t)y'y(\mu(\mathbb{R})) = \lim_{n \to \infty} \int_E f_n(t)y'y(\mu(\mathbb{R})).$

Since for each $\beta \in J$ and $y' \leq q_\beta$ we have $v(y'y, E) \leq v_\beta(\mu, E)$, and since $\lim_{n, m \to \infty} \|f_n(t) - f_m(t)\| v_\beta(\mu, dt) = 0$, it is clear that

$$\left\{ \int_E f_n(t) \mu(dt) \right\}_{n \in \mathbb{N}}$$

is Cauchy for each $E \in C(\gamma)$. Hence $f$ is $\mu$-integrable.

If $A(E) = \int_E f(t) \mu(dt)$ then $y'\lambda(E) = \int_E f(t)y'y(\mu(\mathbb{R}))$ and

$$q_\beta(\lambda(E)) = \sup_{y' \leq q_\beta} |y'\lambda(E)|$$

$$\leq \sup_{y' \leq q_\beta} \int_E \|f(t)\| v(y'y, dt)$$

$$\leq \int_E \|f(t)\| v_\beta(\mu, dt)$$

for each $\beta \in J$.

$$v_\beta(A, E) \leq \sup_{1 \leq i \leq n} \sum_{i=1}^{n} \int_{E_i} \|f(t)\| v_\beta(\mu, dt)$$

$$\leq \int_E \|f(t)\| v_\beta(\mu, dt).$$
4. **Application to weakly compact operators.** The purpose of this section is to show how the theory of vector integration developed here can be applied to represent a weakly compact operator on a continuous functions space and in particular, how the properties of the weakly compact operators are reflected in the behaviour of its representing measure and conversely. This theory was first initiated by Grothendieck [42] on the space of scalar valued continuous functions. The theory of integral representation can be found in [3], [36], [25], [4], [51], [27], [48], [15] and [22]. Also the relationship between an operator and its representing measure has been studied in [15],[65],[6],[7] and [13].

In this section we assume that $T$ is a compact Hausdorff topological space and $C(\mathfrak{B})$ is the smallest $\sigma$-algebra containing $\mathfrak{B}$, where $\mathfrak{B}$ denotes the $\delta$-ring of all compact subsets of $T$. Let us recall that $X$ is a normed linear space and $Y$ a locally convex Hausdorff linear topological space generated by the family of continuous semi-norms $\{q_\beta\}_{\beta \in J}$. Let $X''$ and $Y''$ denote the biduals of $X$ and $Y$ respectively. Here $C(T,X)$ represents the space of all continuous functions from $T$ to $X$ endowed with the topology $\mathcal{J}$ of the usual supremum norm. We shall write $C(T)$ in place of $C(T,X)$ when $X = C$.

**Definition 4.1.** A measure $\mu : C(\mathfrak{B}) \to L(X,Y)$ is said to be regular if for each $\varepsilon > 0$ and $E \in C(\mathfrak{B})$ there exists a compact set $A$ and an open set $B$ such that $A \subset E \subset B$ and $\hat{\mu}_\beta(B \setminus A) < \varepsilon$ for all $B \in J$. 
We recall that a linear operator is weakly compact if it maps bounded subsets into relatively weakly compact subsets.

**Theorem 4.2** Let a continuous linear operator $U : C(T, X) \to Y$ be weakly compact. Then there is a unique measure $\mu : C(f, Y) \to L(X, Y)$ such that

(i) $\mu$ is $X'$-regular, that is, $y'\mu$ is regular for each $y' \in Y$,

(ii) the set $Q = \sum_{i \in I} \mu(E_i) x_i$, $I$ finite, $E_i \in C(f)$ disjoint, $x_i \in X$, $\|x_i\| \leq 1$} is relatively weakly compact,

(iii) every bounded Borel function defined on $T$ is $\mu$-integrable,

(iv) $Uf = \int_T f(t) \mu(dt)$ for $f \in C(T, X)$, and

(v) $U'y' = y'\mu$ for $y' \in Y$.

Conversely, if $\mu$ is an $L(X, Y)$-valued measure which satisfies (i), (ii) and (iii), then (iv) defines a weakly compact operator which satisfies (v).

**Proof**: Since the dual of $C(T, X)$ is isometrically isomorphic to $rcavv(C(f, X'))$, the space of all regular $X'$-valued measures of finite variations defined on $C(f)$, the equation

$$g''(m) = \int_T g(t) m(dt)$$
defines an element of $C(T,X)''$ for each bounded Borel function $g$.

Now, if $U : C(T,X) \to Y$ is weakly compact then $U''$, the second adjoint of $U$, maps $C(T,X)''$ into $Y$. Let us define

$$
\mu (E) x = U'' (x_E)'' \quad \text{for each } E \in C(\emptyset).
$$

It is clear that $\mu (E) : X \to Y$ is linear for each $E \in C(\emptyset)$.

Also for each $y' \in Y'$, $U'y' = \mu_y'$, is a measure in $rcabv(C(\emptyset),X')$.

If $y' \in Y'$ and $x \in X$ then

$$
y' \mu (E) x = y' (U''(x_E)'' ) = (U'y') (x X_E'' ) = \mu_y(E)x
$$

for each $E \in C(\emptyset)$. Thus $y' \mu \in rcaov (C (\emptyset), X')$ for each $y' \in Y'$.

For each $E \in C(\emptyset)$ and $\beta \in J$, $q\beta (\mu(E)x) \leq \hat{\mu}_\beta(E) \| x \|$ shows that $\mu(E) : X \to Y$ is continuous. It is also clear that $\mu$ is countably additive and $U'y' = \mu_y'$, $= y'\mu$, which implies that $\mu$ is $X'$-regular.

Let $V = \{ f : f \in C(T,X) \text{ and } \| f \|_T \leq 1 \}$. Then $V$ is a $J$-bounded subset of $C(T,X)$ and $V^0$ is a neighbourhood of zero in $C(T,X)'$ with respect to the strong topology. Let $G = \{ ( \sum_{i \in I} x_i E_i ) : \}$

I finite, $E_i \cap E_j = \emptyset$, $i \neq j$, $E_i \in C(\emptyset)$, $\| x_i \| \leq 1 \}$. We shall show that $G \subset V^0$. Let $( \sum_{i \in I} x_i X_{E_i} )'' \in G$. For each $f' \in V^0$, let

$$
f' \leftrightarrow m \quad \text{where } m \in rcaov (C (\emptyset), X') . \quad \text{Then we have } |( \sum_{i \in I} x_i X_{E_i} )''(f')| \leq 1
$$

So $( \sum_{i \in I} x_i X_{E_i} )'' \in V^0$ and consequently $G \subset V^0$. Hence $G$ is
equicontinuous (Proposition 6, [45], p. 200) and therefore by (2a) of Theorem 9.3.1 of [35], Q is relatively weakly compact.

Let \( g = \sum_{i=1}^{k} x_i X_{E_i} \) be any \( X \)-valued simple function defined on \( T \). Then \( U^n g = \int_{T} g(t) \mu \, dt \). Thus \( y'(U^n f) = \int_{T} f(t) y' \mu \, dt \)
holds for every bounded Borel function \( f \) and for each \( y' \in Y' \) and therefore it is \( \mu \)-integrable.

Since \( U^n \) is the extension of \( U \), \( Uf = U^n f = \int_{T} f(t) \mu \, dt \) for all \( f \in C(T,X) \).

As concerns the uniqueness of \( \mu \), suppose that \( \mu_1 \) is another \( L(X,Y) \)-valued measure such that \( \int_{T} f(t) \mu_1 \, dt = Uf \) for all \( f \in C(T,X) \).

Then for each \( y' \in Y' \) we have

\[
\int_{T} f(t) y' \mu \, dt = \int_{T} f(t) y' \mu_1 \, dt
\]

for all \( f \in C(T,X) \) and consequently, \( y' \mu = y' \mu_1 \). Hence \( \mu = \mu_1 \) since \( Y \) is a locally convex space.

Conversely, if \( \mu \) satisfies (i), (ii) and (iii), then \( U:C(T,X) \to Y \) defined by (iv) is a continuous linear operator, since for each \( \beta \in J \),

\[
q_{\beta}(Uf) = q_{\beta} \left( \int_{T} f(t) \mu \, dt \right) \leq \| f \|_{T} A_{\beta}(T)
\]
Also $U'y' = y'\mu$ for each $y' \in Y'$, since $\mu$ is $X'$-regular.

To prove that $U$ is weakly compact, let $V$ be the closed convex balanced hull of $Q$. Then $V^0$ is a neighbourhood of zero in $Y'$ with respect to the Mackey topology $\mathcal{J}(Y', Y)$. If $W = \{ f \in C(T, X) : \| f \|_T \leq 1 \}$, then $W^0$ is a neighbourhood of zero in $C(T, X)'$ with respect to $\sigma$-topology, where $\sigma$ is the collection of all bounded subsets of $C(T, X)$.

We shall now show that $U'$ is continuous with respect to $\mathcal{J}(Y', Y)$-topology and the $\sigma$-topology. Let $y' \in V^0$. Then

$$|< y', \sum_{i \in I} \mu(E_i) x_i > | \leq 1 \quad \text{for all } \sum_{i \in I} \mu(E_i) x_i \in Q.$$ 

Since $f$ is $\mu$-integrable, there is a Cauchy sequence $\{ f_n \}$ of simple functions such that $f_n \to f$ and $(y'\mu, \cdot)$-a.e. and

$$\int_T f(t) y' \mu (dt) = \lim_{n \to \infty} \int_T f_n(t) y' \mu (dt) \quad \text{for each } y' \in Y'.$$

Let $h = \sum_{i=1}^r x_i \chi_{E_i}$ be any simple function such that $\| h \|_T \leq 1$.

Now

$$| \int_T h(t) y' \mu (dt) | = \left| \sum_{i=1}^r y' \mu(E_i) x_i \right| \leq 1.$$ 

Therefore, for each $f \in W$,

$$|< U'y', f > | = |< y', Uf > | = | \int_T f(t) y' \mu(dt) | \leq 1.$$
which shows that \( U'y' \in W' \). So \( U'(V^0) = W' \) and consequently, \( U' \)
is continuous with respect to \( T(Y', Y) \)-topology and the \( C \)-topology. Hence by Theorem 9.3.1 of [35], \( U \) is weakly compact.

**Theorem 4.3** Let \( U : C (T, X) \rightarrow Y \) be a continuous linear operator with \( \mu \) its representing measure. If \( U \) is weakly compact then \( \mu(E) : X \rightarrow Y \) is weakly compact for each \( E \in C (\mathcal{M}) \).

Conversely, if \( X' \) and \( X'' \) have the Radon-Nikodým property, \( \hat{\mu}_E (\cdot) \) is continuous at \( \emptyset \) and \( \mu(E) : X \rightarrow Y \) is a weakly compact operator for each \( E \in C (\mathcal{M}) \), then \( U \) is a weakly compact operator whenever \( Y \) is quasicomplete.

**Proof** :- For the "necessary" part, we recall that \( \mu(E)x = U''(x \mathcal{X}_E)'' \), where \( \mathcal{X}_E \) denotes the characteristic function of \( E \). To complete the proof it is enough to show that for each \( E \in C (\mathcal{M}) \), the set \( PE = \{ \mu(E)x : \| x \| \leq 1 \} \) is weakly relatively compact in \( Y \).

If \( V = \{ m : m \in \text{rcabv} (C (\mathcal{M}), X'), |m|(T) \leq 1 \} \) then \( V \) is a neighbourhood of zero in \( \text{rcabv} (C (\mathcal{M}), X') \), where \( |m| (\cdot) \) denotes the total variation of \( m \). Let \( G_E = \{ (x x_E)'' : \| x \| \leq 1, x \in X, E \in C (\mathcal{M}) \} \). Then \( G_E \) is a set of continuous linear functionals defined on \( \text{rcabv} (C (\mathcal{M}), X') \).

Now for \( E \in C (\mathcal{M}) \) and \( x \in X \) with \( \| x \| \leq 1 \) we have
This shows that $G_E \subset V^O$ and consequently, $G_E$ is equicontinuous. Hence by Theorem 9.3.1 of [35], $\{ U''(x_E)'' : \| x \| \leq 1 \}$ is relatively weakly compact, and therefore $P_E$ is relatively weakly compact.

Conversely, suppose that $X'$ and $X$ have the Radon-Nikodým property and $\mu(E): X \rightarrow Y$ is a weakly compact operator for each $E \in C(\beta)$. Here $U : C(T,X) \rightarrow Y$ is defined by $Uf = \int \int f(t) \mu(dt)$ for all $f \in C(T,X)$.

Since $C(T,X)' \equiv \text{rcabv}(C(\beta), X')$ we have $U'y' \leftrightarrow \mu_y'$ for each $y' \in Y'$, where $\mu_y' \in \text{rcabv}(C(\beta), X')$. Let $M$ be any equicontinuous subset of $Y'$. By Proposition 6 of [45], p. 200 there exists a neighbourhood $V$ of zero in $Y$ such that $M \subset V^O$. Let us consider the set $K = \{ U'y' : y' \in V^O \}$.

Since $U'(M) \subset U'(V^O)$, it is enough to show that $K$ is relatively weakly compact.

Now $K$ is bounded since $\sup_{y' \in V^O} \| \mu_y' \| (T) < \infty$. Since $\hat{\mu}_y'(\cdot)$ is continuous at $\emptyset$, for each $\beta \in J$, Lemma 3.1 and Proposition 3.1 of [15] imply that $\{ \| \mu_y' \| : y' \in V^O \}$ is uniformly countably additive.

Also, since $\mu(E): X \rightarrow Y$ is weakly compact, $\{ \mu(E)'y' : y' \in V^O \} = \{ \mu_y(E) : y' \in V^O \}$ is relatively weakly compact in $X'$. So by Prop.3.1 of [15], $K$ is relatively weakly compact in $\text{ca}(C(\beta), X)$. Thus, by Theorem 9.3.2 of [35], $U$ is weakly compact.

Remark. This theorem generalizes 4.1 of [15] when $Y$ is a locally convex space.